EXISTENCE OF FELLER COCYCLES ON A $C^*$-ALGEBRA

J MARTIN LINDSAY AND STEPHEN J WILLS

Abstract. The quantum stochastic differential equation $dk_t = k_t \circ \theta^\beta_\cdot \ d\Lambda^\beta_\cdot (t)$ is considered on a unital $C^*$-algebra, with separable noise dimension space. Necessary conditions on the matrix of bounded linear maps $\theta$ for the existence of a completely positive contractive solution are shown to be sufficient. It is known that for completely positive contraction processes, $k$ satisfies such an equation if and only if $k$ is a regular Markovian cocycle. Feller refers to an invariance condition analogous to probabilistic terminology if the algebra is thought of as a noncommutative topological space.

0. Introduction

The quantum stochastic differential equation

$$dk_t = k_t \circ \theta^\beta_\cdot \ d\Lambda^\beta_\cdot (t), \quad k_0(a) = a \otimes 1,$$

was introduced by Evans and Hudson ([EvH]) as a noncommutative generalisation of the SDE’s which govern stochastic flows of diffeomorphisms on a manifold ([Kun]). Noncommutativity appears both in the geometry ([Con]) and the noise ([Par]), since the equation is driven by linear maps acting on an operator algebra, and the differentials of the equation are quantum stochastic processes on a symmetric Fock space.

The data for such equations is a triple $(\mathcal{A}, \eta, \theta)$ consisting of a $*$-algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathfrak{h}$, called the initial space, an orthonormal basis $\eta = (e_i)_{i \geq 1}$ for a separable Hilbert space $k$, called the noise dimension space, and a matrix $\theta = [\theta^\beta_\cdot]_{\alpha, \beta \geq 0}$ of linear maps on $\mathcal{A}$. Viewing $k$ as a subspace of $\hat{k} = \mathbb{C}^\oplus k$, the basis $\eta$ is extended to a basis $\hat{\eta} = (e_\alpha)_{\alpha \geq 0}$ of $\hat{k}$ by putting $e_0 = 1$. With respect to this basis, $\theta$ is the coefficient matrix of a (densely defined) sesquilinear map $\Theta : \hat{k} \times \hat{k} \to L(\mathcal{A})$ and for us $\Lambda = [\Lambda^\beta_\cdot]_{\alpha, \beta \geq 0}$ is the quantum noise matrix of Boson Fock stochastic calculus

$$\theta^\beta_\cdot = \Theta(e_\alpha, e_\beta); \quad \begin{bmatrix} \Lambda^\beta_\alpha (t) & \Lambda^\beta_j (t) \\ \Lambda^\beta_i (t) & \Lambda^\beta_j (t) \end{bmatrix} = \begin{bmatrix} t^1 & a^*(e_j 1_{[0,t]}) \\ a(e_i, 1_{[0,t]}) & d\Gamma(|e_j| \otimes 1_{[0,t]}) \end{bmatrix},$$

$a^*$, $d\Gamma$ and $a$ being respectively Fock space creation, differential second quantisation and annihilation, and $M_{[0,t]}$ being the multiplication operator by the indicator function $1_{[0,t]}$ on $L^2(\mathbb{R}_+)$ ([Par]). In most of the work so far on this equation each $\theta^\beta_\cdot$ is assumed to belong to $B(\mathcal{A})$ and, when $k$ is infinite dimensional, $\mathcal{A}$ is assumed to be a von Neumann algebra and $\theta^\beta_\cdot$ is assumed to be normal. In the current work the former assumption stands, but the latter does not: for us $\mathcal{A}$ is a unital $C^*$-algebra.

The existence of a solution of (0.1) was established by Evans, for finite dimensional $k$, using Picard iteration ([Eva]). The estimates used there were refined by Mohari and Sinha who extended the results to infinite dimensional noise space, and matrices $\theta$ for which each column $\theta^\beta_\cdot := [\theta^\beta_\cdot]_{\alpha \geq 0}$ defines an operator $\mathcal{A} \to B(\mathfrak{h}; h \otimes \hat{k})$ which satisfies the relative boundedness conditions $\|\theta^\beta_\cdot (a) u\| \leq \|(a \otimes 1_h) B^\beta_\cdot a u\|$, for

