ON THE RIEMANN–HILBERT–PROBLEM

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Contents

1. Introduction

The objective of the first sections is the relationship between the factorisation of scalar functions and the theory of singular integral equations as far as they are relevant to boundary value problems for holomorphic functions. By Cauchy's integral theorem, a continous function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ posesses an integral representation in its domain of holomorphicity, bounded by a contour Γ :

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} dt \quad , \quad z \in \hat{\mathbb{C}} \backslash \Gamma \tag{1}$$

If Γ divides the Riemann sphere $\hat{\mathbb{C}}$ into two disjoint domains $\Gamma_\pm$, then the above equation defines two holomorphic functions of the complex variable z, depending on whether $z \in \Gamma_+$ or $z \in \Gamma_-$. We shall investigate the behaviour of $f(z)$ for $z \in \Gamma$. If $0 \in \Gamma_+$ and $\infty \in \Gamma_-$ then a holomorphic function f_+ in Γ_+ has only positive powers in its power series expansion around 0, while a function f_- holomorphic in Γ_- only has negative powers in its expansion around ∞. Conversely the search for analytic functions leads to the investigation of integral equations. Since the kernel has uncountably many singularities, the corresponding operator fails to be compact, but its boundedness can be secured on certain classes of functions and the boundary values are described by the formulas of Sokhotski/Plemelj. All this is a short survey of classical results obtained by Muskelishvili [11], Vekua [12] and

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Gakhov [4], standard references, where missing details may be found. After some preliminaries, these problems will be generalized to matrix valued functions. The matrix Riemann as well as the Riemann-Hilbert problems will be dealt with followed by applications in the theory of differential equations.

2. Singular integral equations

Definition 2.1. A contour Γ is a rectifiable simply closed smooth [1] path of finite length that bounds a simply connected domain Γ_+ in a positively oriented sense. Denote by $\Gamma_- := \hat{\mathbb{C}}\backslash\overline{\Gamma_+}$ its unbounded compliment that contains ∞. Furthermore, since this is rarely a restriction, let $0 \in \Gamma_+$.

Set $\mathcal{C}(\Gamma) := \{\varphi : \Gamma \to \mathbb{C} \text{ continous}\}$ and define thereon the Cauchy Integraloperator

$$S_\Gamma\varphi(z) := \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-z}\,dt,\; z \in \mathbb{C}\backslash\Gamma.$$

The function φ is called the density and $\frac{1}{t-z}$ the Cauchy–kernel of the operator S_Γ. Integration along Γ is to be performed positively. By Cauchy's integral theorem the function $f := \frac{1}{2}S_\Gamma\varphi$, $\varphi \in \mathcal{C}(\Gamma)$ is holomorphic in the domains Γ_+ and Γ_- and is therefore called piecewise holomorphic. Denote the restrictions:

$$f_{|\Gamma_-} =: f_-,$$
$$f_{|\Gamma_+} =: f_+.$$

The power series expansion of f_- is

$$\frac{1}{t-z} = -\frac{1}{z}\cdot\frac{1}{1-\frac{t}{z}} = -\sum_{k=0}^{\infty}\frac{t^k}{z^{k+1}},\; z \in \Gamma_-$$

and thus

$$f_-(z) = \frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{1}{z^k}\int_\Gamma t^{k-1}\varphi(t)\,dt$$

In particular $f_-(\infty) = \lim_{\|z\|\to\infty} f_-(z) = 0$ so f_- is of order $\frac{1}{z}$ for large $\|z\| := \sqrt{z\bar{z}}$

Nonconstant holomorphic functions $f_\pm : \Gamma_\pm \to \mathbb{C}$ posess poles when extended continously to Γ due to their integral representation, since the Cauchy–kernel vanishes.

Definition 2.2. For $z_0 \in \Gamma$, $\epsilon > 0$ and $\Gamma_\epsilon(z_0) := \{t \in \Gamma : \|t - z_0\| \le \epsilon\}$, the limit

$$\lim_{\epsilon\to 0}\int_{\Gamma\backslash\Gamma_\epsilon(z_0)}\frac{\varphi(t)}{t-z_0}\,dt$$

is called the principal value of the integral $\int_\Gamma \frac{\varphi(t)}{t-z}\,dt$ in z_0.

Example 2.1. : $(\varphi \equiv 1)$ Let $z \in \Gamma$ be arbitrary and define

$$H(z;\varrho) := \{\xi \in \Gamma_+ : \|\xi - z\| = \varrho\},\; \varrho > 0$$

$H(z;\varrho)$ is homotopic to $\Gamma\backslash\Gamma_\varrho(z)$ and

$$\int_{\Gamma\backslash\Gamma_\varrho(z)}\frac{d\xi}{\xi - z} = \int_{H(z;\varrho)}\frac{d\xi}{\xi - z}$$

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[1]For our purposes a C^2 - parametrisation of Γ suffices.

and in polar coordinates $\xi = z + \varrho e^{i\vartheta}$ have $d\xi = i\varrho e^{i\vartheta} d\vartheta$ and

$$\lim_{\varrho \to 0} \int_{H(z;\varrho)} \frac{d\xi}{\xi - z} = \lim_{\varrho \to 0} \int_{H(z;\varrho)} i d\vartheta = i\pi$$

since in the limit $\Gamma_\varrho(z)$ straightens into a line, and thus the argument ϑ varies by π. Alltogether with Cauchy's integral theorem we get:

$$\frac{1}{2i\pi} \int_\Gamma \frac{1}{t - z} dt = \begin{cases} 1 & \text{for } z \in \Gamma_+ \\ \frac{1}{2} & \text{for } z \in \Gamma \\ 0 & \text{for } z \in \Gamma_- \end{cases} \tag{2}$$

the winding number of a point $z \in \hat{\mathbb{C}}$ with respect to Γ. Therefore the integral operator S_Γ is defined at least for constant densities.

From the decomposition for a $z_0 \in \Gamma$:

$$\int_\Gamma \frac{\varphi(t)}{t - z_0} dt = \int_\Gamma \frac{\varphi(t) - \varphi(z_0)}{t - z_0} dt \, + \, \varphi(z_0) \int_\Gamma \frac{1}{t - z_0} dt \tag{3}$$

it follows that we need to investigate only the first integral of the right hand side of (3) for the existence of the principal value for a given density. To achieve some estimate for the line integral of $\frac{\varphi(t) - \varphi(z_0)}{t - z_0}$ we will need more than mere continuity. Functionspaces, on which S_Γ turns out to be bounded are C^α - and L^p - spaces. Hölder continous densities are most commonly dealt with in the classical literature.

Definition 2.3. A complex-valued map φ on a set $U \subset \mathbb{C}$ is called uniformly Hölder continous with Hölder exponent $0 < \alpha \leq 1$, if there exists a constant $K \in \mathbb{R}$ such that

$$\|\varphi(s) - \varphi(t)\| \leq K \|s - t\|^\alpha \text{ for all } s, t \in U.$$

Denote by $C^{0,\alpha}(\Gamma)$ the Banach-algebra of uniformly Hölder-continous bounded functions on Γ w.r.t the norm $\|\cdot\|_\alpha = \|\cdot\|_\infty + |\cdot|_\alpha$:

$$\|\varphi\|_\alpha := \sup_{t \in \Gamma} \|\varphi(t)\| + \sup_{s,t \in \Gamma, s \neq t} \frac{\|\varphi(s) - \varphi(t)\|}{\|s - t\|^\alpha}$$

In the following let H-continous be synonymous with bounded uniformly Hölder continous. Of technical significance is the fact that H-continuity is a local property

Lemma 2.1. *Let $U \subset \mathbb{C}$, $0 < \alpha \leq 1$ and $\varphi : U \to \mathbb{C}$ satisfy:*

1. *$\|\varphi(t)\| \leq K$ for all $t \in U$*
2. *There exist $\gamma > 0$ and $C \in \mathbb{N}$ such that $\|\varphi(s) - \varphi(t)\| \leq C\|s - t\|^\alpha$ for all s, $t \in U$ for which $\|s - t\| \leq \gamma$.*

Then φ is H-continous with $\|\varphi\|_\alpha \leq K + \; max(C, \frac{2K}{\gamma^\alpha})$

Proof. If $\|s - t\| > \gamma$, then

$$\|\varphi(s) - \varphi(t)\| \leq 2K \leq 2K \left(\frac{\|s - t\|}{\gamma} \right)^\alpha$$

and thus $| \, \varphi \, |_\alpha \leq \max \left\{ C, \frac{2K}{\gamma^\alpha} \right\}$. □

Uniformly Hölder-continous functions 'live' between the continous and the differentiable functions and guarantee the existence of the principal value.

Theorem 2.2. *The principal value of the Cauchy integral*

$$f(z) = \frac{1}{i\pi}\int_\Gamma \frac{\varphi(t)}{t-z}dt \quad \textit{for all } z\in\Gamma$$

exists for all densities $\varphi \in C^{0,\alpha}(\Gamma)$ *with* $0<\alpha\leq 1$.

Proof. Pick an $r\in(0,1]$ such that the normal $n:\Gamma\to S^1$ satisfies :

1. $\langle n(t),n(s)\rangle \geq \frac{1}{2}$ for the scalar product $\langle\cdot,\cdot\rangle$ and for all $s,t\in\Gamma, \|s-t\|\leq r$
2. $\{\xi\in\Gamma \mid \|\xi-z\|\leq r\} =: \Gamma_r(z)$ is connected for all $z\in\Gamma$.

Project $\Gamma_r(z)$ onto the tangent $T_z\Gamma$ in z along the normal (in general a non-tangential approach suffices [4]). Then the line elements dt on Γ and $d\tau$ on $T_z\Gamma$ satisfy :

$$dt = \frac{d\tau}{\langle n(t),n(s)\rangle} \leq 2d\tau$$

With this we obtain the following estimates :

$$\int_{\Gamma_r(z)} \frac{\|\varphi(t)-\varphi(z)\|}{\|t-z\|}\|dt\| \leq 2\mid\varphi\mid_\alpha \int_{-r}^{r}\|t-z\|^{\alpha-1}d\tau \leq 4\mid\varphi\mid_\alpha\int_0^r \tau^{\alpha-1}d\tau \leq \frac{4r^\alpha}{\alpha}\mid\varphi\mid_\alpha$$

$$\int_{\Gamma\setminus\Gamma_r(z)} \frac{\|\varphi(t)-\varphi(z)\|}{\|t-z\|}\|dt\| \leq \mid\varphi\mid_\alpha \int_{\Gamma\setminus\Gamma_r(z)}\|t-z\|^{\alpha-1}\|dt\| \leq r^{\alpha-1}\mid\Gamma\mid\mid\varphi\mid_\alpha$$

and thus the integral $\int_\Gamma \frac{\varphi(t)-\varphi(z)}{t-z}dt$ exists improperly for all $z\in\Gamma$ □

