ON STABILITY OF EQUIVARIANT MINIMAL TORI IN THE 3-SPHERE

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Abstract. We prove that amongst the equivariant constant mean curvature tori in the 3-sphere, the Clifford torus is the only local minimum of the Willmore energy. All other equivariant minimal tori in the 3-sphere are local maxima of the Willmore energy.

Introduction

Integrable systems methods have played an important role in exhibiting the complexity and wealth of examples of constant mean curvature (cmc) surfaces in 3-dimensional space forms \([7, 1, 4]\). Associated to a cmc torus is a hyperelliptic Riemann surface called its spectral curve, whose genus \(g\) is called the spectral genus. For every genus there are infinitely many examples, and in fact even restricting to minimal tori in the 3-sphere, there are infinitely many examples when \(g \geq 1\), see \([3]\). For equivariant cmc tori the spectral genus is either \(g = 0\) or \(g = 1\), see \([2]\). For \(g = 0\) the cmc tori are all flat and thus homogeneous, and the Clifford torus is the only minimal example. When \(g = 1\), the conformal factor of the metric is an elliptic function of one real variable, and the corresponding cmc tori all have a 1-parameter family of symmetries. Here we consider equivariant minimal tori and their stability under the Willmore energy. We show that while the Clifford torus is stable, all the spectral genus \(g = 1\) minimal tori in the 3-sphere \(S^3\) are unstable. In a previous paper \([6]\) we studied the moduli of equivariant cmc tori in \(S^3\) by flowing through cmc tori using the isoperiodic deformation \([5]\). Using this flow we here obtain the following

Theorem. Amongst the equivariant cmc tori in the 3-sphere, the Clifford torus is the only local minimum of the Willmore energy. All other equivariant minimal tori are local maxima of the Willmore energy.

1. Homogeneous tori

In order to set notations we first recall how to obtain a cmc surface in the round 3-sphere from a solution of the sinh-Gordon equation. The Maurer-Cartan-equations

\[
2 \, d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0 \quad \text{for all } \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}
\]

are an integrability condition. In our setting we define for concreteness

\[
\alpha_\lambda = \frac{1}{2} \begin{pmatrix}
  u_z \, dz - u_\bar{z} \, d\bar{z} & i \lambda^{-1} e^u \, dz + i e^{-u} \, d\bar{z} \\
  i e^{-u} \, dz + i \lambda e^u \, d\bar{z} & -u_z \, dz + u_\bar{z} \, d\bar{z}
\end{pmatrix}
\]

for some smooth function \(u : \mathbb{C} \to \mathbb{R}\). The matrix 1-form \(\alpha_\lambda\) takes values in \(su_2\) for \(\lambda \in S^1\). Decomposing \(\alpha_\lambda = \alpha'_\lambda \, dz + \alpha''_\lambda \, d\bar{z}\) into \((1, 0)\) and \((0, 1)\) parts, we compute

\[
\partial \alpha'_\lambda = \frac{1}{2} \begin{pmatrix}
  u_{zz} & i \lambda^{-1} u_z e^u \\
  -i u_{\bar{z}z} e^{-u} & -u_{\bar{z}z}
\end{pmatrix}, \quad \partial \alpha''_\lambda = \frac{1}{2} \begin{pmatrix}
  -u_{zz} & -i u_z e^{-u} \\
  i \lambda u z e^u & u_{\bar{z}z}
\end{pmatrix},
\]