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some supplementary Hilbert space $K$ and bounded operator $B_\beta$ ([MoS]). In Meyer’s exposition of this result ([Mey]) the validity of the existence theorem is extended to matrices $\theta$ which satisfy the growth conditions

$$
sup\{(n!)^{-1}\gamma^n C^\theta_n(a,u,d)^2 : n \geq 1\} < \infty
$$

for $\gamma > 0$, $a \in A$, $u \in \mathfrak{h}$ and $d \geq 0$, where

$$
C^\theta_n(a,u,d)^2 = \sum_{0 \leq \beta \leq d} \sum_{\alpha_i \geq 0} ||\theta^\alpha_{\beta n} \circ \cdots \circ \theta^\alpha_{\beta 1}(a)||^2.
$$

All of this work focused on the construction of $\ast$-homomorphic solutions to (0.1), and the characterisation of those $\theta$ which satisfy such a solution. Motivated by quantum filtering theory, the search for a noncommutative analogue of the theory of measure-valued diffusion, and the desire to see both flows on a $C^\ast$-algebra and their Markov semigroups from a common viewpoint, attention has broadened to the class of equations whose solutions are completely positive, first for finite dimensional noise ([LiP]), and subsequently for infinite dimensional noise ([LW1]); see also [Bel].

The characterisation of those matrices $\theta$ that generate completely positive contraction flows obtained in [LW1] starts by assuming the existence of a (weakly regular, weak) solution, and it is shown that $\theta$ must be completely bounded. When $A$ is a von Neumann algebra and $\theta$ is normal, if $\theta$ is also completely bounded then it necessarily satisfies the Mohari-Sinha regularity conditions ([LW1], Proposition 1.3), and so the existence assumption holds. Here we address the $C^\ast$-situation.

Weak solutions of (0.1) (which are weakly regular) form Markovian cocycles with respect to the right shift semigroup $(\hat{\sigma}_s)_{s \geq 0}$ on the quantum noise ([LW2], Proposition 5.2). When $A$ is a von Neumann algebra and each $k_i$ is completely bounded and normal, the cocycle identity reads simply

$$
k_{s+t} = \hat{k}_s \circ \sigma_s \circ k_t, \quad k_0(a) = a \otimes 1
$$

where $\hat{k}_s = k_s \otimes \text{id}_{\text{Ran}(\sigma_s)}$ and $\sigma_s = \text{id}_A \otimes \hat{\sigma}_s$ ([Bra]). However, when $A$ is just assumed to be a $C^\ast$-algebra the interpretation of (0.4) must be refined. The resulting definition (see (2.4) below) includes invariance of the $C^\ast$-algebra under semigroups associated with the process — this may be viewed as a Feller type condition if $A$ is thought of as a noncommutative topological space. In ([LW2]) all such Feller cocycles on a unital $C^\ast$-algebra that are Markov regular, completely positive and contractive are shown to strongly satisfy a QSDE of the form (0.1).

The purpose of this note is to prove that matrices $\theta$ whose columns are completely bounded necessarily satisfy (a strong form of) the regularity condition (0.2) identified by Meyer, so that (0.1) has a strong solution. The above mentioned characterisation of completely positive contraction flows is therefore freed of the existence assumption, and thereby attains a more satisfactory form which moreover is manifestly basis independent.

**Notations and conventions.** Greek indices vary over the set $0 \leq \alpha < D$ whereas roman indices vary over $1 \leq i < D$, where $D = 1 + \dim k$; Einstein’s summation convention for repeated indices is used throughout. Algebraic tensor products are denoted $\otimes$. For a function $f$ defined on $\mathbb{R}_+$, and subset $I$ of $\mathbb{R}_+$, $f_I := f1_I$ where $1_I$ is the indicator function of $I$; $L$ is used for spaces of linear maps, and $B$ (respectively $CB$) is used for spaces of bounded (resp. completely bounded) linear operators.

For a vector $e$ in a Hilbert space $\mathfrak{h}$ we use the notation $E_e$ for each bounded Hilbert space operator $u \mapsto u \otimes e$, and $E^e$ for its adjoint. Thus $E_e \in B(\mathfrak{h}; \mathbb{H} \otimes \mathfrak{h})$ and $E^e$ may be thought of as a coordinate map. The Hilbert space $\mathbb{H}$ will always be made clear by the context. Note that the linear maps $e \mapsto E_e$ are isometric $\mathfrak{h} \to B(\mathfrak{h}; \mathbb{H} \otimes \mathfrak{h})$. When an orthonormal basis $(e_i)_{i \in I}$ for $\mathfrak{h}$ is understood, $E^{(i)}$ will denote $E^{e_i}$ for $u = e_i$, and the tensor product $\mathbb{H} \otimes \mathfrak{h}$ will be identified with the
orthogonal direct sum $\bigoplus_{i \in I} H$ by the prescription $\xi \mapsto (E^{(i)} \xi)_{i \in I}$. The resulting partial Parseval identity is useful
\[
\|\xi\|^2 = \sum_{i \in I} \|E^{(i)} \xi\|^2 \quad \forall \xi \in H \otimes h.
\] (0.5)

