THIRTY FIRST IRISH MATHEMATICAL OLYMPIAD

Saturday, 12 May 2018

Second Paper

Solutions

6. Proposed by Steve Buckley.

   Find all real-valued functions f satisfying

   $$f(2x + f(y)) + f(f(y)) = 4x + 8y$$

   for all real numbers x and y.

   Solution 1

   Letting $y = 0$, we see that

   $$f(2x + f(0)) = 4x - f(f(0))\,. \qquad (1)$$

   Replacing x by $(x - f(0))/2$ above, we get

   $$f(x) = 2x + c\,, \qquad (2)$$

   where $c = -2f(0) - f(f(0))$. Thus,

   $$f(f(x)) = 2(2x + c) + c = 4x + 3c\,. \qquad (3)$$

   Using (2) and (3) in the original functional equation, we get

   $$2(2x + 2y + c) + c + (4y + 3c) = 4x + 8y\,,$$

   and so $c = 0$. Thus, $f(x) \equiv 2x$ is the only solution.

   Solution 2

   Letting $x = y = 0$, we see that $f(f(0)) + f(f(0)) = 0$, thus

   $$f(f(0)) = 0\,. \qquad (4)$$

   Substituting $x = -\frac{1}{2}f(0)$ and $y = 0$ in the original equation and using (4), we get $f(0) = -2f(0)$, which implies

   $$f(0) = 0\,. \qquad (5)$$

   Replacing x by $\frac{1}{2}x$ and letting $y = 0$ in the original equation and using (5), we get

   $$f(x) = 2x\,.$$

   Because $2(2x + 2y) + 2(2y) = 4x + 8y$ this indeed is a solution.

7. Proposed by Finbarr Holland.

Let a, b, c be the side lengths of a triangle. Prove that

$$2\left(a^3+b^3+c^3\right) < (a+b+c)\left(a^2+b^2+c^2\right) \leq 3\left(a^3+b^3+c^3\right).$$

Solution 1

Consider the inequality on the left. We prove this using the triangle inequality according to which $a < b+c$, $b < c+a$, and $c < a+b$. Applying this,

$$\begin{aligned} &(a+b+c)(a^2+b^2+c^2) - 2(a^3+b^3+c^3) \\ &= a^2(b+c)+b^2(c+a)+c^2(a+b)-a^3-b^3-c^3 \\ &= a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) > 0. \end{aligned}$$

This establishes the inequality on the left.

The inequality on the right holds more generally for any triple of positive numbers a, b, c, and can be handled by Hölder's inequality, or by strict convexity of the function $t \mapsto t^p$ with $p \geq 1$. Either of these tells us that: if $x, y, z > 0$ and $p \geq 1$, then

$$(x+y+z) \leq (x^p+y^p+z^p)^{1/p}3^{(p-1)/p} \quad \text{or equivalently} \quad \left(\frac{x+y+z}{3}\right)^p \leq \frac{x^p+y^p+z^p}{3},$$

with equality iff $x=y=z$. Apply this first with $x=a$, $y=b$, $z=c$ and $p=3$, to get

$$a+b+c \leq (a^3+b^3+c^3)^{1/3}3^{2/3},$$

and next with $x=a^2$, $y=b^2$, $z=c^2$ and $p=3/2$ to get

$$a^2+b^2+c^2 \leq (a^3+b^3+c^3)^{2/3}3^{1/3}.$$

Equality holds in both iff $a=b=c$. In any event, forming their product, we get

$$(a+b+c)(a^2+b^2+c^2) \leq 3(a^3+b^3+c^3).$$

This establishes the second inequality, with equality iff the triangle is equilateral.

Solution 2.

