THIRTY FIRST IRISH MATHEMATICAL OLYMPIAD

Saturday, 12 May 2018

First Paper

Solutions and Marking Schemes

1. Proposed by Bernd Kreussler.

Mary and Pat play the following number game. Mary picks an initial integer greater than 2017. She then multiplies this number by 2017 and adds 2 to the result. Pat will add 2019 to this new number and it will again be Mary's turn. Both players will continue to take alternating turns. Mary will always multiply the current number by 2017 and add 2 to the result when it is her turn. Pat will always add 2019 to the current number when it is his turn. Pat wins if any of the numbers obtained by either player is divisible by 2018.

Mary wants to prevent Pat from winning the game. Determine, with proof, the smallest initial integer Mary could choose in order to achieve this.

Solution

Let m be the initial integer Mary has picked. Because

$$2017x + 2 \equiv 2 - x \pmod{2018} \text{ and}$$
$$x + 2019 \equiv x + 1 \pmod{2018},$$

the numbers produced $\pmod{2018}$ are

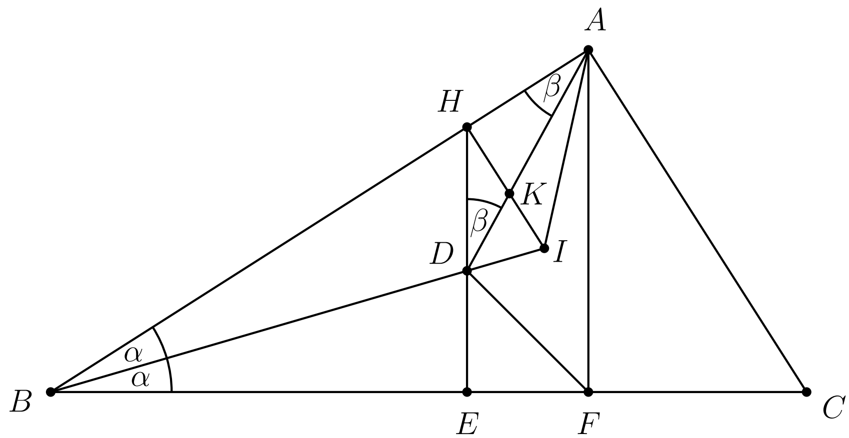
$$m \xrightarrow{M} 2 - m \xrightarrow{P} 3 - m \xrightarrow{M} 2 - (3 - m) = m - 1 \xrightarrow{P} m$$

where $\xrightarrow{M}$ indicates Mary's turn and $\xrightarrow{P}$ indicates Pat's turn. We see that the values produced $\pmod{2018}$ form a cycle of length 4. Hence, Mary wins exactly when none of the numbers $m, 2-m, 3-m, m-1$ is divisible by 2018, i.e., if m does not have remainder $0, 1, 2$ or 3 on division by 2018. The smallest integer greater than 2017 that satisfies this condition is $2018 + 4 = 2022$.

2. Proposed by Jim Leahy.

The triangle ABC is right-angled at A. Its incentre is I, and H is the foot of the perpendicular from I on AB. The perpendicular from H on BC meets BC at E, and it meets the bisector of $\angle ABC$ at D. The perpendicular from A on BC meets BC at F. Prove that $\angle EFD = 45°$.

Solution 1



Step 1: Let $\angle ABI = \alpha$ and $\angle BAD = \beta$. Then $\angle HDI = \angle BDE = 90° - \alpha$ and therefore $\angle ADI = 90° - \alpha - \angle HDA$.

Step 2: Now $\angle HAI = 45°$ since AI is the bisector of $\angle BAC = 90°$. Therefore $\angle AIH = 45°$, and so $|HA| = |IH|$.

Step 3: $\angle BHI = 90°$ implies $\angle HIB = 90° - \alpha = \angle HDI$ from Step 1 above. Therefore $|HD| = |IH|$.

Step 4: Combining the results of Steps 2 and 3, we have $|HD| = |HA|$ and thus $\angle HDA = \angle HAD = \beta$. This implies $\angle ADI = 90° - \alpha - \beta$.

