Complex Numbers — Basic Definitions and Properties

A complex number is specified by a pair of real numbers \(x, y \in \mathbb{R}\) — we write \(z = x + iy\). The set of all such numbers is denoted \(\mathbb{C}\), where, by definition, \(x + iy = u + iv\) if, and only if, \(x = u\) and \(y = v\). Since \(z\) is uniquely specified by the pair \((x, y)\), we can identify the set \(\mathbb{C}\) with the plane \(\mathbb{R}^2\), with \(z = x + iy\) corresponding to the point \((x, y) \in \mathbb{R}\). The modulus of \(z\) is defined to be \(|z| = \sqrt{x^2 + y^2}\), i.e. the length of the corresponding position vector in \(\mathbb{R}^2\). If \(z = x + iy\), then the numbers \(x\) and \(y\) are referred to as the real and imaginary parts of \(z\), respectively, and denoted by \(\text{Re} \ z\) and \(\text{Im} \ z\). We think of \(\mathbb{R}\) as being a subset of \(\mathbb{C}\), where \(x \in \mathbb{R}\) is used as shorthand for the complex number \(x + 0i\). Similarly, \(iy\) is shorthand for \(0 + iy\). Note that

\[
|z| > 0 \iff \text{Re} \ z = 0 \text{ and } \text{Im} \ z = 0 \iff |z| = 0.
\]

We also have a polar representation for \(z = x + iy\). We can write \(z = r \cos \theta + ir \sin \theta\) where \(r = |z|\) and, if \(z = 0\), \(\theta\) is arbitrary, otherwise it is determined up to a multiple of \(2\pi\). The algebraic structure on \(\mathbb{C}\) is defined as follows:

\[(x + iy) + (u + iv) = (x + u) + i(y + v); \quad (x + iy)(u + iv) = (xu - yv) + i(xv + yu).
\]

If we put \(z = u = 0\) and \(y = v = 1\) in the second of these we get \(i^2 = -1\). If we define \(e^{i \theta} := \cos \theta + i \sin \theta\), then the product of \(z = re^{i \theta}\) and \(w = Re^{i \phi}\) can conveniently be expressed as \(zw = rR e^{i(\theta + \phi)}\). This follows from standard trigonometric identities. It is a routine matter to check the following properties:

**Commutativity** \(z_1 + z_2 = z_2 + z_1\), \(z_1 z_2 = z_2 z_1\)

**Associativity** \(z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3\), \(z_1 (z_2 + z_3) = (z_1 z_2) z_3\)

**Distributivity** \(z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3\)

**Additive and Multiplicative Identities** \(0 + z = z\), \(1 \cdot z = z\)

**Existence of Inverses** \(z + (-z) = 0\) where \(-z := (-x) + i(-y)\), and, if \(z \neq 0\), \(zz^{-1} = 1\), where \(z^{-1} := x/(x^2 + y^2) - iy/(x^2 + y^2) = \overline{z}/|z|^2\).

We write \(z - w\) for \(z + (-w)\) and \(z/w\) for \(zw^{-1}\). A set with two operations (an ‘addition’ and a ‘multiplication’) satisfying the above conditions is known as a field. Other common examples of fields are \(\mathbb{R}\) and \(\mathbb{Q}\).

The complex conjugate of \(z = x + iy\) is \(\overline{z} := x - iy\). If we write \(z = re^{i \theta}\) then \(\overline{z} = re^{-i \theta}\), since \(\cos(-\theta) = \cos(\theta)\) and \(\sin(-\theta) = -\sin \theta\). The following properties of the modulus and conjugate operations are easy to check:

\[
\begin{align*}
& (i) \quad \overline{\overline{z}} = z \quad (ii) \quad 2 \text{Re} \ z = z + \overline{z}, \quad 2i \text{Im} \ z = z - \overline{z} \quad (iii) \quad |z| = |\overline{z}|, \quad z\overline{z} = |z|^2 \\
& (iv) \quad \overline{z + w} = \overline{z} + \overline{w}, \quad \overline{zw} = z\overline{w} \quad (v) \quad |\text{Re} \ z| \leq |z|, \quad |\text{Im} \ z| \leq |z| \\
& (vi) \quad |z + w| \leq |z| + |w|, \quad ||z| - |w|| \leq |z - w|, \quad |zw| = |z||w|
\end{align*}
\]

For (vi), note that the equality follows from the second half of (iii), and then

\[
|z + w|^2 = (z + w)(\overline{z + w}) = |z|^2 + 2 \text{Re} \ z \overline{w} + |w|^2 \leq |z|^2 + 2|z||w| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2,
\]

where we have used (iii), (ii), (v) and the equality in (vi). Since \(|z + w|\) and \(|z| + |w|\) are both nonnegative real numbers, the first inequality (the triangle inequality) in (vi) follows by taking square roots. For the second inequality note that

\[
|z| = |z - w + w| \leq |z - w| + |w| \Rightarrow |z| - |w| \leq |z - w|,
\]

using the triangle inequality. But also \(|z - w| = |-(w - z)| = |w - z| \geq |w| - |z|\), from which the second inequality then follows. Note that the inequalities in (v) and (vi) are inequalities between real numbers. Indeed, it can be shown that it is impossible to put a sensible order structure on \(\mathbb{C}\) that behaves in a similar way to the order structure on \(\mathbb{R}\).