A standard ploy used to deal with inequalities involving the side lengths a, b, c of a triangle is to reformulate them in terms of the positive variables

$$x = a+b-c, \quad y = b+c-a, \quad z = c+a-b,$$

so that

$$a = \frac{z+x}{2}, \quad b = \frac{x+y}{2}, \quad c = \frac{y+z}{2},$$

whence $a+b+c = x+y+z$,

$$a^2+b^2+c^2 = \frac{1}{2}(x^2+y^2+z^2+xy+yz+zx) = \frac{1}{2}\left(\sum x^2 + \sum xy\right),$$

$$(a+b+c)(a^2+b^2+c^2) = \frac{1}{2}\Big(\sum x^3 + 2\sum xy(x+y) + 3xyz\Big)$$

and

$$a^3+b^3+c^3 = \frac{1}{8}\Big(2\sum x^3 + 3\sum xy(x+y)\Big)\,.$$

To establish the left inequality, notice that

$$\begin{aligned}
&(a+b+c)(a^2+b^2+c^2) - 2(a^3+b^3+c^3)\\
&= \frac{1}{2}\Big(\sum x^3 + 2\sum xy(x+y) + 3xyz\Big) - \frac{1}{4}\Big(2\sum x^3 + 3\sum xy(x+y)\Big)\\
&= \frac{1}{4}\Big(\sum xy(x+y) + 6xyz\Big)\,,
\end{aligned}$$

which is clearly positive. Hence the left inequality follows.

Also,

$$\begin{aligned}
&3(a^3+b^3+c^3) - (a+b+c)(a^2+b^2+c^2)\\
&= \frac{1}{8}\Big(2(x^3+y^3+z^3) - 12xyz + x^2(y+z) + y^2(z+x) + z^2(x+y)\Big)\\
&= \frac{1}{8}\Big(x^3+y^3+z^3 - 12xyz + (x^2+y^2+z^2)(x+y+z)\Big)\\
&= \frac{1}{8}\Big([x^3+y^3+z^3-3xyz] + [(x^2+y^2+z^2)(x+y+z) - 9xyz]\Big)\\
&\geq 0,
\end{aligned}$$

by three applications of the AM-GM inequality, with equality iff $x=y=z$. In other words, the second inequality holds.

Solution 3.

To establish the inequality on the left, multiply the three inequalities $a<b+c$, $b<c+a$, and $c<a+b$ by a^2, b^2, and c^2, respectively. This results in

$$a^3 < a^2(b+c), \quad b^3 < b^2(c+a), \quad c^3 < c^2(a+b)\,.$$

Adding these together and then adding $a^3+b^3+c^3$ to both sides of the resulting inequality gives the desired inequality. This is essentially the same proof as in Solution 1.

The inequality on the right holds for any triple of positive numbers a,b,c. To establish this, similar to Solution 2, but not switching to x,y,z, we multiply out the middle term and subtract it from $3(a^3+b^3+c^3)$, to see that we are left to show that

$$2(a^3+b^3+c^3) - ab^2 - a^2b - bc^2 - b^2c - ca^2 - c^2a \geq 0\,. \tag{6}$$

But

$$\begin{aligned}
&2(a^3+b^3+c^3) - ab^2 - a^2b - bc^2 - b^2c - ca^2 - c^2a\\
&= \big(a^3+b^3-a^2b-b^2a\big) + \big(b^3+c^3-b^2c-c^2b\big) + \big(c^3+a^3-c^2a-a^2c\big)\\
&= (a^2-b^2)(a-b) + (b^2-c^2)(b-c) + (c^2-a^2)(c-a)\\
&= (a+b)(a-b)^2 + (b+c)(b-c)^2 + (c+a)(c-a)^2 \geq 0\,,
\end{aligned}$$

since a, b, c are positive, with equality iff $a = b = c$. Hence the second inequality holds.

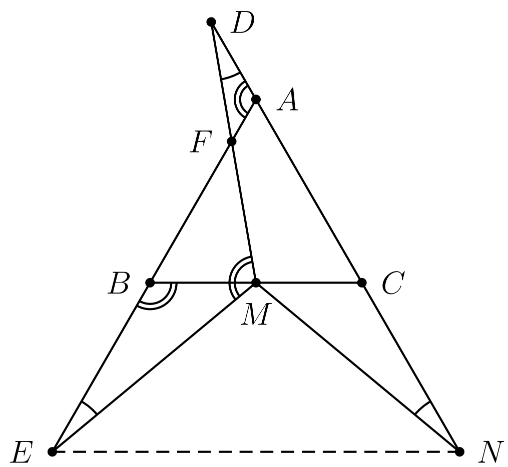
Remark. Inequality (6) is a special case of Muirhead's inequality. In the notation used to formulate Muirhead's inequality, (6) can be written as $6 \cdot [3, 0, 0] - 6 \cdot [2, 1, 0] \geq 0$, which is true, because $(3, 0, 0)$ dominates $(2, 1, 0)$.