\[
[\alpha'_\lambda, \alpha''_\lambda] = \frac{1}{4} \begin{pmatrix}
  -e^{2u} + e^{-2u} & 2 i u_z \lambda^{-1} e^u + 2 i u_{\bar{z}} e^{-u} \\
  -2 i \lambda u z e^u - 2 i u_{\bar{z}} e^{-u} & e^{2u} - e^{-2u}
\end{pmatrix}.
\]
Now \(2\,d\alpha + [\alpha \land \alpha] = 0\) is equivalent to \(\partial\partial'\alpha - \partial\partial''\alpha = [\alpha', \alpha'']\), which holds if and only if \(u\) solves the sinh-Gordon equation
\[
\partial\partial'2u + \sinh(2u) = 0.
\]
For a smooth solution \(u\) of the sinh-Gordon equation, we can integrate the initial value problem
\[
\begin{cases}
    dF_{\lambda} = F_{\lambda} \alpha_{\lambda} \\
    F_{\lambda}(0) = 1
\end{cases}
\]
to obtain a map \(F_{\lambda} : \mathbb{C} \times \mathbb{C}^\times \to \text{SL}(\mathbb{C})\), which is called an extended frame for \(u\). Note that \(F_{\lambda} \in \text{SU}_2\) for \(\lambda \in S^1\). For distinct \(\lambda_1, \lambda_2 \in S^1\), define the map \(f : \mathbb{C} \to \text{SU}_2 \cong S^3\) by
\[
f = F_{\lambda_1} F_{\lambda_2}^{-1}.
\]
We refer to the two distinct unimodular numbers \(\lambda_1, \lambda_2\) as the sym points. Straightforward computations reveal that \(f\) is a conformal immersion with constant mean curvature
\[
H = i \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}.
\]
The induced metric \(f_\ast \langle \cdot, \cdot \rangle = v^2 dz \otimes d\bar{z}\) has conformal factor
\[
v^2 = \frac{e^{2u}}{H^2 + 1}.
\]
and the immersion has constant Hopf differential \(Q \, dz^2\) with
\[
Q = i \frac{(\lambda_1^{-1} - \lambda_2^{-1})}{4}.
\]
We in fact get an \(S^1\)-family of isometric conformal \(\text{cmc}\) immersions, called an associated family, which is obtained by simultaneously rotating \(\lambda_1, \lambda_2\) while keeping the angle between them fixed. Thus each member of an associated family has the same mean curvature, but different Hopf differential.

Let \(F_{\lambda}\) be an extended frame for a \(\text{cmc}\) immersion \(f : \mathbb{C} \to S^3\), so that \(f = F_{\lambda_1} F_{\lambda_2}^{-1}\) for two distinct unimodular numbers \(\lambda_1, \lambda_2\). Suppose we have a lattice
\[
\Gamma = \gamma_1 Z \oplus \gamma_2 Z.
\]
Periodicity \(f(z + \gamma_j) = f(z)\) in terms of the extended frame then reads
\[
F_{\lambda_1}(\gamma_j) = F_{\lambda_2}(\gamma_j) = \pm 1.
\]
Homogeneous tori have constant mean curvature, and up to isometry there is exactly one homogeneous torus for each value of the mean curvature \(H \in (-\infty, \infty)\). The next proposition singles out a simple representative for each value of the mean curvature.

**Proposition 1.1.** Let \(\lambda = \exp(it), t \in (0, \pi)\). The map \(f : \mathbb{C}/\Gamma \to S^3, f = F_{\lambda} F_{1/\lambda}^{-1}\) with
\[
F_{\lambda} = \left( \begin{array}{cc} \cos \mu_{\lambda} & i \lambda^{-1/2} \sin \mu_{\lambda} \\ i \lambda^{1/2} \sin \mu_{\lambda} & \cos \mu_{\lambda} \end{array} \right)
\]
and
\[
\mu_{\lambda} = \mu_{\lambda}(z) = \frac{\pi}{2} \left( z \lambda^{-1/2} + \bar{z} \lambda^{1/2} \right)
\]
is a homogeneous torus. Its period lattice is generated by
\[
\gamma_1(t) = \pi \sec(t/2), \quad \gamma_2(t) = \pi \csc(t/2)
\]
and it has mean curvature \(H(t) = -\cot(t)\).
Proof. Solving $dF_\lambda = F_\lambda \alpha_\lambda$ with $u \equiv 0$ in $\alpha_\lambda$ of (1.2) provides $F_\lambda$ as in (1.10). Suppose $f$ has constant mean curvature $H$. Pick distinct $\lambda_1, \lambda_2 \in S^1$ that satisfy (1.6). Rotating the $\lambda$-plane we may arrange that $\lambda_2 = 1/\lambda_1$, and we can absorb this rotation by an appropriate rotation in the $z$-plane by (1.11). The periods are obtained by solving the periodicity conditions (1.9).