1. Column spaces

A closed subspace $V$ of $B(H_1; H_2)$, for Hilbert spaces $H_1$ and $H_2$, is called an operator space in $B(H_1; H_2)$. For each $n \geq 1$, $M_n(B(H_1; H_2))$ is naturally identified with $B(H_1^{(n)}; H_2^{(n)})$, where $H^{(n)}$ denotes the $n$-fold orthogonal sum of copies of $H$. In this way a norm is induced on $M_n(V)$. A bounded operator $\phi : V \rightarrow W$ into an operator space $W$ in $B(K_1; K_2)$ induces bounded operators
\[
\phi^{(n)} : M_n(V) \rightarrow M_n(W); \quad [T^*_j] \mapsto [\phi(T^*_j)];
\]
$\phi$ is completely bounded (CB) if its complete bound $\|\phi\|_{cb} := \sup_{n \geq 1} \|\phi^{(n)}\|$ is finite. Standard references for operator spaces and complete boundedness are Paulsen’s book ([Pau1]) and the review article by Christensen and Sinclair ([ChS]); two new books are about to appear ([EfR],[Pis]). However, the proof of our main result is elementary, depending only on the simple lemmas below.

For an operator space $V$ in $B(H_1; H_2)$ and a Hilbert space $h$, we define two operator spaces, the column space $C_h(V)_h$ and the square-matrix space $M_h(V)_h$:
\[
C_h(V)_h := \{ T \in B(H_1; H_2 \otimes h) : E^* T \in V \ \forall e \in h \},
\]
\[
M_h(V)_h := \{ T \in B(H_1 \otimes h; H_2 \otimes h) : E^* T E_e \in V \ \forall d, e \in h \}. \quad (1.1)
\]
The column space $C_h(V)_h$ is an operator space in $B(H_1; H_2 \otimes h)$ lying between the minimal tensor product $V \otimes_{\min} h := V \otimes h$ (closure in the norm topology, [BIP]), and the ultraweak tensor product $V \otimes_{uw} h := \overline{V \otimes h^{uw}}$. It equals the first if $V$ or $h$ is finite dimensional and equals the second if and only if $V$ is ultraweakly closed. The analogous inclusions also hold for the square-matrix space. In particular, $h = C^n$ provides the identifications $M_h(V)_h = V \otimes_{\min} B(h) = V \otimes M_h = M_n(V)$.

Remark. Since the map $e \mapsto E^* e$ is linear and isometric, $T \in B(H_1; H_2 \otimes h)$ belongs to $C_h(V)_h$ provided that $E^* T \in V$ for all vectors $e$ from any total subset of $h$.

When $\mathcal{A}$ is a $C^*$-algebra the operator space $M_h(\mathcal{A})_h$ is typically not an algebra, in particular the inclusion $\mathcal{A} \otimes_{\min} B(h) \subset M_h(\mathcal{A})_h$ is strict. As an example to illustrate this point take $\mathcal{A} = c$, the commutative unital $C^*$-algebra of convergent complex sequences — represented on $l^2$ by diagonal matrices, and let $h = l^2$. Let $T \in B(l^2 \otimes h) = B(\bigoplus_{n=1}^\infty l^2)$ be given by the matrix
\[
\begin{bmatrix}
e_1 & 0 & 0 & \ldots \\
e_3 & 0 & 0 & \ldots \\
e_5 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]
where $e_k = \text{diag}[0, \ldots, 0, 1, 0, \ldots]$ with 1 in the $k$th place and zeros elsewhere, so that $T^* T$ has matrix
\[
\begin{bmatrix}
a & 0 & \cdots \\
0 & 0 & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix},
\]
where $a = \text{diag}[1, 0, 1, 0, \ldots]$. Then $T \in M_h(\mathcal{A})_h$ since $e_k \in c$ for each $k$, but $T^* T \notin M_h(\mathcal{A})_h$ since $a \notin c$. In particular $T \in M_h(\mathcal{A})_h \setminus \mathcal{A} \otimes_{\min} B(h)$. 
Lemma 1.1. Let $V$ be an operator space in $B(H_1; H_2)$ and let $h$ and $k$ be Hilbert spaces. Then $C_b(C_k(V))_{h,b} \cong C_k\otimes h(V)_{h,b}$.