3. The formula of Sokhotski–Plemelj

The formula of Sokhotski–Plemelj renders the uniqueness of the boundary values that f_+ and f_- aquire when extended from Γ_+ or Γ_- respectively onto Γ. The Cauchy integral $f(z)=\frac{1}{2}S_\Gamma\varphi(z)$ is itself H-continous on $\overline{\Gamma_+}$ and $\overline{\Gamma_-}$, except for a jump when crossing Γ. To show this, put a parallel-strip around Γ: Pick $h>0$ such that every point z in $D_h := \{s+\eta h n(s) : s\in\Gamma, \eta\in[-1,1]\}$, n the normal on Γ, posesses a unique representation $z=s+\eta h n(s)$. To use Lemma 1.1, decompose f as follows:

$$\begin{aligned} f(z) &= \frac{1}{2i\pi}\int_\Gamma \frac{\varphi(t)-\varphi(s)}{t-z}dt + \frac{\varphi(s)}{2i\pi}\int_\Gamma\frac{1}{t-z}dt \\ &= \frac{1}{2i\pi}\int_\Gamma \frac{\varphi(t)-\varphi(s)}{t-z}dt + \begin{cases} \varphi(s)\,, z\in\Gamma_+\cap D_h \\ \frac{\varphi(s)}{2}\,, z\in\Gamma \\ 0\ \,, z\in\Gamma_-\cap D_h \end{cases} \end{aligned}$$

and use rigourous estimates to show that the integral $\frac{1}{2i\pi}\int_\Gamma\frac{\varphi(t)-\varphi(s)}{t-z}dt$ is bounded and locally H-continous in D_h, and thus with the Lemma H-continous in all of D_h, in particular it thus exists improperly in the sense of the principal value. Taking the limit ($s\to z$):

$$\begin{aligned} f_+(z) &= \frac{1}{2\pi i}\int_\Gamma\frac{\varphi(t)-\varphi(z)}{t-z}dt\ +\varphi(z) \\ f(z) &= \frac{1}{2\pi i}\int_\Gamma\frac{\varphi(s)-\varphi(z)}{t-z}dt\ +\frac{1}{2}\varphi(z) \\ f_-(z) &= \frac{1}{2\pi i}\int_\Gamma\frac{\varphi(t)-\varphi(z)}{t-z}dt. \end{aligned}$$

Elimination of the integrals gives

$$f_+(z) = f(z) + \frac{1}{2}\varphi(z)$$
$$f_-(z) = f(z) - \frac{1}{2}\varphi(z)$$

and finally another application of the Lemma gives the H-continuity of f in all $\overline{\Gamma_+}$ and $\overline{\Gamma_-}$. Alltogether we have proven

Theorem 3.1. *(Sokhotski,Plemelj) Let $\varphi \in C^{0,\alpha}(\Gamma)$ with $0 < \alpha < 1$. The holomorphic function $f := \frac{1}{2}S_\Gamma\varphi : \mathbb{C}\backslash\Gamma \to \mathbb{C}, z \mapsto \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)}{t-z}dt$ can be H-continously extented from $\Gamma_\pm$ to Γ with boundary values for $z \in \Gamma$:*

$$f_\pm(z) = \frac{1}{2}(S_\Gamma \pm Id)\varphi(z) \tag{4}$$

Corollary 3.2. *The Cauchy integral operator $S_\Gamma : C^{0,\alpha}(\Gamma) \to C^{0,\alpha}(\Gamma)$ is bounded.*

Corollary 3.3. *The operators S_Γ und $-S_\Gamma$ are adjoint with respect to the dual system $\langle C^{0,\alpha}(\Gamma), C^{0,\alpha}(\Gamma)\rangle$, induced by the non-degenerate bilinear form*

$$\langle\varphi,\psi\rangle := \int_\Gamma \varphi(z)\psi(z)\,dz\ ;\ \varphi,\,\psi \in C^{0,\alpha}(\Gamma)$$

Proof. For two functions $\varphi, \psi \in C^{0,\alpha}(\Gamma)$ set :

$$f(z) := \frac{1}{2i\pi}\int_\Gamma \frac{\varphi(t)}{t-z}dt,\ g(z) := \frac{1}{2i\pi}\int_\Gamma \frac{\psi(t)}{t-z}dt,\ z \in \mathbb{C}\backslash\Gamma$$

With (4) and Cauchy's integral theorem we have

$$\begin{aligned}\langle S_\Gamma\varphi,\psi\rangle + \langle\varphi, S_\Gamma\psi\rangle &= \langle f_+ + f_-, g_+ - g_-\rangle + \langle f_+ - f_-, g_+ + g_-\rangle \\ &= 2\langle f_+, g_+\rangle - 2\langle f_-, g_-\rangle = 2\int_{\|z\|=r} f(z)g(z)dz.\end{aligned}$$

The last integral vanishes for large r, since f, g are of order $\frac{1}{z}$ for large $\|z\|$. □

Note: From (4) we obtain

$$f_+ + f_- = S_\Gamma\varphi \tag{5}$$
$$f_+ - f_- = \varphi \tag{6}$$

As an application of the formulas of Sokhotski/Plemelj three boundary value problems will be solved :

1. Prescribed jump: $f_+ - f_- = \varphi$
2. Prescribed boundary values: $f\,|_\Gamma \equiv \varphi$
3. Prescribed glueing: $f_+ = gf_-$ (Riemann-Problem)

For the first boundary value problem we have the following existence and uniqueness result as a direct consequence of (5)

Corollary 3.4. *For every $\varphi \in C^{0,\alpha}(\Gamma)$ there exists a unique $f : \hat{\mathbb{C}} \to \mathbb{C}$ holomorphic in Γ_+ and Γ_-, which is*

1. *Continously extendable from Γ_+ to $\overline{\Gamma_+}$ as well as from Γ_- to $\overline{\Gamma_-}$*
2. *Vanishes for large $\|z\|$*
3. *The boundary values satisfy $f_+ - f_- = \varphi$ on Γ.*

Proof. $f(z) := \frac{1}{2i\pi}\int_\Gamma \frac{\varphi(t)}{t-z}dt$ is the desired function and satisfies all three conditions. If f^1 and f^2 are two solutions, set $g := f^1 - f^2$. By hypothesis $f^1_+ - f^1_- = \varphi$ as well as $f^2_+ - f^2_- = \varphi$ so $(f^1_+ - f^1_-) - (f^2_+ - f^2_-) = 0$. Hence for the boundary values of g we have $g_+ - g_- = 0$ on Γ. Thus g is entire by Morera's theorem and since f^1_- and f^2_- vanish at ∞, g is bounded, so by Liouville's theorem identically zero. Hence the solution is uniquely determined up to addition of entire functions. □

We adapt the following notation :

$$\mathcal{H}^+ := \{\text{holomorphic maps in } \Gamma_+ \text{ and continous on } \overline{\Gamma_+}\ \}$$

$$\mathcal{H}^-_0 := \{\text{in } \infty \text{ vanishing, holomorphic maps in } \Gamma_- \text{ and continous on } \overline{\Gamma_-}\}$$

Thus every H-continous $\varphi : \Gamma \to \mathbb{C}$ can be decomposed $\varphi = f_+ - f_-$. With

$$\mathcal{H}^+_\Gamma := \mathcal{H}^+ \cap C^{0,\alpha}(\Gamma) \subseteq \text{Image}\{\tfrac{1}{2}(S_\Gamma + \text{Id})\},$$

$$\mathcal{H}^-_\Gamma := \mathcal{H}^-_0 \cap C^{0,\alpha}(\Gamma) \subseteq \text{Image}\{\tfrac{1}{2}(S_\Gamma - \text{Id})\}$$

the next step is to show that this sum is direct: $\mathcal{H}^+_\Gamma \cap \mathcal{H}^-_\Gamma = \{0\}$. It turns out that the only way to reconstruct a holomorphic function from its boundary values is if one of the summands is identically zero:

Corollary 3.5.

$$\varphi \in \mathcal{H}^+_\Gamma \iff \varphi \in kernel\{\tfrac{1}{2}(S_\Gamma - Id)\}$$

$$\varphi \in \mathcal{H}^-_\Gamma \iff \varphi \in kernel\{\tfrac{1}{2}(S_\Gamma + Id)\}$$

or to phrase it in terms of the second boundary value problem : For $\varphi \in C^{0,\alpha}(\Gamma)$ *there exists exactly one holomorphic function* f *in* Γ_+ *with* $f\,|_\Gamma \equiv \varphi$ *iff* $\frac{1}{2}(S_\Gamma - Id)\varphi = 0$*, and for a* $\varphi \in C^{0,\alpha}(\Gamma)$ *there exists exactly one holomorphic function* f *in* Γ_- *that vanishes at* ∞*, with* $f\,|_\Gamma \equiv \varphi$ *when* $\frac{1}{2}(S_\Gamma + Id)\varphi = 0$.

Proof. $'\Rightarrow'$: Let $f : \Gamma_+ \to \mathbb{C}$ holomorphic and $f(t) = \varphi(t)$ for all $t \in \Gamma$. By Cauchy's integral theorem, f can be written:

$$f(z) = \frac{1}{2i\pi}\int_\Gamma \frac{f(t)}{t-z}dt = \frac{1}{2i\pi}\int_\Gamma \frac{\varphi(t)}{t-z}dt \quad , z \in \Gamma_+$$

which by Sokhotski/Plemelj has the boundary values

$$f_+ = \frac{1}{2}S_\Gamma\varphi + \frac{1}{2}\varphi = \varphi \Leftrightarrow \frac{1}{2}S_\Gamma\varphi - \frac{1}{2}\varphi = 0$$

so $\varphi \in \text{kernel}\{\frac{1}{2}(S_\Gamma - \text{Id})\}$

$'\Leftarrow'$: Let $\varphi \in \text{kernel}\{\frac{1}{2}(S_\Gamma - \text{Id})\}$, then $f(z) := \frac{1}{2}S_\Gamma\varphi$ is holomorphic in Γ_+ with boundary values $f_+ = \frac{1}{2}S_\Gamma\varphi + \frac{1}{2}\varphi = \varphi$. Hence $f\,|_\Gamma \equiv \varphi$. The second equivalence is shown analogously. □

The operator S_Γ thus leaves functions holomorphic in Γ_+ respectively Γ_- invariant.

$$S_{\Gamma_{|_{\mathcal{H}^+}}} = \text{Id} \qquad S_{\Gamma_{|_{\mathcal{H}^-_0}}} = -\text{Id} \tag{7}$$

Corollary 3.6. *The Cauchy Integraloperator is unitary.*

Proof. $S^2_\Gamma\varphi = S^2_\Gamma(f_+ - f_-) = S_\Gamma f_+ + S_\Gamma f_- = f_+ - f_- = \varphi$. □

The fact that $S_\Gamma^2 = \mathrm{Id}$ means that

$$\mathcal{P}_\Gamma = \frac{1}{2}(S_\Gamma + \mathrm{Id}) \quad \mathcal{Q}_\Gamma = \frac{1}{2}(S_\Gamma - \mathrm{Id})$$

are complementary projections (i.e $\mathcal{P}^2 = \mathcal{P}, Q^2 = \mathcal{Q}, PQ = QP = 0$) and furthermore

$$\mathrm{im}(\mathcal{P}_\Gamma) = \ker(\mathcal{Q}_\Gamma) = \mathcal{H}_\Gamma^+ \qquad \ker(\mathcal{P}_\Gamma) = \mathrm{im}(\mathcal{Q}_\Gamma) = \mathcal{H}_\Gamma^-$$

are closed subalgebras and render the decomposition of $C^{0,\alpha}(\Gamma)$ into the direct sum

$$C^{0,\alpha}(\Gamma) = \mathcal{H}_\Gamma^- \oplus \mathcal{H}_\Gamma^+$$

Note that the operator S_Γ cannot be compact and that the subspaces $\mathcal{H}_\Gamma^+$ and $\mathcal{H}_\Gamma^-$ are not finite dimensional.

4. The Riemann-problem

The following boundary value problem was first formulated by Riemann in his inaugural dissertation but since Hilbert first attempted to solve it using integral equations it is sometimes refered to in the literature as the Riemann-Hilbert-Problem. As a prepatory notion for the solution we'll need the following

Definition 4.1. For $g : \Gamma \to \mathbb{C}\backslash\{0\}$ define the index, $ind_\Gamma(g)$, as the integer, divided by 2π, by which the argument of g, $\arg(g)$, changes after going around Γ once positively:

$$ind_\Gamma(g) := \frac{1}{2\pi}\int_\Gamma d\arg(g)$$

where the integral is to be understood in the sense of Stieltjes and for two functions $f, g : \Gamma \to \mathbb{C}\backslash\{0\}$ have

$$ind_\Gamma(f \cdot g) = ind_\Gamma(f) + ind_\Gamma(g)$$

as well as

$$ind_\Gamma(\frac{f}{g}) = ind_\Gamma(f) - ind_\Gamma(g)$$

and for a polynomial $p : \Gamma \to \mathbb{C}\backslash\{0\}$:

$$ind_\Gamma(p) = \#\{\text{roots of } p \text{ in } \Gamma_+ \}$$

where $\#\{\cdot\}$ denotes the cardinality of the set.