We are now in a position to compute the Willmore energy $W = \int (H^2 + 1) \, dA$ of the family of homogeneous tori. The family is parametrized by $H(t) = -\cot(t) \in (-\infty, \infty)$, or equivalently by $t \in (0, \pi)$. Using the simple periods in (1.12) yields that embedded homogeneous tori have Willmore energy
\[ W = 2\pi^2 \csc^3(t) \]
so that the Clifford torus has Willmore energy $2\pi^2$, since $H(t) = 0$ when $t = \pi/2$. Taking the first two derivatives with respect to $t$ gives
\[ \dot{W} = -6\pi^2 \cot(t) \csc^3(t) \quad \text{and} \quad \ddot{W} = 3\pi^2 (5 + 3 \cos(2t)) \csc^5(t). \]
Hence $\dot{W} = 0$ if and only if $H = 0$ and we then have $\dot{W} \big|_{H=0} = 6\pi^2$. This yields

**Proposition 1.2.** The Clifford torus is the only minimal homogenous torus in the $3$-sphere. It is a local minimum of the Willmore energy for the flow through homogeneous tori.

The moduli space of equivariant CMC tori in the 3-sphere is connected [6]. For each covering type $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ there is an $\mathbb{R}$-family of homogeneous tori. In each such an $\mathbb{R}$-family of homogeneous tori there is a discrete infinite subfamily of homogeneous tori that allow a bifurcation into spectral genus $g = 1$ tori - we call these Delaunay tori. The rotational tori amongst these have a lobe number $\ell$, the number of intrinsic periods it takes to close one extrinsic period. It is shown in [6] that $\ell \geq 2$.

Figure 1.1 shows a small part of the tree through the Clifford torus. Of separate interest are the values of the Willmore energy of these bifurcating homogeneous tori.
Proposition 1.3. For every integer $\ell \geq 2$ the homogeneous torus with mean curvature
\[ H_{\ell} = \frac{\ell^2 - 2}{2\sqrt{\ell^2 - 1}} \]
bifurcates into the $\ell$-lobed family of Delaunay tori. It has rectangular conformal type $(1, \tau_{\ell})$ with
\[ \tau_{\ell} = i\sqrt{\ell^2 - 1}, \]
and Willmore energy
\[ W_{\ell} = \frac{\pi^2 \ell^2}{\sqrt{\ell^2 - 1}}. \]

Proof. Evaluating the function $\mu_{\lambda}$ in (1.11) at the two periods $\gamma_1(t), \gamma_2(t)$ in (1.12) gives
\[ \mu_{\lambda}(\gamma_1(t)) = \frac{\pi i(\lambda + 1) \sec \left( \frac{t}{2} \right)}{2\sqrt{\lambda}}, \quad \mu_{\lambda}(\gamma_2(t)) = \frac{\pi(\lambda - 1) \csc \left( \frac{t}{2} \right)}{2\sqrt{\lambda}}. \]

To bifurcate into a spectral genus 1 tori, we need to determine the spectral curves of homogeneous tori that possess a double point on the unit circle. Such a double point can then be opened up to flow into spectral genus $g = 1$. We thus determine those values of $t$ for which $\mu_{\pm 1}(\gamma_j(t)) \in \pi i \mathbb{Z}$. Solving the system
\[ \mu_{\lambda}(\gamma_1(t)) = \frac{\pi i \kappa_1}{2\sqrt{\kappa_1^2 - 1}}, \quad \mu_{\lambda}(\gamma_2(t)) = \frac{\pi(\lambda - 1) \csc \left( \frac{t}{2} \right)}{2\sqrt{\lambda}}. \]
yields the eight solutions
\[ \lambda = \frac{\kappa_1^2 + \kappa_2^2 - 2\kappa_1^2 \kappa_2^2 \pm 2\kappa_1 \kappa_2 \sqrt{(\kappa_1^2 - 1)(\kappa_2^2 - 1)}}{\sqrt{\kappa_1^2 - \kappa_2^2}}, \]
\[ t = \pm 2 \sec^{-1} \left( \pm \sqrt{\frac{\kappa_1^2 - \kappa_2^2}{1 - \kappa_2^2}} \right). \]