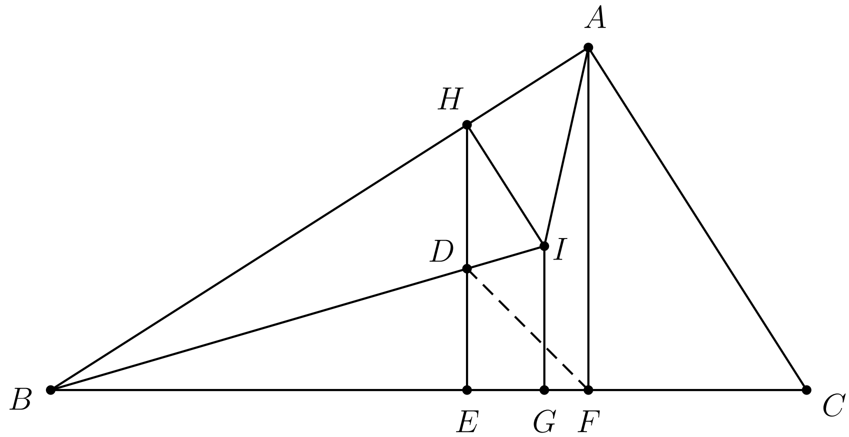
Step 5: Let K be the point of intersection of AD and HI. Then $\angle DKI + \angle KDI + \angle KID = 180°$, i.e., $\angle HKA + \angle ADI + \angle HID = 180°$ and therefore $(90° - \beta) + (90° - \alpha - \beta) + (90° - \alpha) = 180°$, and so $2\alpha + 2\beta = 90°$.

Step 6: Since AF is perpendicular to BC, we have $\angle ABF + \angle BAF = 90°$, i.e., $2\alpha + \beta + \angle DAF = 2\alpha + 2\beta$ and so $\angle DAF = \beta$.

Step 7: This implies that DA bisects $\angle BAF$, and so D is the incentre of triangle BAF. Therefore, DF bisects $\angle AFB$ and so $\angle BFD = 45°$. It follows that $\angle EFD = 45°$.

Solution 2

Let G on BC be the foot of the perpendicular from I on BC and $r = |IG|$ the radius of the incircle of $\triangle ABC$. Because AI is the angle bisector of the right angle at A, $\angle AIH = \angle HAI = 45°$, hence $|HA| = |IH| = r$.



Because BH and BG are tangent to the incircle of triangle ABC, we have

$$|BG| = |BH|\,. \tag{1}$$

Because DE is parallel to IG, the Intercept Theorem implies

$$\frac{|BE|}{|BG|} = \frac{|DE|}{|IG|}\,. \tag{2}$$

Because $EH \| AF$, the Intercept Theorem or the similarity of $\triangle BEH$ and $\triangle BFA$ implies

$$\frac{|BE|}{|BH|} = \frac{|EF|}{|HA|}\,. \tag{3}$$

Combining (2), (1) and (3), we obtain

$$|DE| = r \cdot \frac{|BE|}{|BG|} = r \cdot \frac{|BE|}{|BH|} = r \cdot \frac{|EF|}{|HA|} = |EF|\,.$$

Because $\angle FED$ is a right angle, we see now that $\angle EDF = \angle EFD = 45°$.

3. Proposed by Eugene Gath.

Find all functions $f(x) = ax^2 + bx + c$, with $a \neq 0$, such that

$$f(f(1)) \;=\; f(f(0)) \;=\; f(f(-1))\,.$$

Solution 1

First note that $f(0) = c$, $\quad f(1) = a + b + c$, $\quad f(-1) = a - b + c$. Second, note that if $f(\alpha) = f(\beta)$, then $a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$, so $(\alpha - \beta)(a(\alpha + \beta) + b) = 0$. Third, a key observation is that for fixed k, there are at most two distinct real numbers x for which $f(x) = k$.

It follows from this that since $f(f(1)) = f(f(0)) = f(f(-1))$, the set $\{f(1), f(0), f(-1)\}$ can contain at most two distinct numbers. This leads us to three cases.

Case 1: $f(1) = f(-1)$.

This gives $b = 0$, so $f(1) = f(-1) = a + c$ and $f(0) = c$. Therefore

$$0 = f(f(1)) - f(f(0)) = (f(1) - f(0))(a)(f(1) + f(0)) = a(a)(a + 2c).$$

So $c = -\dfrac{a}{2}$, which yields $f(x) = a(x^2 - \frac{1}{2})$. Let us verify that it works.