If a function f is to be composed with any branch of the logarithm, its argument mustn't vary by more than 2π, for the composition $\ln \circ f$ to be unique. On the set

$$\{f \in \mathcal{C}(\Gamma) : \text{ There exists } g : \exp(g) = f\} = \{f \in \mathcal{C}(\Gamma) : ind_\Gamma f = 0\}$$

we can uniquely apply exp and log and can therefore transform multiplicative problems like the Riemann-Problem into additive ones. Let

$$GC^{0,\alpha}(\Gamma) := \{\varphi \in C^{0,\alpha} : \varphi(t) \neq 0 \text{ for all } t \in \Gamma\}$$

be the subalgebra of invertible H-continous functions on Γ. The contour Γ divides the Riemann sphere $\hat{\mathbb{C}}$ into two disjoint domains Γ_+ and Γ_-, in which holomorphic functions are to be found that can be glued together along Γ. Thus we obtain a global function on $\hat{\mathbb{C}}$ from local holomorphic data.

Theorem 4.1. *Let $g \in GC^{0,\alpha}(\Gamma)$ with $ind_\Gamma(g) = \kappa$. Then there exists exactly one piecewise holomorphic function f with root of order κ at ∞, whose boundary values satisfy $f_+ = gf_-$ and f_+ resp. f_- non-zero in Γ_+ resp. Γ_-.*

Proof. To take the lorarithm of g, set $G(t) := \frac{g(t)}{t^\kappa}$, $t \in \Gamma$. Now $ind_\Gamma(G) = 0$ and thus $\ln \circ G$ is H-continous and single valued. Define

$$F(z) := \frac{1}{2i\pi}\int_\Gamma \frac{\ln \circ G(t)}{t-z}dt$$

By Sokhotski/Plemelj F has boundary values F_- and F_+ which satisfy :

$$F_+ - F_- = \ln \circ G$$

exponentiating gives

$$e^{F_+(t)} = \frac{g(t)}{t^\kappa} e^{F_-(t)} \tag{8}$$

$e^{F_-(z)}$ for large $\|z\|$ goes to 1, since F_- vanishes for these. The function

$$f(z) := \begin{cases} e^{F_+(z)} & \text{for } z \in \Gamma_+ \\ z^{-\kappa}e^{F_-(z)} & \text{for } z \in \Gamma_- \end{cases}$$

solves the Riemann-problem $f_+ = gf_-$ and has a zero of order κ at ∞. Concerning the uniqueness, let $\tilde{f}$ be another solution with zero of order κ at ∞ and $\tilde{f}_+ = g\tilde{f}_-$. Then $q := \frac{\tilde{f}}{f}$ is piecewise holomorphic and the boundary-values satisfy :

$$q_+ = \frac{\tilde{f}_+}{f_+} = \frac{g\tilde{f}_-}{gf_-} = \frac{\tilde{f}_-}{f_-} = q_- \text{ on } \Gamma$$

so with Morera's theorem q is entire going towards 1 for large $\|z\|$, thus $q \equiv 1$ by Liouville and hence $\tilde{f} = f$. □

The equation (8) can be rewritten as

$$g(t) = e^{F_+(t)} \cdot t^\kappa \cdot e^{-F_-(t)}$$

and in the above proof, the function g was factorized, split into three factors, of which the right-hand side one was holomorphic in the interior, the left one holomorphic in the exterior of the contour, both invertible in their domains of holomorphicity and the middle factor was a monomial. Since merilly the knowledge of $ind_\Gamma(g) = \kappa$ for a given $g \in GC^{0,\alpha}(\Gamma)$ was required, $GC^{0,\alpha}(\Gamma)$ posses the following decomposition :

Set $g_0(t) := \frac{g(t)}{t^\kappa}$ and since $ind_\Gamma(g_0) = 0$ there's a function $f \in C^{0,\alpha}(\Gamma)$ with $e^f = g_0$. With the projections $\mathcal{P}_\Gamma = \frac{1}{2}(S_\Gamma + \text{Id})$ and $\mathcal{Q}_\Gamma = \frac{1}{2}(S_\Gamma - \text{Id})$ and the fact that $\mathcal{P}_\Gamma - \mathcal{Q}_\Gamma = \text{Id}$ we get

$$g_0 = e^f = e^{(\mathcal{P}_\Gamma - \mathcal{Q}_\Gamma)f} = e^{\mathcal{P}_\Gamma f} \cdot e^{-\mathcal{Q}_\Gamma f} = \frac{g(t)}{t^\kappa}$$

since $e^{\mathcal{P}_\Gamma f} \in G\mathcal{H}^+$ and $e^{-\mathcal{Q}_\Gamma f} \in G\mathcal{H}^-$ have

$$\begin{aligned} g(t) &= e^{-\mathcal{Q}_\Gamma f(t)} \cdot t^\kappa \cdot e^{\mathcal{P}_\Gamma f(t)} \\ &=: g_-(t) \cdot t^\kappa \cdot g_+(t) \end{aligned}$$

a factorisation of $g \in C^{0,\alpha}(\Gamma)$ and for the Riemann problem the solution :

$$f_+ := e^{\mathcal{P}_\Gamma f}, \quad f_- := \frac{e^{\mathcal{Q}_\Gamma f}}{t^\kappa}$$

if the boundary values are to satisfy $f_+ = gf_-$. Furthermore

$$GC^{0,\alpha}(\Gamma) = G\mathcal{H}_\Gamma^- \odot M(\Gamma) \odot G\mathcal{H}_\Gamma^+$$

where $M(\Gamma) := \{t \mapsto t^\kappa \mid \kappa \in \mathbb{Z}\}$. For meromorphic functions $r : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ without essential singularities, this factorisation becomes algebraic :

A meromorphic function r without zeros and poles on a contour Γ can be written as a rational function $r = \frac{p}{q}$, with p, q polynomials of the form $p(\lambda) = \prod_{i=1}^{S}(\lambda - \lambda_i)$ and $q(\lambda) = \prod_{j=1}^{L}(\lambda - \mu_j)$ with $\lambda_i, \mu_j \in \hat{\mathbb{C}}\backslash\Gamma$. With the aid of partial fractions r can be written as a finite Laurent series :

$$\begin{aligned} r(\lambda) &= \sum_{i=-a}^{-1} r_i\lambda^i + \sum_{i=0}^{b} r_i\lambda^i \\ &=: r_-(\lambda) + r_+(\lambda) \end{aligned}$$

and thus an additive decomposition.

Definition 4.2. Denote by $\mathcal{R}(\Gamma)$ the algebra of rational maps on $\hat{\mathbb{C}}$ with poles away from Γ and with

$$\mathcal{R}^-(\Gamma) := \{r \in \mathcal{R}(\Gamma) \text{ pole-less on } \Gamma_-\}$$

and

$$\mathcal{R}^+(\Gamma) := \{r \in \mathcal{R}(\Gamma) \text{ pole-less in } \Gamma_+\}$$

the subalgeras of the direct sum.

Again, this decomposition is achieved by the projectors $\mathcal{P}_\Gamma = \frac{1}{2}(S_\Gamma + \mathrm{Id})$ and $\mathcal{Q}_\Gamma = \mathcal{P}_\Gamma - \mathrm{Id}$. This statement is independent of the previous existence claims :

Lemma 4.2. *For the decomposition $r = r_- + r_+$ of $r \in \mathcal{R}(\Gamma)$ with $r_- \in \mathcal{R}^-(\Gamma)$ and $r_+ \in \mathcal{R}^+(\Gamma)$ we have for all $t \in \Gamma$:*

$$S_\Gamma r(t) = r_+(t) - r_-(t)$$

Proof. With equation (3) we have :

$$S_\Gamma r_+(t) = \frac{1}{i\pi}\int_\Gamma \frac{r_+(\tau) - r_+(t)}{\tau - t}d\tau + \frac{r_+(t)}{i\pi}\int_\Gamma \frac{1}{\tau - t}d\tau = r_+(t)$$

since the integrand $\frac{r_+(\tau)-r_+(t)}{\tau-t}$ is holomorphic in Γ_+. If r_- is of the form $r_-(t) = \frac{1}{(t-\mu)^n}$, $n \in \mathbb{N}$ und $\mu \in \Gamma_+$, then :

$$\frac{r_-(\tau) - r_-(t)}{\tau - t} = -\sum_{i=0}^{n-1}(t-\mu)^{i-n}(\tau-\mu)^{-i-1}$$

and thus

$$S_\Gamma r_-(t) = -r_-(t) \ , t \in \Gamma$$

for general $r_- \in \mathcal{R}^-(\Gamma)$ the claim follows from the fact that these posess an expansion $r_-(t) = \sum_{i=1}^{l} \frac{\alpha_i}{(t-\mu_i)^{n_i}}$ with $\alpha_i \in \mathbb{C}, \mu_i \in \Gamma_+, n_i \geq 0$. □

In the multiplicative decomposition of $G\mathcal{R}(\Gamma)$ the individual factors should be invertible in their domains of holomorphicity. We therefore have to seperate the roots of p and q such that $\lambda_1, \ldots\lambda_s,\ \mu_1, \ldots\mu_l \in \Gamma_+$ and $\lambda_{s+1}, \ldots\lambda_S,\ \mu_{l+1}, \ldots\mu_L \in \Gamma_-$.

$$r(\lambda) = \frac{\prod_{j=1}^{l} 1 - \frac{\mu_j}{\lambda}}{\prod_{i=1}^{s} 1 - \frac{\lambda_i}{\lambda}} \cdot \lambda^N \cdot \frac{\prod_{i=s+1}^{S} \lambda - \lambda_i}{\prod_{j=l+1}^{L} \lambda - \mu_j} \tag{9}$$

$$= r_-(\lambda) \cdot \lambda^N \cdot r_+(\lambda) \tag{10}$$

where $N = l - s$ is the difference of zeroes and poles of r in Γ_+. (Logarithmic residue $\int_\Gamma \frac{\acute{r}(\lambda)}{r(\lambda)} d\lambda$.) Since we restricted our attention on H-continous functions up to this point, we sketch two further classes where these methods are successful :

Example 4.1. (The Wiener algebra) Consider absolutly convergent Fourier series on the unit circle S^1 :

$$W(S^1) := \{\varphi : S^1 \to \mathbb{C} \mid \varphi(e^{i\nu}) = \sum_{n=-\infty}^{+\infty} \varphi_n e^{ni\nu} \ \ , \|\varphi\|_W := \sum_{n\in\mathbb{Z}} \mid \varphi_n \mid < \infty\}$$

with the obvious decomposition

$$W(S^1) = W^-(S^1) \oplus W^+(S^1)$$

where

$$W^-(S^1) := \{\varphi_-(e^{i\nu}) = -\sum_{n=-\infty}^{-1} \varphi_n e^{ni\nu}\}$$

and

$$W^+(S^1) := \{\varphi_+(e^{i\nu}) = \sum_{n=0}^{\infty} \varphi_n e^{ni\nu}\}$$

and the corresponding projections

$$\mathcal{Q} : W(S^1) \to W^-(S^1)$$
$$\mathcal{P} : W(S^1) \to W^+(S^1)$$

The continuity of the singular integraloperator S_Γ is a consequence of the fact that $\mathcal{R}(\Gamma)$ lies densly in $W(S^1)$. All calculations and formulas transfer.