There are eight solutions, as for each sign of choice in $\lambda$ there are four choices of signs in $t$. We only care for the two cases $\lambda = \pm 1$, and compute that
\[ \lambda = 1 \iff \kappa_2 = 0 \text{ or } \kappa_1 = \pm 1 \]
\[ \lambda = -1 \iff \kappa_1 = 0 \text{ or } \kappa_2 = \pm 1 \]
In the cases $\kappa_1 = \pm 1$ and $\kappa_2 = \pm 1$ the value for $t$ is undefined, so it suffices to consider the two remaining cases $\kappa_1 = 0$ or $\kappa_2 = 0$. When $\kappa_1 = 0$ we get $\lambda = -1$ and the four solutions
\[ t_{\kappa_2} = \pm 2 \sec^{-1} \left( \pm \sqrt{\frac{\kappa_2^2}{\kappa_2^2 - 1}} \right), \]
and the mean curvature computes to
\[ H_{\kappa_2} = \frac{\kappa_2^2 - 2}{2\sqrt{\kappa_2^2 - 1}}. \]
When $\kappa_2 = 0$ we get $\lambda = 1$ and the four solutions
\[ t_{\kappa_1} = \pm 2 \sec^{-1} \left( \pm \sqrt{\kappa_1^2} \right), \]
and the mean curvature computes to
\[ H_{\kappa_1} = -\frac{\kappa_1^2 - 2}{2\sqrt{\kappa_1^2 - 1}}. \]
Hence the square of the mean curvature coincides in the two cases, and we denote by $\ell$ the non-zero integer $\kappa_1$ or $\kappa_2$. 
When $\kappa_1 = 0$, then for $\ell = \kappa_2$ we have
\[ \gamma_1(t\ell) = \pm \frac{\pi}{\sqrt{1 - 1/\ell^2}} \quad \gamma_2(t\ell) = \pm \pi i \ell, \]
and so conformal type $\tau_\ell = i/\sqrt{\ell^2 - 1}$. When $\kappa_2 = 0$, then for $\ell = \kappa_1$ we have
\[ \gamma_1(t\ell) = \pm \pi i \ell \quad \gamma_2(t\ell) = \pm \frac{\pi i}{\sqrt{1 - 1/\ell^2}}, \]
and so conformal type $\tau_\ell = i\sqrt{\ell^2 - 1}$. Hence after the Möbius transform $z \mapsto -1/z$ both conformal types agree.

Furthermore, the area of the fundamental domain $|\gamma_1(t\ell)\gamma_2(t\ell)|$ coincides in both cases. The induced area element is $dA = \frac{1}{H^2 + 1} \, dx \, dy$, since $u \equiv 0$. Thus the Willmore energy computes to
\[ W_\ell = \int (H^2 + 1) \, dA = \int dx \, dy = |\gamma_1(t\ell)\gamma_2(t\ell)| = \frac{\pi^2 \ell^2}{\sqrt{\ell^2 - 1}}. \]

2. Instability

As in [6] we describe spectral genus $g = 1$ CMC tori in $S^3$ by elliptic curves
\[ 4u^2 = (\lambda - k)(\lambda^{-1} - k), \quad k \in [-1, 1]. \]
A spectral genus $g = 1$ CMC surface in $S^3$ is parameterized by its elliptic modulus $k$, its mean curvature $H$, and its associated family parameter. We will use the three coordinates
\[ (k, q, h) \in [-1, 1]^3 \]
where $q, h$ are defined in terms of the sym points as
\[ q := \frac{1}{2} \left( \sqrt{\lambda_1 \lambda_2} + 1/\sqrt{\lambda_1 \lambda_2} \right) \quad \text{and} \quad h := \frac{1}{2} \left( \sqrt{\frac{\lambda_1}{\lambda_2}} + \sqrt{\frac{\lambda_2}{\lambda_1}} \right). \]
The mean curvature $H$ in (1.6) can be expressed as $H = h/\sqrt{1 - k^2}$ so that $H = 0$ if and only if $h = 0$.