Proof. By the associativity of the Hilbert space tensor product, both are subspaces of $B(H_1; H_2 \otimes k \otimes h)$. For $S \in B(H_1; H_2 \otimes k \otimes h)$

$$E^eS \in C_b(V)_{h,b} \forall e \in h \iff E^{d,e}S \in V \forall e \in h, d \in k,$$

where $E^{d,e} = E^dE^e = E^{d\otimes e}$ implies $E^fS \in V \forall f \in k \otimes h$, by the above remark. The result follows.

By the Parseval identity (0.5), a choice of orthonormal basis $(e_i)_{i \in I}$ for $h$ gives rise to an identification of elements $T$ of $C_b(V)_h$ with bounded operators $H \to \bigoplus_{i \in I} H$ defined by column matrices $[E^{(i)}_e]_{i \in I}$.

Lemma 1.2. Let $V$ and $W$ be operator spaces in $B(H_1; H_2)$ and $B(K_1; K_2)$ respectively, and let $\phi : V \to W$ be a completely bounded map. Then for each Hilbert space $h$ there is a unique map $\phi^{(h)} : C_b(V)_h \to C_b(W)_h$, satisfying

$$E^e\phi^{(h)}(T) = \phi(E^eT) \forall e \in h, T \in C_b(V)_h.$$ (1.2)

Moreover $\phi^{(h)}$ is linear and bounded, and $\|\phi^{(h)}\| \leq \|\phi\|_{cb}$.

Proof. Pick an orthonormal basis $(e_i)_{i \in I}$ of $h$ and for each $T \in C_b(V)_h$, set $T^i = E^{(i)}_T \in V$, so that $T$ is identified with the column matrix $[T^i]_{i \in I}$. Let $h_{00}$ be the following dense subspace of $h$: $\{e \in h : \langle e_i, e \rangle \neq 0 \text{ for only finitely many } i\}$. For each finite subset $F = \{i_1, \ldots, i_n\}$ of $I$, define the $n \times n$ operator matrix

$$T^F := \begin{bmatrix} T^{i_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T^{i_n} & 0 & \cdots & 0 \end{bmatrix};$$

thus $T^F \in M_n(V)$ and $\|T^F\| \leq \|T\|$. For $w \in K_1$, letting $w \in K_1^{(n)}$ denote the column vector associated to $(w, 0, \ldots, 0)$, we have

$$\sum_{F \in I} \|\phi(T^F)w\|^2 = \|\phi^{(n)}(T^F)w\|^2 \leq (\|\phi\|_{cb}\|T\|\|w\|)^2.$$ (1.3)

Hence $(\phi(T^F)w)_{F \in I}$ is a well-defined element of $\bigoplus_{i \in I} K_2 = K_2 \otimes h$. Let $\phi^{(h)}(T)$ be the linear map $K_1 \to K_2 \otimes h$ defined by $w \mapsto (\phi(T^F)w)_{F \in I}$. Then for all $e \in h_{00}$,

$$\langle u \otimes e, \phi^{(h)}(T)w \rangle = \sum_{F \in I} \langle (e_i, e)u, \phi(T^F)w \rangle = \langle u, \phi(E^nT)w \rangle$$

which shows that $\phi^{(h)}$ is unique, and that (1.2) holds for such $e$. By continuity of the maps $e \mapsto E^e$ and $\phi$, (1.2) holds for all $e \in h$, and (1.3) implies that $\phi^{(h)}$ is bounded and satisfies $\|\phi^{(h)}\| \leq \|\phi\|_{cb}$. \qed

Remarks. $\phi^{(h)}$ clearly extends the algebraic tensor product $\phi \circ \text{id} : V \otimes h \to W \otimes h$, and thus also the canonical extension of this map to the minimal tensor products of these spaces. In view of the natural (completely isometric) identification of $M_n(C_b(V)_h)$ and $C_b(M_n(V)_h)$, $\|(\phi^{(h)})^{(n)}\| = \|(\phi^{(n)})^{(h)}\|$ and so in fact $\phi^{(h)}$ is completely bounded and $\|\phi^{(h)}\|_{cb} = \|\phi\|_{cb}$.

