Polynomials problems

1. For every real number a, find the roots (if any) of $(x^2+1)(x-1)^2 - ax^2$.

2. Find all roots (real or complex) of $(3x+1)(4x+1)(6x+1)(12x+1) = 2$.

3. Let $\frac{3}{4} < a < 1$. Prove that the equation $x^3(x+1) = (x+a)(2x+a)$ has four distinct real roots and find these root explicitly.

4. Determine the largest real number z such that $x+y+z=5$, $xy+yz+zx=3$ and x, y are also real.

5. Suppose $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients a_i and assume that $a_n a_0 \neq 0$. Prove that if $r = \frac{p}{q}$ is a rational root of $P(x)$ (in lowest terms), then p is a divisor of a_0 and q is a divisor of a_n. Hence identify potential rational roots of $8x^3 - 18x^2 + 13x - 3$.

6. Locate intervals that contain the real roots of each the following polynomials: (a) $x^5 - 3x^4 - x^2 - 4x + 14$; (b) $2x^4 + 5x^3 + x^2 + 5x + 2$.

7. Find the greatest common divisor of $x^6 + x^5 + 89x + 34$ and $x^6 + 14x^3 + 32x + 13$.

8. The product of two of the roots of $P(x) = x^4 - 77x^3 + kx^2 + 308x + 1992$ equals 166. Find k.

9. L is a line that intersects the quartic curve $y = 2x^4 + 7x^3 + 3x - 5$ at the four distinct points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). Find $x_1 + x_2 + x_3 + x_4$.

10. Suppose that the roots of $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ are $r_1, \ldots r_n$, which are all non-zero. Write the following sums in terms of the coefficients: (i) $\displaystyle\sum_{i=1}^{n} \frac{1}{r_i}$, (ii) $\displaystyle\sum_{i=1}^{n} \frac{1}{r_i^2}$.

11. Given a polynomial $p(x) = x^3 - ax^2 + bx - 1$ with real coefficients having 3 real positive roots (not necessarily distinct) find the minimum value of $a+b$.

12. Suppose a and b are two distinct roots of $x^4 + x^3 - 1$. Prove that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1$.

13. Suppose p, q, r are three distinct numbers that satisfy the three simultaneous equations: $q = p(4-p)$, $r = q(4-q)$, $p = r(4-r)$. Find all possible values of $p+q+r$.

14. Determine all solutions of the system of simultaneous equations

$$\begin{aligned} x+y+z &= -1 \\ x^2+y^2+z^2 &= 29 \\ x^3+y^3+z^3 &= 29. \end{aligned}$$

15. Find all nth degree polynomials for which $P(x^2) = P(x)^2$.

16. Determine all polynomials for which $P(x^2+1) = (P(x))^2 + 1$ and $P(0) = 0$.

17. IrishMO 1993 The real numbers α, β satisfy $\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0$ and $\beta^3 - 3\beta^2 + 5\beta + 11 = 0$ respectively. Find $\alpha + \beta$.

18. Find all pairs of integers (a, b) such that the polynomial $ax^{17} + bx^{16} + 1$ is divisible by $x^2 - x - 1$.

19. Suppose $P(x)$ is a polynomial with integer coefficients such that $P(k)$, $P(k+1)$ and $P(k+2)$ are divisible by 3. Prove that $P(m)$ is divisible by 3 for every integer m.

20. Let $p(x)$ and $q(x)$ be non-constant polynomial functions with integer coefficients. It is known that the polynomial $p(x)q(x) - 2015$ has at least 33 distinct integer roots. Prove that neither $p(x)$ nor $q(x)$ can be a polynomial of degree less than three.

21. Find all positive integers m, n such that there exist a quadratic function $P(x)$ and a quartic function $Q(x)$, both with positive integer coefficients, such that $(mx^2 - nx + n)P(x) = Q(x)$.

22. Write $x^{16} - 1$ as a product of real irreducible linear and quadratic factors.
Hence list the values of $\cos\left(\frac{k\pi}{8}\right)$ for $k = 0, 1, \ldots, 8$.

23. Suppose $n \geq 2$ and consider the polynomial $z^n - z^{n-2} - z + 2$. For each n, find all complex numbers z that are roots of the polynomial and have $|z| = 1$.

24. Let $g(x)$ be a fixed polynomial and define $f(x)$ by $f(x) = x^2 + g(x^3)$. Prove that $f(x)$ is not divisible by $x^2 - x + 1$.

25. The polynomials $A(x), B(x), C(x)$ satisfy the equation $A(x^3) + xB(x^3) = (1 + x + x^2)C(x)$. Prove that $A(1) = 0$.

26. Find all cubic polynomials $x^3 + ax^2 + bx + c$ with rational roots a, b, c.