Example 4.2. (Square integrable functions) On the Hilbert space of square integrable almost nowhere vanishing functions on Γ we can also solve the Riemann problem :

$$f_+ = g f_- \text{ allmost everywhere on } \Gamma$$

Lemma 4.3. *$S_\Gamma : L^2(\Gamma) \to L^2(\Gamma), \varphi \mapsto \frac{1}{i\pi} \int_\Gamma \frac{\varphi(t)}{t - \cdot} dt$ is bounded.*

Proof. $\langle C^{0,\alpha}(\Gamma), C^{0,\alpha}(\Gamma)\rangle$ is a positive dualsystem, generated by the scalar product

$$\langle f, g\rangle := \int_\Gamma f(z)\bar{g}(z)dz \quad f, g \in C^{0,\alpha}(\Gamma)$$

The adjoint S_Γ^* is bounded since $S_\Gamma : C^{0,\alpha}(\Gamma) \to C^{0,\alpha}(\Gamma)$ is bounded. Hence the claim follows from the Lax-Milgram-Lemma and the fact that $C^{0,\alpha}(\Gamma)$ is dense in $L^2(\Gamma)$. □

The induced projections $\mathcal{P}_\Gamma$, $\mathcal{Q}_\Gamma$ are orthogonal on $L^2(\Gamma)$ only when Γ is a circle [6]. In the course of solving the Riemann problem, the following sufficient conditions on function spaces $\mathcal{E}$ were obtained:

(1) Decomposition into a direct sum $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$
(2) Sufficient smoothness to ensure existence of principal value
(3) Closedness of the subalgebras $\mathcal{E}^+, \mathcal{E}^-$ w.r.t inversion

Subalgebras of $C(\Gamma)$ satisfying 1.-3. are thus natural domains of the integral operator $T := \mathcal{P} - g\mathcal{Q}$ whose kernel is a solutionspace of the Riemann problem $f_+ = gf_-$ are called *admissable* in the literature. It should also be mentioned that the preceeding theory has been extended to L^p-spaces for $1 \leq p \leq \infty$ [6].

5. Factorisation of matrix valued functions

Let $A = (a_{ij})_{i,j=1}^n$ be a $n \times n$-matrix whose entries $a_{ij} : \Gamma \to \mathbb{C}$ are continous functions of the variable $\lambda \in \Gamma$. Alternatively we can think of $A : \Gamma \to gl(n, \mathbb{C})$ as a loop in the Lie-algebra $gl(n, \mathbb{C})$. Operators on $n \times n$-matrix-valued functions operate on the individual entries, so for example

$$\int_\Gamma A(\lambda) d\lambda = \left(\int_\Gamma a_{ij}(\lambda) d\lambda \right)_{i,j=1}^n \in gl(n, \mathbb{C})$$

for a continous $A \in gl(n, \mathcal{C}(\Gamma))$. For a vector-valued function $x : \Gamma \to \mathbb{C}^n$ and two matrix-valued functions $A(\lambda)$, $B(\lambda)$ the matrix-vector and matrix-matrix multiplication are defined pointwise: $(Ax)(\lambda) = A(\lambda)x(\lambda)$ and $(AB)(\lambda) = A(\lambda)B(\lambda)$. We will use the notation of the previous sections $\mathcal{H}^\pm$, $\mathcal{R}(\Gamma)$, $\mathcal{E}(\Gamma)$ etc to denote by $gl(n, \mathcal{A})$ the algebra of $n \times n$-matrices whose entries are elements of $\mathcal{A}$.

Definition 5.1. A matrix $A(z)$ of the algebra $GL(n, \mathcal{C}(\Gamma))$ of invertible $n \times n$-matrices with continous entries posesses a *left factorization w.r.t* Γ if it can be written

$$A(z) = A_+(z) D(z) A_-(z)$$

on Γ where the outer factors $A_\pm(z)$ are restrictions of holomorphic invertible matrices in $\Gamma_\pm$ i.e $A_\pm \in GL(n, \mathcal{H}^\pm)$ and the middle factor is a diagonal matrix:

$$D(z) = \begin{pmatrix} z^{k_1} & & \\ & \ddots & \\ & & z^{k_n} \end{pmatrix}$$

and the integers are ordered $k_1 \geq \ldots \geq k_n$.

Exchanging $+$ and $-$ leads to the notion of right factorisation and the two theories are essentially equivalent modulo transposition. We shall pursue the left theory and will first investigate the polynomial loops.

Definition 5.2. A matrix whose entries $(a_{ij}(\lambda))_{i,j=1}^n$ are polynomials in $\lambda \in \mathbb{C}$ is called a polynomial matrix and can also be written as a polynomial with matrix coefficients :

$$A(\lambda) = A_0 + A_1\lambda + \ldots + A_m\lambda^m, \quad A_i \in gl(n, \mathbb{C})$$

This polynomial is called proper, if the coefficient of the highest power is regular, i.e $det(A_m) \neq 0$. The degree of a polynomial matrix is the highest degree of its entries

$$deg(A(\lambda)) := \max_{i,j} \{deg(a_{ij})\}$$

DIVISION :
$A(\lambda), B(\lambda)$ polynomial matrices of order n, B proper and if $A(\lambda) = Q(\lambda)B(\lambda) +$

$R(\lambda)$ $\,, deg(R) < deg(B)$, then $Q(\lambda)$ is called right quotient, $R(\lambda)$ right remainder of the division of $A(\lambda)$ by $B(\lambda)$
Euclidian Algorithm :
Let $A(\lambda) = \sum_{i=o}^{m} A_i \lambda^i$ and $B(\lambda) = \sum_{j=o}^{p} B_j \lambda^j$ polynomial matrices with

$$deg(A(\lambda)) = m$$

and

$$deg(B(\lambda)) = p$$

(1) (case : $p > m$) Set $Q(\lambda) \equiv 0$, $R(\lambda) := A(\lambda)$

(2) (case : $p \leq m$) Divide $A(\lambda)$ by $B(\lambda)$ from the right, we obtain for the highest term a first iteration $Q(\lambda) = A_m B_p^{-1} \lambda^{m-p} + \ldots$ and hence

$$A(\lambda) = A_m B_p^{-1} \lambda^{m-p} B(\lambda) + R^{(1)}(\lambda)$$

where $R^{(1)}(\lambda) := R_0^{(1)} + \cdots + R_{m^{(1)}}^{(1)}$ and $deg(R^{(1)}(\lambda) =: m^{(1)} < m$
If $m^{(1)} < p$, set $Q(\lambda) = A_m B_p^{-1} \lambda^{m-p}$ and $R(\lambda) = R^{(1)}(\lambda)$,
otherwise $(m^{(1)} \geq p)$:
Divide $R^{(1)}(\lambda)$ by $B(\lambda)$ from the right, i.e

$$R^{(1)}(\lambda) = R_{m^{(1)}}^{(1)} B_p^{-1} \lambda^{m^{(1)}-p} B(\lambda) + R^{(2)}(\lambda)$$

with $R^{(2)}(\lambda) = R_0^{(2)} + \cdots + R_{m^{(2)}}^{(2)} \lambda^{m^{(2)}}$ $\quad deg(R^{(2)}(\lambda) =: m^{(2)} < m^{(1)}$
If $m^{(2)} < p$: set $Q(\lambda) = A_m B_p^{-1} \lambda^{m-p} + R_{m^{(1)}}^{(1)} B_p^{-1} \lambda^{m^{(1)}-p}$ and $R(\lambda) = R^{(2)}(\lambda)$
otherwise $(m^{(2)} \geq p)$:
Divide $R^{(2)}(\lambda)$ by $B(\lambda)$ and proceed as above and since the degrees of the remainders $R^i(\lambda)$ are decreasing, there exists $i \in \mathbb{N}$ such that the degree of the i-th iterated remainder is smaller than the degree of the divisor $B(\lambda)$.
Thus $A(\lambda) = Q(\lambda)B(\lambda) + R(\lambda)$ with $deg(R) < deg(B)$ where

$$Q(\lambda) = A_m B_p^{-1} \lambda^{m-p} + R_{m^{(1)}}^{(1)} B_p^{-1} \lambda^{m^{(1)}-p} + R_{m^{(2)}}^{(2)} B_p^{-1} \lambda^{m^{(2)}-p} + \ldots$$

$$R(\lambda) = R^{(i)}(\lambda)$$

Concerning the uniqueness : If $A(\lambda) = Q(\lambda)B(\lambda) + R(\lambda) = \tilde{Q}(\lambda)B(\lambda) + \tilde{R}(\lambda)$ with

$$deg(R(\lambda)) < deg(B(\lambda)) > deg(\tilde{R}(\lambda)$$

and

$$(Q(\lambda) - \tilde{Q}(\lambda))B(\lambda) = R(\lambda) - \tilde{R}(\lambda)$$

then assuming $Q(\lambda) - \tilde{Q}(\lambda) \neq 0$ we obtain the contradiction $deg(R - \tilde{R}) \geq deg(B)$.

6. Diagonalisation

Analog to constant matrices there also exist normal forms for polynomial matrices into which they can be transformed by means of elementary transformations.

Definition 6.1. Two polynomial matrices are called equivalent, if a combination of the following operations transforms one into the other.

(1) Multiplication of a row or column by a non-zero scalar.
(2) Addition of a polynomial multiple of a row or column with a row or column.
(3) Interchanging of rows resp. columns.

Note that matrix multiplication with the above elementary matrices from the left effect lines, from the right columns. The corresponding matrices to the operations 1.-3. have λ-independent non-zero determinant.

DIAGONALISATION (SMITH - Form) :
Let $A(\lambda) = (a_{ik}(\lambda))_{i,k=1}^{n}$ be a polynomial matrix of order $n \geq 0$

(1) Choose $a_{ik}(\lambda)$ of least order that's not identically zero and use elementary transformations to exchange it with $a_{11}(\lambda)$
(2) Divide all other entries of the first row and first column by $a_{11}(\lambda)$ i.e :

$$a_{i1}(\lambda) = a_{11}(\lambda)q_{i1}(\lambda) + r_{i1}(\lambda)$$

$$a_{1k}(\lambda) = a_{11}(\lambda)q_{1k}(\lambda) + r_{1k}(\lambda)$$

For $a_{i1}(\lambda)$ to be replaced by $r_{i1}(\lambda)$ resp. $a_{1k}(\lambda)$ by $r_{1k}(\lambda)$, subtract from this row resp. column the corresponding multiple of the first row resp. column . If not all remainders vanish, a new pivot element will be at service to succesivly reduce the orders of the remainders until finally all entries of the first column and first row except the current diagonal element $\tilde{a}_{11}(\lambda)$ will have vanished, that is $A(\lambda)$ is equivalent to a matrix of the form

$$\begin{pmatrix} \tilde{a}_{11}(\lambda) & 0 \\ 0 & \tilde{A}(\lambda) \end{pmatrix} \quad (*)$$

In the matrix $\tilde{A}(\lambda)$ there might be entries $\neq 0$, whose order are smaller than $\tilde{a}_{11}(\lambda)$. If this is the case, bring the element of least order that doesn't vanish identically into pivot position and start with step 2. A repetition of this order reduction will finally transform $A(\lambda)$ into a matrix of the form $(*)$, in which $\tilde{a}_{11}(\lambda)$ is the smallest order entry not identically zero.
(3) If $\tilde{a}_{11}(\lambda)$ divides an entry $\tilde{a}_{ij}(\lambda)$ of $\tilde{A}(\lambda)$ with remainder, add the j-th column to the first column, determine quotients and remainders of this new column from the division by $\tilde{a}_{11}(\lambda)$ and repeat steps 1. and 2. until the form $(*)$ is achieved. This process can only be repeated finitely many times before we obtain a matrix of the form :

$$\begin{pmatrix} d_1(\lambda) & 0 \\ 0 & A_1(\lambda) \end{pmatrix}$$

in which $d_1(\lambda)$, after a possible rescaling, divides all non-zero entries of $A_1(\lambda)$ without remainder. Now proceed analogously with the submatrix $A_1(\lambda)$.