8. Proposed by Jim Leahy.

Let M be the midpoint of side BC of an equilateral triangle ABC. The point D is on CA extended such that A is between D and C. The point E is on AB extended such that B is between A and E, and $|MD| = |ME|$. The point F is the intersection of MD and AB. Prove that $\angle BFM = \angle BME$.

Solution.

Let N on AC extended be such that $EN \| BC$. Join M to N.



Triangles $\triangle ABC$ and $\triangle AEN$ are similar, hence $|AE| = |AN|$ and so $|BE| = |CN|$. Alternatively, we may define N to be the point on AC extended such that $|BE| = |CN|$ and C is between A and N.

Since $|BM| = |CM|$, $|BE| = |CN|$ and $\angle EBM = \angle NCM = 120°$, triangles $\triangle MBE$ and $\triangle MCN$ are congruent by SAS. This implies $|MN| = |ME| = |MD|$.

This means that M is the centre of the circle through D, E and N. Using the Central Angle Theorem, this implies that $\angle DME = 2 \cdot \angle DNE = 2 \cdot 60° = 120°$.

Alternatively, we may observe that the congruence of $\triangle MBE$ and $\triangle MCN$ and the equality $|MN| = |MD|$ imply that $\angle AEM = \angle ANM = \angle ADM$. This implies that the four points E, M, A, D are concyclic. Therefore, the two inscribed angles $\angle DME$ and $\angle DAE = 120°$ are equal. This can also be seen from $\angle AEM = \angle ADM$ by considering the two triangles $\triangle DAF$ and $\triangle EMF$.

Triangles $\triangle MFE$ and $\triangle BME$ are similar, because $\angle MBE = 120° = \angle DME = \angle FME$ and both triangles share an angle at E. Therefore, $\angle BFM = \angle MFE = \angle BME$.

9. Proposed by Bernd Kreussler.

The sequence of positive integers $a_1, a_2, a_3, \ldots$ satisfies

$$a_{n+1} = a_n^2 + 2018 \qquad \text{for } n \geq 1.$$

Prove that there exists at most one n for which a_n is the cube of an integer.

Solution 1 (Anna Mustata, Lucas Bachmann)

Suppose there is at least one cube in the given sequence and let n be the smallest positive integer for which a_n is the cube of an integer. Because the cube of an integer can only be congruent to 0, 1 or $-1 \pmod 9$, we see that $a_n^2 \equiv 0$ or $1 \pmod 9$. Hence,

$$\begin{aligned} a_{n+1} &\equiv a_n^2 + 2 \equiv 2 \text{ or } 3 \pmod 9 \\ a_{n+2} &\equiv a_{n+1}^2 + 2 \equiv 6 \text{ or } 2 \pmod 9 \\ a_{n+3} &\equiv a_{n+2}^2 + 2 \equiv 2 \text{ or } 6 \pmod 9 \end{aligned}$$

and we see that $a_{n+k} \equiv 2$ or $6 \pmod 9$ for all $k > 1$. This implies that a_{n+k} cannot be a cube for any $k > 0$, hence a_n was the only cube in the sequence.

Solution 2

If $n \geq 3$ we can write

$$a_n = a_{n-1}^2 + 2018 = \left(a_{n-2}^2 + 2018\right)^2 + 2018 \equiv \left(a_{n-2}^2 + 2\right)^2 + 2 \pmod 9.$$

The following table shows the possible remainders of $\left(a^2+2\right)^2 + 2$ on division by 9.

a	0	± 1	± 2	± 3	± 4
a^2	0	1	4	0	-2
a^2+2	2	3	-3	2	0
$\left(a^2+2\right)^2+2$	-3	2	2	-3	2

It is well known that any cube of an integer is congruent to 0, 1 or $-1 \pmod 9$. Therefore, a_n cannot be the cube of an integer when $n \geq 3$.