Thus any completely bounded map between two operator spaces lifts to a completely bounded map between their column spaces. We now apply this construction to completely bounded maps $V \to C_b(V)_h$. 
Lemma 1.3. Let \( V \) be an operator space and \( \mathfrak{h} \) a Hilbert space with orthonormal basis \((e_i)_{i \in I} \). Let \((\psi_i)_{i \geq 1}\) be a sequence of completely bounded maps \( V \to C_b(V)_\mathfrak{h} \), and for each \( i \in I \) and \( n \geq 1 \) let \( \psi_n^i : V \to V \) be the completely bounded map defined by

\[
\psi_n^i(T) = E_i^n \psi_n(T).
\]

Then, for each \( n \geq 1 \),

\[
\psi_n^{i_1} \circ \cdots \circ \psi_1^{i_1} = E^{(i_n \cdots i_1)} \psi_n \circ \cdots \circ \psi_1,
\]

where \( E^{(i_n \cdots i_1)} = E^u \) for \( u = e_{i_n} \otimes \cdots \otimes e_{i_1} \), and \( \psi_k = \psi_{k}(H) \) for \( H = \otimes^{(k-1)} \mathfrak{h} \).

Proof. For \( S \in C_{\otimes(n-1)}^0(V)_\mathfrak{h} \),

\[
E^{(i_n \cdots i_1)}(S) = E^{(i_n)}(E^{(i_{n-1} \cdots i_1)}(S))
\]

Iterating this identity gives (1.4). \( \square \)

2. Existence of Feller Cocycles

Let \( A \) be a unital \( C^* \)-algebra acting on \( \mathfrak{h} \) and let \( \eta = (e_i)_{i \geq 1} \) be an orthonormal basis for a separable noise dimension space \( k \), which is extended to a basis \( \tilde{\eta} = (e_{i_0} e_{i_1}) \). After introducing the relevant classes of mapping matrix \( \theta \), quantum stochastic process on \( A \), and Markovian cocycle on \( A \), we use the properties of column spaces established in the previous section to prove the existence of completely positive contractive solutions of (0.1) by showing that necessary conditions on \( \theta \) for a solution to be completely positive and contractive are sufficient for Picard iteration to yield a solution. Basis independence is then confirmed, and this allows us to frame existence of solutions and characterisation of cocycles as a bijective correspondence between a class of square-matrix space valued maps on \( A \) (i.e. coordinate-free mapping matrices) and the space of regular Markovian CP contraction cocycles on \( A \), for a given noise dimension space.

Mapping matrices. The extended basis \( \tilde{\eta} \) of \( \hat{k} \) is used to freely identify the Hilbert spaces \( \mathfrak{h} \otimes \hat{k} \) and \( \bigoplus_{n \geq 0} \mathfrak{h} \) as in (0.5). A matrix \( \theta = [\theta_{ij}^a] \) consisting of bounded linear maps on \( A \) is called a mapping matrix. Such a matrix is regular if (0.2) holds, and strongly regular if furthermore the suprema over \( n \geq 1 \) and \( \|a\| = \|u\| = 1 \) are finite for each \( d; \theta \) is (completely) bounded if there is a (completely) bounded operator \( \tilde{\theta} : A \to B(\mathfrak{h} \otimes \hat{k}) \) such that

\[
E^{(a)} \tilde{\theta}(a) E_{(b)} = \theta_{ij}^a(a).
\]

In the latter case, by the argument in the remark before Lemma 1.1, \( \tilde{\theta}(A) \subset M_k(A)_\mathfrak{h} \) and so \( \tilde{\theta} \) may be viewed as an element of \( B(A; M_k(A)_\mathfrak{h}) \). Finally \( \theta \) has completely bounded columns if for each \( \beta \geq 0 \) there is a completely bounded map \( \theta_{\beta} : A \to B(\mathfrak{h}; \mathfrak{h} \otimes \hat{k}) \) satisfying

\[
\theta_{\beta}(a) u = (\theta_{\beta}^a(a) u)_{\alpha \geq 0}.
\]