Alltogether we have proven

Theorem 6.1. *For every polynomial matrix $A(\lambda)$ there exist two matrices $E(\lambda), F(\lambda)$ with non-zero constant, λ-independent, determinant that diagonalize $A(\lambda)$:*

$$E(\lambda)A(\lambda)F(\lambda) = diag[d_1(\lambda), ..., d_n(\lambda)]$$

$$d_{i-1}(\lambda) \mid d_i(\lambda) \quad , i = 2, ..., n$$

The Smith form is not unique, since there where choices involved in constructing the matrices $E(\lambda), F(\lambda)$, but it turns out that the exponents of the diagonal entries can be computed.

Lemma 6.2. *The entries of the diagonal matrix $d_i(\lambda)$ are given by*

$$d_i(\lambda) = \frac{\gcd\{i \times i\text{-subdeterminants of } A(\lambda)\}}{\gcd\{(i-1) \times (i-1)\text{-subdeterminants of } A(\lambda)\}}$$

The $d_i(\lambda)$'s are called invariant divisors and comprise all invariants under elementary transformations.

Proof. Let $E(\lambda)A(\lambda)F(\lambda) = diag[d_1(\lambda), ..., d_n(\lambda)]$. The rows of $E(\lambda)A(\lambda)$ are a liner combination of the rows of $A(\lambda)$ and hence $i \times i$-subdeterminants of $E(\lambda)A(\lambda)$ are linear combinations of $i \times i$-subdeterminants of $A(\lambda)$.
Likewise $i \times i$-subdeterminants of $E(\lambda)A(\lambda)F(\lambda)$ are linear combinations of $i \times i$-subdeterminants of $E(\lambda)F(\lambda)$, and hence also of $i \times i$-subdeterminants of $A(\lambda)$ i.e

$$\gcd\{i \times i\text{-subdet. of } E(\lambda)A(\lambda)F(\lambda)\} = \gcd\{i \times i\text{-subdet. of } A(\lambda)\}$$

Since $d_{i-1} \mid d_i$ for $i = 2, ..., n$ the elements $d_1(\lambda) \cdot ... \cdot d_i(\lambda)$ are always the gcd's of all $i \times i$-subdeterminants of $E(\lambda)A(\lambda)F(\lambda)$ that don't identically vanish, and so the conclusion follows with

$$d_1(\lambda) = \gcd\{a_{ij}(\lambda) \mid i, j = 1, ..., n\}$$

□

Since every complex polynomial factorizes, the Smith form is a convenient means to convey the zeroes of the determinant of $A(\lambda)$.

Definition 6.2. The individual factors $(\lambda - \lambda_i)^{k_i}$ of the diagonal elements of the Smith form are called elementary divisors of multiplicity k_i.

The obove can be used to prove

Corollary 6.3. *Let $A(z) = (a_{ij}(z))$ be analytic in z_0 with $det(A(z))$ not identically zero. Then $A(z)$ can be written :*

$$A(z) = E(z)diag[(z - z_0)^{k_1}, \dots, (z - z_0)^{k_n}]F(z)$$

with matrices $E(z)$, $F(z)$ analytic in and invertible in a neighborhood of z_0 and integers $k_1 \geq \dots \geq k_n$.

Proof. Let m_{ij} be the multiplicity of z_0 as a root of a_{ij} having set $m_{ij} = \infty$ if $a_{ij} \equiv 0$ and $m_{ij} = 0$ if $a_{ij}(z_0) \neq 0$. Let $p := min\{m_{ij}\}$ and since $det(A(z))$ is not identically zero, must have $p < \infty$. W.l.o.g let $p = m_{11}$. Now multiply the first row of $A(z)$ with $\frac{(z-z_0)^p}{a_{11}(z)}$ to obtain a matrix $A_1(z)$, whose $(1,1)-$ entry is $(z - z_0)^p$. Write $A_1(z) = (b_{ij}(z))_{i,j=1}^n$ with $b_{ij}(z) = (z - z_0)^p c_{ij}(z)$ analytic functions $c_{ij}(z)$ and $c_{ij}(z_0) \neq 0$ and now proceed as is the polynomial case. □

7. Spectral theory of polynomial matrices

A root $\hat{\lambda}$ of $det(L(\lambda))$ of a polynomial matrix $L(\lambda)$ is also called an eigenvalue of $L(\lambda)$ and a vector $y \in \mathbb{C}^n \backslash \{0\}$ resp. $x \in \mathbb{C}^n \backslash \{0\}$ a corresponding left resp. right eigenvector, if $yL(\hat{\lambda}) = 0$ resp. $L(\hat{\lambda})x = 0$. In particular, there are k corresponding left resp. right eigenvectors $y_1, ..., y_k$ resp. $x_1, ..., x_k$ to a root of multiplicity k of $det(L(\lambda))$. For aesthetic and habitual reasons we'll pursue the right theory and omit the reference 'right'

Definition 7.1. The vectors $x_1, ..., x_k$,$x_1 \neq 0$ form an eigenchain to an eigenvalue $\hat{\lambda}$ of lenght k, if the following equations are satisfied :

$$\sum_{p=0}^{j} \frac{1}{p!} L^{(p)}(\hat{\lambda}) x_{j+1-p} = 0 \quad ; j = 0, 1, ..., k-1$$

where $L^{(p)}(\hat{\lambda})$ denotes the p-th derivative of L w.r.t λ in $\hat{\lambda}$.

A Jordan chain to an eigenvalue $\hat{\lambda}$ can be easily obtained from a Smith form in the following way :
Let $diag[d_1(\lambda), ..., d_n(\lambda)] = E(\lambda)L(\lambda)F(\lambda)$ be a Smith form of a polynomial matrix $L(\lambda)$. Denote with $f_i(\lambda)$ the i-th column of $F(\lambda)$ and with $e_i(\lambda)$ the i-th column of $E^{-1}(\lambda)$. Then $L(\lambda)f_i(\lambda) = e_i(\lambda)d_i(\lambda), i = 1, ..., n$.

If $d_i(\lambda) = (\lambda - \hat{\lambda})^k r(\lambda)$ with $r(\hat{\lambda}) \neq 0, \quad k \geq 1$, then $\hat{\lambda}$ is null-eigenvalue of $L(\lambda)$ with elementary divisor $(\lambda - \hat{\lambda})^k$. Differentiating $L(\lambda)f_i(\lambda) = e_i(\lambda)(\lambda - \hat{\lambda})^k r(\lambda)$ $(k-1)$-times in $\hat{\lambda}$ then the r.h.s vanish and addition of all k-1 equations renders :

$$\sum_{p=0}^{j} \binom{j}{p} L^{(p)}(\hat{\lambda}) f_i^{(j-p)}(\hat{\lambda}) = 0 \text{ for j = 0,1,...,k-1}$$

Thus the vectors

$$f_i(\hat{\lambda}), \frac{1}{1!} f_i^{(1)}(\hat{\lambda}), ..., \frac{1}{(k-1)!} f_i^{(k-1)}(\hat{\lambda})$$

form an eigenchain of lenght k to the eigenvalue $\hat{\lambda}$. With the aid of eigenchains one can also obtain a fundamental system of the following linear system of differential equations with constant coefficients.

Lemma 7.1. *The vectors $u_1, \ldots, u_k$ in $\mathbb{C}^n$ with $u_1 \neq 0$ form an eigenchain to the eigenvalue $\hat{\lambda}$ of the polynomial matrix $A(\lambda) = \sum_{i=0}^{m} A_i \lambda^i$ if and only if the functions*

$$v_{j+1}(\tau) = e^{\hat{\lambda}\tau} \sum_{p=0}^{j} \frac{\tau^p}{p!} u_{j+1-p} \quad , j = 0, \ldots, k-1$$

solve the homogenous differential equation :

$$\sum_{i=0}^{m} A_i \tfrac{d^i}{d\tau^i} v(\tau) = 0$$

Proof. The simple computation is achieved by writing $A(\lambda)$ into its Taylor series in $\hat{\lambda}$:

$$A(\lambda) = \sum_{i=0}^{m} \tfrac{1}{i!} A^{(i)}(\hat{\lambda})(\lambda - \hat{\lambda})^i$$

and replacing λ with $\frac{d}{d\tau}$. □

The number of roots λ_i of $det(A(\xi))$ of a polynomial matrix $A(\xi) = \sum_{j=0}^{l} A_j \xi^j, A_j \in gl(n;\mathbb{C})$ counted with multiplicities (c.w.m) is at most the product of the highest power with the order n of the matrix. The polynomial $det(A(\xi))$ can thus be written

$$det\, A(\xi) \; = k \prod_{i=1}^{ln} (\xi - \lambda_i) \quad k \in \mathbb{C}$$

To each non-constant elementary divisor $d_{\lambda_i}(\xi) = (\xi - \lambda_i)^{r_i}$ of degree r_i assign a pair $\left(X_{d_{\lambda_i}}, J_{d_{\lambda_i}}\right)$, where $X_{d_{\lambda_i}}$ is the $n \times r_i$-matrix whose columns $x_1, \ldots, x_r$ are the eigenchain to the eigenvalue λ_i and $J_{d_{\lambda_i}} = \lambda_i \delta_{s,t} + \delta_{s+1,t}$ is the usual $r_i \times r_i$-Jordan block :

$$X_{d_{\lambda_i}} = col[x_1, \ldots, x_r] \quad J_{d_{\lambda_i}} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{pmatrix}$$

If $d_1(\lambda), \ldots, d_s(\lambda)$ is an enumeration of all elementary divisors of the polynomial matrix $A(\lambda)$, set

$$X := col[X_{d_1}, \ldots, X_{d_s}] \in gl(\underbrace{ord(A)}_{n} \times \underbrace{deg(detA)}_{ln}; \mathbb{C})$$

where the matrix X is arranged such that the first columns of the X_{d_i}'s span the kernel of $A(\lambda_i)$ and

$$J := J_{d_1} \oplus \ldots \oplus J_{d_s} = \begin{pmatrix} J_{d_1} & & \\ & \ddots & \\ & & J_{d_s} \end{pmatrix}$$

Definition 7.2. The pair (X, J) is called (finite) spectral pair and contains all spectral data.

Due to the block structure the k-th power of J is obtained by taking the k-th power of each individual Jordan cell. In particular

$$J_{d_i}^k = (\lambda_i Id + \delta_{i+1,j})^k = \sum_{n=0}^{k} \begin{pmatrix} k \\ n \end{pmatrix} \lambda_i^{k-n} \delta_{i+n,j}$$

All in all the spectral data of a polynomial matrix $A(\lambda) = \sum_{i=0}^{l} A_i \lambda^i$ with $A_i \in gl(n, \mathbb{C})$ are encoded in the solution of the matrix equation of the size $n \times ln$:

$$A_0 X + A_1 X J + A_2 X J^2 + \ldots + A_l X J^l = 0 \tag{11}$$

Note that the matrix

$$Q := col[X \cdot J^i]_{i=0}^{l-1}$$

has maximal rank and in the special case $det(A_l) \neq 0$ have $Q \in GL(ln; \mathbb{C})$ and can assign the polynomial matrix an $ln \times ln$-matrix :

$$L(\lambda) = \sum_{i=0}^{l} A_i \lambda^i \mapsto \mathcal{L} := \begin{pmatrix} 0 & I & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & I \\ A_l^{-1} A_0 & \cdots & \cdots & \cdots & A_l^{-1} A_{l-1} \end{pmatrix}$$

Definition 7.3. $\mathcal{L}$ is called the first companion matrix, the transposed $\mathcal{L}^T$ the second companion of the polynomial matrix $L(\lambda)$.