In other words, only a_1 and a_2 could be perfect cubes. Suppose both of them are, so that there are integers b, c for which

$$a_1 = b^3 \quad \text{and} \quad a_2 = c^3.$$

From $a_2 = a_1^2 + 2018$ we obtain $c^3 = b^6 + 2018$, which can be rewritten as

$$\left(c - b^2\right)\left(c^2 + cb^2 + b^4\right) = 2 \cdot 1009.$$

Because $c - b^2 < c^2 + cb^2 + b^4$ and 1009 is a prime number, we obtain

$$c - b^2 = 2, c^2 + cb^2 + b^4 = 1009 \quad \text{or} \quad c - b^2 = 1, c^2 + cb^2 + b^4 = 2018.$$

Using $\left(c^2 + cb^2 + b^4\right) - \left(c - b^2\right)^2 = 3cb^2$ we can rule out the second case, because $2018 - 1$ is not divisible by 3. In the first case, we obtain from this identity $cb^2 = (1009-4)/3 = 335$. Using $b^2 = c - 2$ this leads to $c^2 - 2c = 335$, or equivalently, $(c-1)^2 = 336 = 2^4 \cdot 3 \cdot 7$, which has no integer solution. Hence, it is not possible that a_1 and a_2 are both perfect cubes. This concludes the proof.

10. Proposed by Steve Buckley.

The game of *Greed* starts with an initial configuration of one or more piles of stones. Player 1 and Player 2 take turns to remove stones, beginning with Player 1. At each turn, a player has two choices:

- take one stone from any one of the piles (a *simple move*);
- take one stone from each of the remaining piles (a *greedy move*).

The player who takes the last stone wins.

Consider the following two initial configurations:

(a) There are 2018 piles, with either 20 or 18 stones in each pile.

(b) There are four piles, with 17, 18, 19, and 20 stones, respectively.

In each case, find an appropriate strategy that guarantees victory to one of the players.

Solution

The winning strategy in (a) is easily found: all piles are even in number so Player 2 maintains this all-even status by parroting Player 1's every move. This eventually guarantees a win for Player 2.

Configuration (b) is trickier. We will see that there is a guaranteed win for Player 1.

Below, I've chosen to first rewrite the problem in a more formal way suitable for all stone removal games. Students would not be expected write it this way, but the ideas can also be communicated clearly without such formalisms.

We first mathematically encode the game as a system operating on the set S of sequences $s = (s_i)_1^\infty$ of non-negative integers with only finitely many non-zero entries. For each $i \in \mathbf{N}$, the function f_i is defined on the subset of S of sequences s such that $s_i > 0$: $f_i(s) = t$, where $t_i = s_i - 1$ and $t_i = s_i$ otherwise. The function $g : S \to S$ is defined by $g(s) = t$, where $t_j = \max\{0, s_j - 1\}$, $j \in \mathbf{N}$. Below, we write F for the set of functions consisting of g and every f_i.

A particular game is given by a sequence $(s^k)_{k=0}^\infty$ of elements of S, where $s^{k+1} = f(s^k)$ for some $f \in F$. The winner is determined by the parity of the minimal k for which s^k is the zero element, denoted z below.

Given $s \in S$, the notation $f(s)$ will always refer to $f(s)$ for some $f \in F$; the number of positive and odd entries in s are denoted by $n(s)$ and $o(s)$, respectively.

We call $W \subset S$ a (winning) *strategy* if the following conditions hold:

(1) If $s \in W$, and if $f(s)$ is defined, then $f(s) \notin W$, but there exists $f' \in F$ such that $f'(f(s)) \in W$.

(2) $z \in W$.