In this case effectively \( \theta_{\beta} \in CB(A; C_k(A)_\mathfrak{h}) \). When \( \theta \) is bounded

\[
\tilde{\theta}^d : a \mapsto E^d \tilde{\theta}(a) E_d, \quad d, e \in \hat{k}
\]

reverses the sesquilinear map \( \Theta : (d, e) \mapsto \tilde{\theta}^d_e \) mentioned in the introduction. The collections of regular, strongly regular, bounded and completely bounded mapping matrices are denoted respectively \( M_D(B(A))_R \), \( M_D(B(A))_S \), \( M_D(B(A))_B \) and \( M_D(B(A))_CB \). Thus \( B(A; M_k(A)_\mathfrak{h}) \) and \( CB(A; M_k(A)_\mathfrak{h}) \) are the coordinate-free counterparts to the last two classes of mapping matrix.
Processes on $A$ ([Par],[LW1]). For a Borel subset $I$ of $\mathbb{R}_+$ let $\mathcal{F}_I$ denote the symmetric Fock space over $L^2(I;k)$, and write $\mathcal{F}$ for $\mathcal{F}_{\mathbb{R}_+}$. The factorisation $\mathcal{F} = \mathcal{F}_I \otimes \mathcal{F}_I$ is best seen through exponential vectors: $\varepsilon(f) = \varepsilon(f) \otimes \varepsilon(f)$; $\mathcal{F}_I$ is also identified with the following subspace of $\mathcal{F}$: $\{\xi \otimes \varepsilon(0) : \xi \in \mathcal{F}_I\}$. The definition of processes requires the notions of admissible sets and adaptedness. For a subset $S$ of $L^2(\mathbb{R}_+;k)$, $\mathcal{E}_S$ denotes $\text{Lin}(\varepsilon(f) : f \in S)$; $S$ is admissible if $\mathcal{E}_S$ is dense in $\mathcal{F}$ and $f(0,t) \in S$ whenever $f \in S$ and $t \geq 0$. A useful admissible set is

$$M[n] = \{f \in (L^2 \cap L^\infty_{w^*})(\mathbb{R}_+;k) : f^\epsilon \neq 0 \text{ for only finitely many } n\},$$

where $f^\epsilon(\cdot) := E^{0\epsilon}f(\cdot) \in L^2(\mathbb{R}_+)$. An $\mathcal{E}_S$-process defined on $A$ is then a family of linear maps $k_t : A \rightarrow L(\mathfrak{h} \otimes \mathcal{E}_S; \mathfrak{h} \otimes \mathcal{F})$ satisfying the adaptedness and measurability conditions

1. $k_t(u)\varepsilon(f) \in \mathfrak{h} \otimes \mathcal{F}_{[0,t]}$: $k_t(u)\varepsilon(f) = [k_t(u)\varepsilon(f_{[0,t]})] \otimes \varepsilon(f_{[t,\infty]});$ 
2. $t \mapsto k_t(u)\varepsilon(f)$ is weakly measurable; 
3. $k_t(u)$ is affiliated to $A' \otimes B(\mathcal{F})$.

A bounded process is a process $k$ for which $k_t(A) \subset B(\mathfrak{h} \otimes \mathcal{E}_S; \mathfrak{h} \otimes \mathcal{F})$ and $a \mapsto k_t(a)$ is bounded for each $t$. When the $C^*$-algebra is not a von Neumann algebra and $k$ is bounded we are interested in processes satisfying the stronger Feller-type condition:

(iii) $k_t(A) \subset M_\mathcal{F}(A)_{\mathfrak{h}}$.

The linear space of $\mathcal{E}_S$-processes on $A$ is denoted $\mathbb{P}(A, \mathcal{E}_S)$; processes being identified when they weakly agree almost everywhere. The subspace of processes satisfying the strong regularity condition

$$\sup\{\|k_t(u)\varepsilon(f)\| : \|u\| = 1, t \in [0,T]\} < \infty$$

for all $f \in S$ and $T > 0$ is denoted $\mathbb{P}_{\mathcal{SR}}(A, \mathcal{E}_S)$. For later use we now collect an amalgam of results from [LW1].