$x_0,\dots,x_{k-1}$ is eigenchain of L to the eigenvalue λ iff the vectors

$$\begin{pmatrix} x_0 \\ \lambda x_0 \\ \lambda^2 x_0 \\ \vdots \\ \lambda^{l-1} x_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ \lambda x_1 + x_0 \\ \lambda^2 x_1 + 2x_0 \\ \vdots \\ \lambda^{l-1} x_1 + (l-1)\lambda^{l-2} x_0 \end{pmatrix}, \dots, \begin{pmatrix} x_{k-1} \\ \lambda x_{k-1} + x_{k-2} \\ \vdots \\ \vdots \\ \lambda^{l-1} x_{k-1} + \dots + x_{k-l} \end{pmatrix} \quad (*)$$

are an eigenchain of $\mathcal{L}$ to the eigenvalue λ. $\mathcal{L}$ is similar to a Jordanmatrix which is diagonal iff all elementary divisors are simple. It is worth noting that the chain in $(*)$ is linearly independent and that two chains to different eigenvalues of $\mathcal{L}$ are linearly independent so that the set of all chains form a basis of $gl(ln;\mathbb{C})$. In contrast to this, neither the vectors $x_0,\dots,x_{k-1}$ need be linearly independent nor are chains to different eigenvalues linearly independent. In the case $det(A_l)=0$ the polynomial matrix $L(\lambda)$ has the linearisation :

$$C_L(\lambda) := \lambda \begin{pmatrix} I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & & \ddots & & \\ \vdots & & & I & 0 \\ 0 & \dots & \dots & 0 & A_l \end{pmatrix} - \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & I \\ -A_0 & -A_1 & \dots & \dots & -A_{l-1} \end{pmatrix}$$

with $ln \times ln$-matrices such that we also have $det(L(\lambda)) = 0$ iff $det(C_L(\lambda)) = 0$. If $det(L(\lambda))$ not identically zero, it can only have finitely many, so that the inverse, where it exists can be written :

$$L^{-1}(\lambda) = [Id_n, 0, \dots, 0] C_L(\lambda)^{-1} [0, \dots, 0, Id_n]^T$$

and for $L^{\#}(\lambda) := \lambda^l L(\frac{1}{\lambda})$ have

$$L^{\#}(\lambda)^{-1} = [0, \dots, 0, \mathrm{Id}_n] C_L^{\#}(\lambda)^{-1} [0, \dots, 0, \mathrm{Id}_n]$$

8. Rational matrices

If a polynomial matrix $A(\lambda) = \sum A_i \lambda^i$ is pointwise invertible on a contour Γ i.e $det(A(\lambda)) \neq 0$ for all $\lambda \in \Gamma$, then we can arrange the roots of $det(A(\lambda))$ with the aid of the Smith form $E(\lambda)D(\lambda)F(\lambda)$ on the middle diagonal term $D(\lambda)$ in such a way that $D(\lambda) = D_+(\lambda)D_-(\lambda)$ and $det(D_+(\lambda)) \neq 0$ for $\lambda \in \Gamma_+$ and $det(D_-(\lambda)) \neq 0$ for $\lambda \in \Gamma_-$. Hence a factorisation of the polynomial matrix

$$A(\lambda) = A_+(\lambda) A_-(\lambda)$$

is achieved, where A_+ in Γ_+ holomorphic and invertible and A_- is invertible in Γ_- but of course A_- cannot be holomorphic there, since the entries are polynomials and have poles at ∞. To remove these poles of the negative factor we need negative powers of λ and thus leave the class of polynomials and move on to the rationals.

Definition 8.1. Let Γ as previously be a contour on the Riemann sphere. Denote by

$$gl(n, \mathcal{R}(\Gamma)) := \{ \left(r_{ij}(\lambda) = \frac{p(\lambda)}{q(\lambda)} \right)_{i,j=1}^{n} \; p, q \text{ polynomials, } q(\lambda) \neq 0 \text{ for all } \lambda \in \Gamma \}$$

the rational matrix functions with poles away from Γ. The invertible elements in $gl(n,\mathcal{R}(\Gamma))$ are denoted by

$$GL(n,\mathcal{R}(\Gamma)) := \{R(\lambda) \in gl(n,\mathcal{R}(\Gamma)) | det(R(\lambda)) \neq 0 \text{ for all } \lambda \in \Gamma\}$$

and with

$$GL(n,\mathcal{R}^{\pm}(\Gamma)) := \{R(\lambda) \in GL(n,\mathcal{R}(\Gamma)) | det(R(\lambda)) \neq 0 \text{ for all } \lambda \in \Gamma_{\pm}$$
$$\text{und } R(\lambda) \text{ holomorphically extendable to } \Gamma_{\pm}\}$$

The projections $\mathcal{P} = \frac{1}{2}(S_\Gamma + \mathrm{Id})$ and $\mathcal{Q} = \frac{1}{2}(S_\Gamma - \mathrm{Id})$ realize the decomposition

$$gl(n,\mathcal{R}(\Gamma)) = gl(n,\mathcal{R}^{-}(\Gamma)) \oplus gl(n,\mathcal{R}^{+}(\Gamma))$$

The next aim is to show that any regular $R \in GL(n,\mathcal{R}(\Gamma))$ can be decomposed multiplicatively. Analogously to (9) there appears a middle term that swallows the poles of the outer factors.

Theorem 8.1. *Every regular $R \in GL(n,\mathcal{R}(\Gamma))$ allows a factorisation :*

$$R(\lambda) = R_{+}(\lambda) D(\lambda) R_{-}(\lambda)$$

with $R_{\pm} \in GL(n,\mathcal{R}^{\pm}(\Gamma))$ and

$$D(\lambda) = \begin{pmatrix} \lambda^{k_1} & & \\ & \ddots & \\ & & \lambda^{k_n} \end{pmatrix}$$

and integers $k_1 \geq \ldots \geq k_n$, called the partial indices.

Proof. (In [3] the proof is for $R = R_{-} D R_{+}$)

(1) Step : Pull out the gcd of all entries of $R(\lambda)$, which is a rational function $r(\lambda)$ and factorise it as in (9) w.r.t Γ. Hence

$$R(\lambda) = r_{+}(\lambda) \cdot \lambda^{n} \cdot P(\lambda) \cdot r_{-}(\lambda)$$

with a polynomial matrix $P(\lambda)$ and the rational functions r_{+} , r_{-} that are invertible in their domains of holomorphicity and so it remains to factorise the polynomial matrix $P(\lambda)$. (λ^n will be dealt with later.)

(2) Step : (Construction of the positive factor $R_{+}(\lambda)$)
Denote the set of zeroes of a map F by $\mathcal{Z}(F)$. Now find the Smith form of $P(\lambda) = E(\lambda)D(\lambda)F(\lambda)$ and sort the zeroes of the diagonal entries such that

$$D(\lambda) = D_{+}(\lambda) D_{-}(\lambda)$$

with $\mathcal{Z}(det\, D_{+}) \subset \Gamma_{-}$ und $\mathcal{Z}(det\, D_{-}) \subset \Gamma_{+}$. Since $E(\lambda)$ is polynomial in λ and invertible everywhere have

$$R_{+}(\lambda) := E(\lambda) D_{+}(\lambda) \in GL(n,\mathcal{R}^{+}(\Gamma))$$

(3) Step : (Partial construction of the negative factor $R_{-}(\lambda)$)
The entries of the negative factor to be constructed mustn't all be zero at ∞ and what more, we want R_{-} invertible in ∞, which implies that each column needs to have an entry with a constant coefficient, and these columns need to be linearly independent. This shall be achieved in the following and fourth step. The first goal is to transform the remaining term $D_{-}(\lambda)F(\lambda)$ into a matrix with solely negative powers of λ, whose

determinant only posesses a root at $\lambda^{-1} = 0$. Since $\mathcal{Z}(det\, D_- F) \subset \Gamma_+$ have for $M := deg(D_- F)$:

$$N(det\, \lambda^M D_-(\frac{1}{\lambda})F(\frac{1}{\lambda})\,) \subset \{0\} \cup \Gamma_+^{-1} \backslash \{\infty\}$$

Take the Smith form of $\lambda^M D_-(\frac{1}{\lambda})F(\frac{1}{\lambda}) := \lambda^M E_1(\frac{1}{\lambda})D_1(\frac{1}{\lambda})F_1(\frac{1}{\lambda})$ and again sort the roots of the determinant such that

$$\begin{aligned} D_1(\frac{1}{\lambda}) &= diag[d_1(\frac{1}{\lambda}), \ldots, d_n(\frac{1}{\lambda})] \\ &= diag[f_1(\frac{1}{\lambda}), \ldots, f_n(\frac{1}{\lambda})] \cdot diag[g_1(\frac{1}{\lambda}), \ldots, g_n(\frac{1}{\lambda})] \end{aligned}$$

and $\mathcal{Z}(g_i) \subset \Gamma_+^{-1} \backslash \{\infty\}$. So we have

$$R_-(\frac{1}{\lambda}) := diag[g_1(\frac{1}{\lambda}), \ldots, g_n(\frac{1}{\lambda})] \cdot F_1(\frac{1}{\lambda}) \in GL(n, \mathcal{R}^-(\Gamma))$$

with remaining middle factor

$$\lambda^{K+M} E_1(\frac{1}{\lambda}) \cdot diag[f_1(\frac{1}{\lambda}), \ldots, f_n(\frac{1}{\lambda})] \;=: \mathcal{M}(\lambda)$$

where $K \geq 0$ is chosen so that any positive λ-powers that might have appeared during the last diagonalisation are compensated. Hence the factorisation of rational matrices is reduced to the factorisation of polynomial matrices whose only root of the determinant lies in $\lambda^{-1} = 0$.

(4) Step : (Construction of the monomial factor $D(\lambda)$)
If $x(\lambda) = \sum_{i=-a}^{-b} x_i \lambda^i$ with $a, b \in \mathbb{N}$ is a polynomial vector of negative λ-powers with coefficients $x_i \in \mathbb{C}^n$, then $x(\infty) = 0$ except when $b = 0$ and $x_0 \neq 0$. If the highest power for which the coefficient isn't zero is pulled out, w.l.o.g let this be b, then

$$\sum_{i=-a}^{-b} x_i \lambda^i = \lambda^{-b} \underbrace{\sum_{i=-a+b}^{0} x_{i-b} \lambda^i}_{\neq 0 \; in \; \infty}$$

This procedure will now be applied to the columns of $\mathcal{M}$. Let l be the multiplicity of the root $\lambda^{-1} = 0$ of $det\, \mathcal{M}(\lambda)$ and write

$$\mathcal{M}(\lambda) = col[r_1(\lambda), \ldots, r_n(\lambda)]; \quad r_i(\lambda) \text{ polynomial columns}$$

Let $m_1, \ldots, m_n$ be the multiplicities of the roots $\frac{1}{\lambda} = 0$ of $r_1, \ldots, r_n$ and w.l.o.g ordered such that $m_1 \geq \ldots \geq m_n$. Write $r_i(\lambda^{-1}) = \lambda^{m_i} r_i'(\lambda^{-1})$ and thus

$$\mathcal{M}(\lambda) = diag[\lambda^{m_1}, \ldots, \lambda^{m_n}] \cdot col[r_1'(\frac{1}{\lambda}), \ldots, r_n'(\frac{1}{\lambda})] \quad (*)$$

(5) If $\sum_{j=1}^n m_j = l$ then $(*)$ is the desired factorisation of $\mathcal{M}$, otherwise ($\sum m_j < l$) the $\{r_i'(0)\}_{i=1}^n$ are linearly dependant. Let p be the smallest number for which the $r_1'(0), \ldots, r_p'(0)$ lie in the span of $\{r_1'(0), \ldots, r_{p-1}'(0)\}$. Then there exist $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ not all zero, such that the vectorvalued function $\lambda \mapsto \sum_{j=1}^p \alpha_j r_j'(\lambda)$ has a root at $\lambda = 0$. Replace the p-th column of $\mathcal{M}$ with $\sum \alpha_j r_j'(\lambda)$ by means of some elementary transformation. With this we have increased the multiplicity of the p-th column by at least one and

repeating this procedure until the sum of the multiplicities of the columns $r_j^{'}$ is maximal i.e l. Then again $(*)$ renders the desired diagonalisation, with the λ-powers of step 1 and step 3 included in the diagonal matrix.