Let $K$ and $E$ denote the complete integrals of the first and second kind with modulus $1 - k^2$, so
\[ K := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-(1-k^2)x^2)}}, \quad E := \int_0^1 \sqrt{\frac{1-(1-k^2)x^2}{1-x^2}} \, dx. \]
Then clearly \( E \leq K \) for all \( k \in [-1, 1] \), and in [6] it is shown that \( K \) and \( E \) satisfy
\[
1 \leq \frac{2E}{1 + k^2} < K < \frac{E}{|k|} \quad \text{for} \quad 0 < |k| < 1.
\]
This implies
\[
E^2 - k^2 K^2 > 0.
\]
Since \( K|_{k=\pm 1} = E|_{k=\pm 1} = \pi^2 \), the function \( 2E - (1 + k^2)K \) has zeroes only at \( k = \pm 1 \).

**Proposition 2.1.** For the flow through \( \text{cmc} \) tori, the equivariant minimal tori of spectral genus \( g = 1 \) are all local maxima of the Willmore energy.

**Proof.** The flow equations [6], rescaled by \((1 - k^2)\) on the parameter space \((k, q, h)\) \(\epsilon [−1, 1]^3\) are
\[
\dot{k} = k \left(1 - k^2\right) \left(q E - h k K\right),
\]
\[
\dot{q} = (1 - q^2) \left((k^2 + 1) E - 2 k^2 K\right),
\]
\[
\dot{h} = (1 - h^2) k \left(2E - (k^2 + 1)K\right).
\]
The conformal factor only depends on the \(y\)-direction in the \(z\)-plane, and is given by \(2\langle f_z, f_z \rangle = v^2 = dn^2(y, 1 - k^2)\). It has period \(2K\), and
\[
\int_0^{2K} v^2(t) \, dt = 2E.
\]
By (1.7) the factor \(1/(H^2 + 1)\) in the area element cancels with the integrand of the Willmore energy, so that Willmore energy is \(W = \int (H^2 + 1) \, dA = \int v^2 \, dx \, dy\). A period in the \(y\)-direction is an integer multiple of \(\gamma_2 = 2iK\). The other period is computed in Proposition 3.6 [6], and is an integer multiple of \(\gamma_1 = 2\sqrt{2} \pi/\chi_0\) with
\[
\chi_0 = \sqrt{k^2 - 2k \left(q h - \sqrt{1 - q^2 \sqrt{1 - h^2}}\right) + 1}.
\]
Thus the Willmore energy with respect to the simple generators of the lattice computes to
\[
W = \int_0^{\gamma_1} \int_0^{\gamma_2} v^2(y) \, dy \, dx = \frac{4\pi \sqrt{2} E}{\sqrt{k^2 - 2k \left(q h - \sqrt{1 - q^2 \sqrt{1 - h^2}}\right) + 1}}.
\]
Using the flow equations (2.6), and \(\frac{dE}{dk} = \frac{dE}{dk} \dot{k} \), the first derivative of the Willmore energy is
\[
\dot{W} = \frac{4\pi \sqrt{2} h k \left(E^2 - k^2 K^2\right)}{\sqrt{k^2 - 2q h k + 2k \sqrt{1 - q^2 \sqrt{1 - h^2}}}}.
\]
Since \(k = 0\) gives the degenerate case of bouquets of spheres, we conclude that for equivariant \(\text{cmc}\) tori we have \(\dot{W} = 0\) if and only if \(H = 0\).

The second derivative at these extrema computes to
\[
\dot{W} \bigg|_{H=0} = \frac{4\pi \sqrt{2} k^2}{\sqrt{k^2 + 2k \sqrt{1 - q^2} + 1}} \left(k K - E\right) \left(k K + E\right) \left(k^2 K + K - 2E\right).
\]
The product of the first two factors in negative by inequality (2.5), while for the third factor we have
\[
\left(k^2 K + K - 2E\right) \bigg|_{k^2=1} = 0,
\]
\[
\frac{d}{dk} \left(k^2 K + K - 2E\right) = \frac{E - k^2 K}{2k^2} \geq 0,
\]
and thus
\begin{equation}
\hat{W}
\bigg|_{H=0} < 0 \text{ for } 0 < |k| < 1.
\end{equation}

Hence with respect to the deformation through \textit{cmc} tori, the Willmore energy has local maxima at all the minimal spectral genus \( g = 1 \) tori in \( S^3 \). \qed

Combining Propositions 1.2 and 2.1 proves the theorem stated in the introduction.

\section*{References}


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