**Proposition 2.1.** Let $\theta \in \text{MD}(B(A))_R$. Then Picard iteration yields a process $k^\theta \in \mathbb{P}(A, M_\mathcal{F})$ strongly satisfying (0.1); the map $\theta \mapsto k^\theta$ is injective, and if $\theta \in \text{MD}(B(A))_{\text{su}}$ then $k^\theta \in \mathbb{P}_{\mathcal{SR}}(A, M_\mathcal{F})$. Moreover if $\theta$ is also bounded then $\langle \varepsilon(f), k_t^\theta(u)\varepsilon(g) \rangle$ is given by

$$\langle \varepsilon(f), \varepsilon(g) \rangle \sum_{n \geq 0} \int_0^t \int_0^{t_2} \cdots \int_0^{t_n} dt_n \cdots dt_1 \int_0^{t_2} dt_1 \int_0^{t_1} dt_0 (u, \phi_1 \circ \cdots \circ \phi_n(a)v)$$

for all $u, v \in \mathfrak{h}$ and $f, g \in M[n]$, where $\phi_i = \tilde{\theta}_d^e$ for $d = (1, f(t_i)), e = (1, g(t_i)) \in \tilde{\mathbb{K}}$.

**Existence and basis independence.** Let $(A, \eta, \theta)$ be the data for a QSDE (0.1), with $A$ a unital $C^*$-algebra and each $\theta^\alpha_\beta$ bounded.

**Theorem 2.2.** If the mapping matrix $\theta$ has completely bounded columns then $\theta$ is strongly regular.

**Proof.** Let $\theta^\beta_\alpha : A \rightarrow C_k(A)_\mathfrak{h}$ be the completely bounded maps associated with $\theta$ ($\beta \geq 0$). By the three lemmas

$$\theta^\alpha_\beta \circ \cdots \circ \theta^\alpha_1(a) = E^{\alpha}(\theta^\beta_\alpha)(a)$$

where $E^{\alpha}(\theta) = E^e$ for $e = e_\alpha := e_\alpha_1 \otimes \cdots \otimes e_\alpha_1$ and $\theta^\beta_\alpha = \Psi^\beta_\alpha \circ \cdots \circ \Psi^1$ for $\Psi_k = \theta^k_\gamma$ where $H = \otimes (k-1)\mathbb{K}$ and $\gamma = \beta_k$. Since $\{e_\alpha : \alpha_1, \ldots, \alpha_n \geq 0\}$ is an orthonormal
Remark. When \( B \) is the Parseval identity (0.5) implies that
\[
\sum_{\alpha} \| \theta_{\beta_0} \circ \cdots \circ \theta_{\beta_1}(a) u \|^2 = \| \theta(a) u \|^2
\]
\[
\leq (\| \theta_{\beta_n} \|_{cb} \cdots \| \theta_{\beta_1} \|_{cb} \| a \| u \|)^2.
\]
It follows that the constants (0.3) satisfy
\[
C_n^3(a,u,d) \leq ((d+1) \max_{0 \leq \hat{\beta} \leq d} \| \theta_{\beta} \|_{cb})^n \| a \| u \|^2,
\]
thus \( \theta \) is strongly regular.

Let \( \psi \in CB(\mathcal{A}; M_2(\mathcal{A})_b) \). The following admissible set is clearly basis independent:
\[
\mathcal{M}(k) := \{ f \in (L^2 \cap L^\infty_M)(\mathbb{R}_+; k) : \dim \text{Lin } R_f < \infty \},
\]
where \( R_f \) is the essential range of \( f \). Suppose that \( \eta_1 \) and \( \eta_2 \) are orthonormal bases of \( k \), then there is a third basis \( \eta_3 \) such that \( M[\eta_1] \cup M[\eta_2] \subset M[\eta_3] \); let \( \theta_i \) \((i = 1, 2, 3)\) denote the mapping matrix obtained from \( \phi \) and \( \eta_i \) such that \( \phi = \theta_i \) according to (2.1). Then \( k_{\eta}^\theta(a) \) extends both \( k_{\eta_i}^\theta(a) \) and \( k_{\eta_i}^{\theta_i}(a) \), by (2.2). It follows that there is a process \( k_{\phi}^\theta \in \mathcal{P}_SR(\mathcal{A}, \mathcal{E}_M(k)) \) satisfying \( k_{\phi}^\theta(a) \supset k_{\eta_i}^{\theta_i}(a) \) whenever \( \theta \) is the coefficient matrix of \( \phi \) with respect to some basis \( \eta \). This gives rise to an injective map
\[
\kappa : CB(\mathcal{A}; M_2(\mathcal{A})_b) \to \mathcal{P}_SR(\mathcal{A}, \mathcal{E}_M(k)), \quad \phi \mapsto k_{\phi}^\theta.
\]

**Remark.** When \( k \) is finite dimensional, \( \kappa \) is naturally defined as a map \( B(\mathcal{A}; \mathcal{A} \otimes B(\mathcal{h})) \to \mathcal{P}_SR(\mathcal{A}, \mathcal{E}) \), where \( \mathcal{E} = \mathcal{E}_2 \) for \( S = (L^2 \cap L^\infty_M)(\mathbb{R}_+; k) \). (\cite{Eva}).