□

In 1913 Birkhoff obtained the analogous result for matrices holomorphic and invertible in a puntured disc in the attempt to reduce homogenous linear systems of differential equations with holomorphic coefficients to a canonical form. In the 1950's Gantmacher [5] and Masani[10] found counterexamples to this reduction.

9. The partial indices

The partial indices developed in three steps in the preceding proof of Theorem 7.1 . In the first step, the factorisation of the rational scalar function contributed the difference between zeroes and poles, in the third step the change of coordinate $\lambda \mapsto \frac{1}{\lambda}$ and the successive diagonalisation, where the highest occuring λ-power is pulled out, further change the indices and finally in the fourth step the zero $\lambda = \infty$ is divided out of the columns. Unfortunately these manipulations don't lead to an apriori formula for the partial indices, which are further obscured by the fact that the diagonalisation(Smith form) is not unique. Yet it turns out that the diagonal factor is unique, that from one factorisation all others can be obtained and that the dual theory of singular integral operators affords the possibility of finding a formula for the partial indices.

Lemma 9.1. *Let* $A(\lambda) = A_+(\lambda)D(\lambda)A_-(\lambda) = \tilde{A}_+(\lambda)\tilde{D}(\lambda)\tilde{A}_-(\lambda)$ *be two factorisations of* $A(\lambda) \in GL(n, \mathcal{C}(\Gamma))$ *with* $A_+, \tilde{A}_+ \in GL(n, \mathcal{H}^+(\Gamma))$ *and* $A_-, \tilde{A}_- \in GL(n, \mathcal{H}^-(\Gamma))$. *Then* $D = \tilde{D}$

Proof. Let $k_1, \ldots, k_n$ and $\tilde{k}_1, \ldots, \tilde{k}_n$ be the corresponding partial indices.

$$A_+DA_- = \tilde{A}_+\tilde{D}\tilde{A}_- \quad \Leftrightarrow \quad \tilde{A}_+^{-1}A_+D = \tilde{D}\tilde{A}_-A_-^{-1}$$

Assume there's a $1 \leq p \leq n$ such that w.l.o.g $k_p > \tilde{k}_p$. Then

$$k_1 \geq \ldots \geq k_p > \tilde{k}_p \geq \ldots \geq \tilde{k}_p$$

with $k_i - \tilde{k}_j > 0$ and hence

$$\left(\tilde{A}_+^{-1}A_+\right)_{ij} t^{k_i - \tilde{k}_j} = (\tilde{A}_-A_-^{-1})_{ij} \text{ for } i = 1, \ldots, p \quad j = p, \ldots, n$$

and thus the contradiction $det(\tilde{A}_-A_-^{-1}) = 0$. □

Definition 9.1. $k := k_1 + \ldots + k_n$ is called the total index for the partial indices $k_1, \ldots, k_n$ of a factorisation of an $A \in GL(n, \mathcal{C}(\Gamma))$

Note that

$$ind_\Gamma det(A(\lambda)) = ind_\Gamma det(D(\lambda)) = \sum_{i=1}^{n} k_i = k \tag{12}$$

since by the argument principle $ind_\Gamma det(A_+) = ind_\Gamma det(A_-) = 0$ due to the absence of zeroes in the corresponding domains of holomorphicity.

Concerning the connection between factorisations and singular integral operators, let's investigate the operator

$$T := \mathcal{P} - A\mathcal{Q}$$

$A \in GL(n, \mathcal{C}(\Gamma))$, on admissable spaces of vectorvalued functions $gl(n \times 1, \mathcal{E}(\Gamma))$.

Lemma 9.2. *Let $A(\lambda) \in GL(n, \mathcal{C}(\Gamma))$ and $A = A_+ D A_-$ be a factorisation with $A_\pm \in GL(n, \mathcal{H}^\pm(\Gamma))$. Then $T = \mathcal{P} - A\mathcal{Q}$ is invertible iff all partial indices are zero. Furthermore T is Fredholm with indices*

$$\alpha := dim(kernel(T)) = \sum_{k_i \geq 0} k_i \qquad \beta := codim(image(T)) = -\sum_{k_i \leq 0} k_i \tag{13}$$

Proof. $T = \mathcal{P} - A_+ D A_- \mathcal{Q} = \underbrace{A_+}_{=:T_1} \underbrace{(\mathcal{P} - D\mathcal{Q})}_{=:T_2} \underbrace{(A_- \mathcal{Q} + A_+^{-1}\mathcal{P})}_{=:T_3}$

The operators T_1, T_3 are invertible with $T_1^{-1} = A_+^{-1}$ and $T_3^{-1} = A_+\mathcal{P} - A_-^{-1}\mathcal{Q}$. So the product $T_1 T_2 T_3$ is invertible iff $T_2 = \mathcal{P} - D\mathcal{Q}$ invertible which is the case iff D is invertible i.e when $t \mapsto t^{k_i}$ is invertible for all $i = 1, \dots, n$ and thus iff $k_i = 0$ for all $i = 1, \dots, n$. The equations (13) follow immediately from the fact that $dim(\text{kernel}(T)) = dim(\text{kernel}(T_2))$ and $codim(\text{image}(T)) = codim(\text{image}(T_2))$ □

From the first step in the proof of theorem 7.1 it evidently suffices to consider matrix valued Laurentseries for the description of the partial indices

$$R(\lambda) = \sum_{i=-l}^{l} R_i \lambda^i \qquad (\lambda \in \Gamma, \quad R_i \in gl(n, \mathbb{C})) \tag{14}$$

where at least one of the coefficients R_{-l}, R_l is assumed not to vanish identically. Consider for $j \in \mathbb{Z}$ the integral operators

$$T_j := \mathcal{P} - \lambda^{-j} R(\lambda) \mathcal{Q}$$

for which the above Lemma gives

$$\alpha_j := dim(\text{kernel}(T_j)) = \sum_{k_i \geq j} (k_i - j) \tag{15}$$

$$\beta_j := codim(\text{image}(T_j)) = \sum_{k_i \leq j} (j - k_i) \tag{16}$$

In particular the dimension of the kernel of T_l and the codimension of the image of T_{-l} are zero, so for the partial indices this means

$$l \geq k_1 \geq \dots \geq k_n \geq -l \tag{17}$$

Hence it suffices to compute

$$\nu_j := \#\{i \in \mathbb{N} \mid k_i = j\}$$

and since by (13) we additionally have

$$\nu_j = \alpha_{j+1} - 2\alpha_j + \alpha_{j-1} \quad (-l \leq j \leq l)$$

it remains to determine the α_j's. The following theorem relies on the resolvent $L^{-1}(\lambda) = X(\text{Id} - \lambda J)^{-1} Y$ of a polynomial matrix $L(\lambda)$ in a punctured neighbourhood of an eigenvalue $\frac{1}{\lambda}$ with corresponding right and left eigenchains X and Y.

Theorem 9.3. *For a* $R(\lambda) \in GL(n, \mathcal{R}(\Gamma))$ *of the form* (14) *let* (X_1, J_1) *be that part of the spectral data of* $L(\lambda) := \lambda^l R(\frac{1}{\lambda})$ *with respect to the p-null-eigenvalues (c.w.m)* λ_0 *of* $L(\lambda)$*, for which* $\lambda_0^{-1} \in \Gamma_-$*. Let* $\nu_j = \#\{i \in \mathbb{N} \mid k_i = j\}$*. Then*

$$\alpha_j = \begin{cases} p - n(l+j) & \text{for } j \leq -l \\ dim(kernel(col[X_1 J_1^i]_{i=0}^{2l-j+1})) & \text{for } j > -l \end{cases} \tag{18}$$

Proof. We'll first show that

$$\text{kernel}(T_{-l}) = \{f \in gl(n\times 1, \mathcal{E}(\Gamma)) \mid f(\lambda) = X_1(\text{Id}_p - \lambda J_1)^{-1} w,\ w \in \mathbb{C}^p \textit{ arbitrary}\} \quad (*)$$

Since

$$dim(\text{kernel}(\mathcal{P} - \lambda^l R\mathcal{Q})) = ind_\Gamma(det(\lambda^l R(\frac{1}{\lambda}))) = ind_\Gamma(det(L(\lambda))) = p$$

and the map

$$w \mapsto X_1(\text{Id}_p - \lambda J_1)^{-1} w$$

is injective, the sets in $(*)$ have equal cardinality and thus it will suffice to show one inclusion : With a few insertions it's easy to show that

$$\mathcal{Q}\lambda^i f = X_1 J_1^{-i}(\text{Id}_p - \lambda J_1)^{-1} w$$

so that especially for $i = 0$ get $\mathcal{Q}f = f$ and thus $\mathcal{P}f = 0$

$$\begin{aligned} T_{-l}(f) &= \lambda^l(R_{-l}\mathcal{Q}\lambda^{-l} f + \ldots + R_l \mathcal{Q}\lambda^l f) \\ &= R_{-l}\mathcal{Q}\lambda^0 f + \ldots + R_l\mathcal{Q}\lambda^{2l} f \\ &= R_{-l}X_1 J_1^0 w + \ldots + R_l X_1 J_1^{2l} w \\ &= \sum_{i=1}^{2l} R_{i-l} X_1 J_1^i w = 0 \text{ by (11)} \end{aligned}$$

and hence the inclusion '$\supseteq$'. Further

$$T = (\mathcal{P} - \lambda^l R\mathcal{Q})\underbrace{(\mathcal{P} + \lambda^{-l}\mathcal{Q})}_{injective}$$

so that

$$\text{kernel}(T) = \text{kernel}(T_{-l}) \cap \text{image}(\mathcal{P} + \lambda^{-l}\mathcal{Q})$$

With this we get

$$f = X_1(I_p - \lambda J_1)^{-1} w \in \text{image}(\mathcal{P} + \lambda^{-l}\mathcal{Q}) \Leftrightarrow w \in \underbrace{\text{kernel}(col[X_1 J_1^i]_{i=0}^{l-1})}_{=:\mathcal{K}}$$

so finally

$$\text{kernel}(T) = \{f \in \mathcal{E}(\Gamma)_n \mid f(\lambda) = X_1(\text{Id}_p - \lambda J_1)^{-1} w,\ w \in \mathcal{K}\}$$

Concerning (18) : $j \leq -l$, then from (17) $R_j = \lambda^{-j} R$ posesses only positive partial indices and with Lemma 8.2 we get

$$\alpha_j = dim(\text{kernel}(R_j)) = k\,, \text{ the total index of } R_j$$

on the other hand from (12) we have

$$\begin{aligned} k &= ind_\Gamma(det(R_j)) \\ &= ind_\Gamma(det(\lambda^{-j}R)) \\ &= ind_\Gamma(det(\lambda^{-(l+j)}R_l)) \\ &= -n(l+j) + ind_\Gamma(det(R_l)) \\ &= p - n(j+l) \end{aligned}$$

and so

$$\alpha_j = p - n(j+l)$$

If $j > -l$, then

$$\text{kernel}(T_j) = \text{kernel}(T_l) \cap \text{image}(\mathcal{P} + \lambda^{l-j}\mathcal{Q})$$

and analogously to above

$$f(\lambda) = X_1(\lambda \text{Id}_p - J_1)^{-1}w \in \text{image}(\mathcal{P} + \lambda^{l-j}\mathcal{Q}) \Leftrightarrow w \in \text{kernel}(col[X_1 J_1^i]_{i=0}^{l-j-1})$$

which implies

$$\alpha_j = dim(\text{kernel}(T_j)) = dim(\text{kernel}(col[X_1 J_1^i]_{i=0}^{l-j-1}))$$

□

10. The Matrix-Riemann-problem

For a continous matrix valued function $A \in gl(n, \mathcal{C}(\Gamma))$ the matrix Riemann problem consists in finding all w.r.t Γ piecewise holomorphic vectorvalued functions

$$\Phi(z) = \begin{cases} \Phi_+(z) & , z \in \Gamma_+ \\ \Phi_-(z) & , z \in \Gamma_- \end{cases}$$

that have at worst a pole of finite order in 0, can be continously extended to Γ and the boundary values satisfy

$$\Phi_+ = A\Phi_- \tag{19}$$

Note that if $\Phi_1, \ldots, \Phi_l$ solves (19) and $r_1, \ldots, r_l$ are rational and nowwhere zero, then

$$\Psi(z) = r_1(z)\Phi_1(z) + \ldots + r_l(z)\Phi_l(z) \tag{20}$$

is again a solution.