(3) W is *rearrangement invariant*: if $w \in W$, then every rearrangement of w lies in W.

(1) says that a player should strive to leave a configuration in W after his/her move. If this is achieved, every move the other player makes gives a configuration not in W; the first player can then move it back to W, and eventually win because of (2). (3) just encodes the fact that a winning strategy is readily adjusted to handle swapping of piles.

A strategy W tells us how to win from certain configurations, but it might not tell us how to proceed from every winnable configuration. W is *k-complete* if for each $s \in S$, $n(s) \leq k$, either $s \in W$ or $f(s) \in W$ for some $f \in F$. (A k-complete strategy tells us how to win from every configuration involving at most k piles.)

The previous given solution to Configuration (a) is essentially equivalent to the fact that

$$W_0 := \{s \in S \mid o(s) = 0\}$$

is a strategy. To verify (1), it suffices to note that if $s \in W_0 \setminus \{z\}$ and if $f(s)$ is defined, then $o(f(s)) \in \{1, n(s)\} \subset \mathbf{N}$, but $f(f(s)) \in W$.

Clearly, W_0 is 2-complete, but it is not 3-complete. We will enlarge W_0 to first make it 3-complete, and then 4-complete.

In general, if we have a strategy W, and a rearrangement invariant $A \subset S$, then $A \cup W$ is also a strategy if for each $s \in A$:

- there is no $f \in F$ with $f(s) \in W$, but
- if $f(s)$ is defined, then there exists $f' \in F$ such that $f'(f(s)) \in W$.

In the above situation, we call $A \cup W$ a *one-step augmentation* (of W).

The sole reason that W_0 is not 3-complete is because of s with $n(s) = 3$ and $o(s) = 2$. However, if we define

$$A := \{s \in S \mid n(s) = 3 \text{ and } o(s) = 2\},$$

then $W_1 := A \cup W_0$ is a one-step augmentation and also a 3-complete strategy. (Note that if $o(s) = 2$, then $o(f(s))$ is either 3 or 1. In the former case, $g(f(s)) \in W_0$; in the latter, $f_i(f(s)) \in W_0$, where i is the index of the odd entry.)

Finally, let $W_2 := W_1 \cup B$, where B consists of all s such that $n(s) = 4$ and either

- $o(s) = 3$ and the odd entries of s all exceed 1, or
- $o(s) = 2$ and the odd entries of s both equal 1.

We call the former sequences *Type 1* and the latter *Type 2*.

We now prove that W_2 is a strategy. (It is not a one-step augmentation of W_1, so the proof requires a little more effort.)

First, suppose s is of Type 1. Since $o(g(s)) = 1$, we have $f_i(g(s)) \in W_0$, where i is the index of the odd entry in $g(s)$. Alternatively, $o(f_i(s)) \in \{2, 4\}$ whenever $f_i(s)$ is defined. If $o(f_i(s)) = 4$, then $g(f_i(s)) \in W_0$, as desired. If instead $o(f_i(s)) = 2$, then the next step depends on whether or not both of the even entries equal 2. If they do, then $g(f_i(s))$ is of Type 2, as desired. If instead the jth entry is even and exceeds 2, then $f_j(f_i(s))$ is of Type 1.

Suppose instead that s is of Type 2. First, $n(g(s)) = 2$ and $o(g(s)) = 2$, so $g(g(s)) \in W_0$. Next, assume that $f_i(s)$ is defined. It might be that $s_i = 1$, in which case $o(f_i(s)) = 1$ and $n(f_i(s)) = 3$. Writing j for the index of the single odd entry in $f_i(s)$, we have $f_j(f_i(s)) \in W_0$, as required. Suppose instead that $s_i \neq 1$, and so s_i is even. Then $o(f_i(s)) = 3$. Taking j so that $s_j = 1$, we see that $n(f_j(f_i(s))) = 3$ and $o(f_j(f_i(s))) = 2$, so $f_j(f_i(s)) \in W_1$, as required.

Using W_2, we see that Player 1 has a guaranteed win in Configuration (b); the first move should be to remove a stone from one of the even piles.