**Feller cocycles.** For a process \( k \in \mathcal{P}(\mathcal{A}, \mathcal{E}_M(k)) \), each pair \( f, g \in \mathcal{M}(k) \), and \( t \geq 0 \), define a map \( k_{t}^{f,g} : \mathcal{A} \to L(\eta) \) by
\[
k_{t}^{f,g}(a) = E(f_{[0,t])}^* k_t(a) E(g_{[0,t])}
\]
where \( E(h) := E_{\mathcal{E}(h)} \). The process \( k \) is a Markovian cocycle if each \( k_{t}^{f,g} \) is a bounded map \( \mathcal{A} \to B(\mathcal{h}) \), and the family satisfies
\[
k_{t}^{f,g}(A) \subset A; \quad k_{0}^{f,g} = \text{id}_{\mathcal{A}}; \quad k_{t+r}^{f,g} = k_{t}^{f,g} \circ k_{r}^{s+t,g}
\]
where \( (s_t)_{t \geq 0} \) is the one-parameter semigroup of isometric right shifts on \( L^2(\mathbb{R}_+; k) \). When \( \mathcal{A} \) is a von Neumann algebra and \( k \) is CB and normal, (2.4) is equivalent to (0.4) (\cite{LW2}, Section 4). In fact, by the use of square-matrix spaces, when \( k \) is CB there is a natural definition of \( \tilde{k}_t \) for which (2.4) is equivalent to (0.4) together with the condition
\[
k_t(A) \subset M_{\{a \otimes 1\}}(\mathcal{A})_b, \quad k_0(a) = a \otimes 1,
\]
for any \( C^* \)-algebra \( \mathcal{A} \), or indeed any operator space (\cite{LW3}).

When each of the families \( (k_{t}^{f,g})_{t \geq 0} \) is strongly continuous in \( t \), \( k \) is called a Feller cocycle. This is relevant when \( \mathcal{A} \) is a \( C^* \)-algebra rather than a von Neumann algebra. A regular Markovian cocycle is one whose Markov semigroup \( (k_{t}^{0,1})_{t \geq 0} \) is norm continuous in \( B(\mathcal{A}) \). Regular Markovian contraction cocycles on a \( C^* \)-algebra are automatically Feller (\cite{LW2}, Proposition 5.4).

Define the following class of basis independent mapping matrices:
\[
\Phi_{CPc}(\mathcal{A}, k) := \{ \phi \in CB(\mathcal{A}; M_2(\mathcal{A})_b) : \phi(a) + \Delta(a) = \psi(a) + J^* a J + E(0) a J \}
\]
and \( \phi(1) \leq 0 \), for some CP map \( \psi \) and bounded operator \( J \),
where \( \Delta(a) = a \otimes P \) and \( P \) is the orthogonal projection of \( \hat{k} \) onto \( k \).
Theorem 2.3. For a unital $C^*$-algebra $A$ acting on $\mathfrak{h}$, and a separable noise dimension space $k$, the map $\kappa$ defined in (2.3) restricts to a bijection from $\Phi_{\text{CPc}}(A, k)$ to the space of regular Markovian CP contraction cocycles on $A$ with noise dimension space $k$.

Proof. Let $\phi \in CB(A; M_\mathfrak{h}(A)_0)$ and let $k = k^\phi$. Then $k$ is a regular Markovian cocycle by [LW2], Proposition 5.2. If $\phi$ lies in $\Phi_{\text{CPc}}(A, k)$ then $k$ is CP and contractive by Proposition 5.1 and Theorem 5.2 of [LW1]. Thus $\kappa$ restricts to an injection from $\Phi_{\text{CPc}}$ into the space of regular Markovian CP contraction cocycles. Surjectivity follows from [LW2], Theorem 5.10 and [LW1], Proposition 5.1. □

Remark. This result is a stochastic generalisation of a theorem of Christensen and Evans ([ChE]).

REFERENCES


School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD
E-mail address: jml@maths.nott.ac.uk
E-mail address: sjw@maths.nott.ac.uk