Definition 10.1. A set of solutions is called complete, if every other solution is of the form (20).

First we'll show how to obtain a complete solution set from a factorisation of $A \in GL(n, \mathcal{C}(\Gamma))$. So let $A \in GL(n, \mathcal{C}(\Gamma))$ posess a factorisation $A = A_+DA_-$ with $A_\pm \in GL(n, \mathcal{H}^\pm)$ and $D(t) = diag[t^{k_1}, \ldots, t^{k_n}]$ with $k_1 \geq \ldots \geq k_n$ partial indices. Set

$$\Phi(z) := \begin{cases} A_+(z)D(z) & , z \in \Gamma_+ \\ A_-^{-1}(z) & , z \in \Gamma_- \end{cases}$$

Hence the columns of Φ_+ and Φ_- solve (19) and it remains to show that this is indeed a complete solution. So let Ψ solve (19) and define the piecewise holomorphic function

$$\Upsilon(z) := \begin{cases} D^{-1}(z)A_+^{-1}(z)\Psi(z) & , z \in \Gamma_+ \\ A_-(z)\Psi(z) & , z \in \Gamma_- \end{cases}$$

Since Ψ is a solution

$$D^{-1}A_+^{-1}\Psi_+ = A_-\Psi_- \quad \Leftrightarrow \Upsilon_+ = \Upsilon_- \text{ on } \Gamma$$

so that Υ is holomorphic in $\hat{\mathbb{C}}\backslash\{0\}$ with finite order pole at 0 i.e $\Upsilon \in GL(n, \mathcal{R}_0^-(\Gamma))$ and so the columns Ψ form a complete solution set of (19).
Conversely, given a complete solution set $\{\Phi_i\}$ of (19), pick $\Phi_1, \ldots, \Phi_n$ with the following properties :

(1) $det[\Phi_1(\infty), \ldots, \Phi_n(\infty)] \neq 0$

(2) The principal parts of the Laurent series at 0 of the $\Phi_1, \ldots, \Phi_n$ are the shortest possible i.e have the least possible negative powers.

A solution set with these properties will be called a standard set. Set

$$X(z) := [\Phi_1(z), \ldots, \Phi_n(z)] \quad , z \in \mathbb{C}\backslash\Gamma$$

then $X_+ = AX_-$ on Γ. Denote with $ord(\Phi_i)$ the order of the pole 0 of Φ_i and arrange these such that

$$ord(\Phi_1) \geq \ldots \geq ord(\Phi_n)$$

Define

$$\widetilde{X}(z) := \begin{cases} X_+\delta_{ij}z^{ord(\Phi_i)} & , z \in \Gamma_+ \\ X_-(z) & , z \in \Gamma_- \end{cases}$$

then obviously on Γ we have

$$\widetilde{X}_+(z)\delta_{ij}z^{-ord(\Phi_i)} = A(z)X_-(z) \tag{21}$$

Lemma 10.1. *$\widetilde{X}_+$ is invertible in $\Gamma_+\backslash\{0\}$ and X_- is invertible in Γ_-.*

Proof. We'll show that $det(X_-(z)) \neq 0$ for all $z \in \Gamma_-$. If there where a $z_0 \in \Gamma_-$ with $det(X_-(z_0)) = 0$, then there would be a $p(1 \leq p \leq n)$ and $c_1, \ldots, c_p \in \mathbb{C}$, $c_p \neq 0$ with

$$c_p\Phi_p(z_0) + \ldots + c_n\Phi_n(z_0) = 0$$

But then

$$\widetilde{\Phi_p}(z) := (z - z_0)\sum_{i=p}^{n} c_i\Phi_i(z)$$

solves (19) and since $ord(\widetilde{\Phi_p}) < ord(\Phi_p)$ after a possible rearranging we gat the contradiction to the fact that $\Phi_1, \ldots, \Phi_n$ is a standard solution. Analogously the claim is shown for $\widetilde{X}_+$. ☐

A consequence of (21) is that $\widetilde{X}_+$ and X_- are simultanously regular or singular on Γ, so that under the further assumption

$$det(X_-(z)) \neq 0 \quad \forall z \in \Gamma$$

a factorisation of $A \in GL(n, \mathcal{C}(\Gamma))$ w.r.t Γ is obtained :

$$A(z) = \widetilde{X}_+\delta_{ij}z^{-ord(\Phi_i)}X_-^{-1}(z)$$

11. Linear differential equations

The monodromy of a fundamental solution of a system of linear differential equations on $\hat{\mathbb{C}}$ is a representation of the fundamental group, that assigns to each homotopy class $[\sigma]$ that regular matrix, with which the fundamental system transforms when analytically continued along σ. The Riemann-Hilbert-Problem consists in finding a system with a fundamental solution whose monodromy coincides with a given representation. Furthermore we can prescribe poles for the coefficient matrix so that the search for the system with the smallest order poles makes sense. So consider a system of linear differential equations :

$$df = \omega f \tag{22}$$

with $\omega(z)$ a $n \times n$-matrix of meromorphic differential forms with poles in $S := \{s_1, \ldots, s_k\}$. By applying conformal transformations of $\hat{\mathbb{C}}$ it is always possible to replace poles in 0 and ∞. Now join these points $s_1, \ldots, s_k$ by a piecewise linear path and denote the half-open line between s_i and s_{i+1}, that contains the point s_i but not the point s_{i+1}, by Γ_i and define a piecewise constant matrix valued function $A(z)$ by

$$A(z) := \mathcal{M}_i \cdot \ldots \cdot \mathcal{M}_1 , \quad z \in \Gamma_i,\, i = 1, \ldots, k$$

where $\mathcal{M}_i$ is the representation of a loop around s_i, that doesn't traverse any point of S. Consider the matrix Riemann problem for $A(z)$:

$$\Phi_+ = A\Phi_- \quad \text{on } \Gamma_i \backslash s_i,\, i = 1, \ldots, k$$

A solution

$$\Phi(z) = \begin{cases} \Phi_+(z),\, z \in \Gamma_+ \\ \Phi_-(z),\, z \in \Gamma_- \end{cases}$$

satisfies

(1) $\det(\Phi(z)) \neq 0$ for all $z \in \hat{\mathbb{C}} \backslash \{0, s_1, \ldots, s_k\}$

(2) $\tilde{\Phi}(z) := \begin{cases} \Phi_+(z) z^D,\, z \in \Gamma_+ \\ \Phi_-(z),\, z \in \Gamma_- \end{cases}$

and is holomorphically invertible at 0 for a suitable diagonal matrix D. $\Phi(z)$ can be analytically continued along any path not going through 0 or any of the points s_i. If Φ traverses along a small loop around any of the points s_i then Φ goes to $\mathcal{M}_i\Phi$. In the same way $\tilde{\Phi}$ transforms, since it posesses the same monodromy, so that the matrix differential form

$$\omega := d\tilde{\Phi}\tilde{\Phi}^{-1} \tag{23}$$

is single valued on $\hat{\mathbb{C}}$ and holomorphic everywhere except at the prescribed $s_1, \ldots, s_k$. Hence system (22) with matrix form ω of (23) has the given monodromy and posesses poles at the prescibed points $s_1, \ldots, s_k$.

This result implies that to presribed poles and monodromy there always exists a regular system, that is a system with poles of finite order. The Riemann-Hilbert problem asks for the existence of a Fuchsian system, that is a system with poles of order at most 1. Hilbert was convinced of the existence of such a system, but in 1989 A.A.Bolibruch [1] found a counter example so that currently it is being investigated under which conditions the problem has a positive answer.

12. The Dressing method

A number of physically important non-linear systems of diferential equations have the property that one can assign to them a linear system, the Lax pair, in which solutions of the non-linear system appear as coefficients of the linear one [9]. In the 'direct' method one naturally tries to find solutions from given coefficients. The inverse problem consists in determining the coefficients from a certain behaviour of a solution. An essential tool is to introduce an additional rational dependance of a solution on a complex parameter, the so called spectral parameter λ. The dressing method consists in reducing the integration of the non-linear system to a Matrix-Riemann problem. To this end, consider the following pair of differential equations

$$\Psi_x = U\Psi \quad \Psi_y = V\Psi \tag{24}$$

with $U(x,y,\lambda)$, $V(x,y,\lambda)$ complex $n\times n$-matrices that depend rationally on λ and the indices denote the partial derivatives with respect to x resp. y. From $\Psi_{xy} = \Psi_{yx}$ we get the integrability condition

$$U_y - V_x + [U,V] = 0 \tag{25}$$

Given a solution Ψ_0, this determines the corresponding U_0, V_0. The following procedure is due to Zakharov and Shabat [13] and allows to construct new solutions from Ψ_0 via a Matrix-Riemann problem. Let Γ be a contour and $G_0(\lambda) \in GL(n, \mathcal{C}(\Gamma))$. Consider for

$$G(x,y,\lambda) := \Psi_0(x,y,\lambda)G_0(\lambda)\Psi_0^{-1}(x,y,\lambda)$$

pointwise for all x, y the Matrix-Riemann problem :

$$\Phi_+ = G\Phi_- \quad \text{on } \Gamma$$

and if Γ goes through any poles of U_0, V_0, then set $G \equiv 1$ there, so that taking partial derivatives gives on the one hand

$$G_x = (\Phi_+)_x\Phi_-^{-1} + \Phi_+(\Phi_-^{-1})_x$$

and on the other hand from the definition of G

$$\begin{aligned} G_x &= \Psi_{0_x}G_0\Psi_0^{-1} + \underbrace{\Psi_0 G_{0_x}\Psi_0^{-1}}_{=0} + \Psi_0 G_0 (\Psi_0^{-1})_x \\ &= U_0\Psi_0G_0\Psi_0^{-1} + \Psi_0G_0\Psi_0^{-1}U_0^{-1} \\ &= U_0G + GU_0^{-1} \end{aligned}$$

and differentiating w.r.t y gives

$$\Phi_{+_y}\Phi_-^{-1} + \Phi_+(\Phi_-^{-1})_y = V_0G + GV_0^{-1}$$

Now define with the solutions of the Matrix-Riemann problem two new solutions U_1, V_1 by gauge transforming U_0, V_0. Since this transformation is isospectral, all singularities of U_0, V_0 are preserved :

$$U_1 := (\Phi_-^{-1})_x\Phi_- + \Phi_-^{-1}U_0\Phi_-$$

$$V_1 := (\Phi_-^{-1})_y\Phi_- + \Phi_-^{-1}V_0\Phi_-$$

These matrices satisfy the integrability condition and the functions

$$\phi_1 := \Phi_+^{-1}\Psi_0 \qquad \phi_2 := \Phi_-^{-1}\Psi_0$$

solve (24) and are compatible with U_1 and V_1.

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