First, $f(0) = -\frac{a}{2}$ and $f(1) = f(-1) = \frac{a}{2}$. But $f(x)$ is even, so $f(-\frac{a}{2}) = f(\frac{a}{2})$ and hence $f(f(1)) = f(f(0)) = f(f(-1))$ as required.

Case 2: $f(1) = f(0)$.

This gives $b = -a$, so $f(1) = f(0) = c$ and $f(-1) = 2a + c$. Therefore

$$\begin{aligned} 0 = f(f(-1)) - f(f(1)) &= (f(-1) - f(1))\left[(a)(f(-1) + f(1)) + b\right] \\ &= (2a)\left[(a)(2a + 2c) - a\right]. \end{aligned}$$

So $2a + 2c - 1 = 0$, and hence $c = -a + \frac{1}{2}$. This yields $f(x) = a(x^2 - x - 1) + \frac{1}{2}$.

Let us verify that it works. Note that $f(\frac{1}{2} - x) = f(\frac{1}{2} + x)$. Also $f(0) = \frac{1}{2} - \frac{a}{2} = f(1)$ and $f(-1) = \frac{1}{2} + \frac{a}{2}$. So $f(f(1)) = f(f(0)) = f(f(-1))$ as required.

Case 3: $f(-1) = f(0)$.

This gives $b = a$, so $f(-1) = f(0) = c$ and $f(1) = 2a + c$. Therefore

$$\begin{aligned} 0 = f(f(1)) - f(f(-1)) &= (f(1) - f(-1))\left[(a)(f(1) + f(-1)) + b\right] \\ &= (2a)\left[(a)(2a + 2c) + a\right]. \end{aligned}$$

So $2a + 2c + 1 = 0$ and hence $c = -a - \frac{1}{2}$. This yields $f(x) = a(x^2 + x - 1) - \frac{1}{2}$.

Let us verify that it works. Note that $f(-\frac{1}{2} - x) = f(-\frac{1}{2} + x)$. Also $f(0) = -\frac{a}{2} - \frac{1}{2} = f(-1)$ and $f(1) = \frac{a}{2} - \frac{1}{2}$. So $f(f(1)) = f(f(0)) = f(f(-1))$ as required.

In summary, the solutions are $a(x^2 - \frac{1}{2})$, $\quad a(x^2 - x - 1) + \frac{1}{2}$, $\quad a(x^2 + x - 1) - \frac{1}{2}$, for any $a \neq 0$.

Solution 2

First note that $f(f(x))$ is a quartic.

The quartic $g(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon$ has the property that $g(1) = g(0) = g(-1)$ if and only if $\alpha + \gamma = 0$ and $\beta + \delta = 0$.

For the given function

$$\begin{aligned} f(f(x)) &= a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c \\ &= a^3x^4 + 2a^2bx^3 + (ab^2 + 2a^2c + ab)x^2 + (2abc + b^2)x + (ac^2 + bc + c). \end{aligned}$$

So the property $f(f(1)) = f(f(0)) = f(f(-1))$ gives $a^3 + ab^2 + 2a^2c + ab = 0$ and $2a^2b + 2abc + b^2 = 0$.

Since $a \neq 0$, the first equation gives $b^2 + b + a^2 + 2ac = 0$.

The latter equation gives $b(2a^2 + 2ac + b) = 0$.

So $b = 0$ or $b = -2a^2 - 2ac = -2a(a + c)$.
We look at these cases separately.

Case 1: $b = 0$.
Then $a^2 + 2ac = 0$ and hence $c = -\dfrac{a}{2}$ (since $a \neq 0$).

Then $f(x) = a(x^2 - \frac{1}{2})$.
Let us verify that it works.
First, $f(0) = -\frac{a}{2}$ and $f(1) = f(-1) = \frac{a}{2}$. But $f(x)$ is even, so $f(-\frac{a}{2}) = f(\frac{a}{2})$ and hence $f(f(1)) = f(f(0)) = f(f(-1))$ as required.

Case 2: $b = -2a^2 - 2ac = -2a(a + c)$.

Then the first equation gives $4a^2(a + c)^2 - 2a^2 - 2ac + a^2 + 2ac = 0$.

Since $a \neq 0$, this gives $(a + c)^2 = \frac{1}{4}$. Thus $c = -a \pm \frac{1}{2}$ and hence $b = \pm a$.

So we get two solutions $f(x) = a(x^2 - x - 1) + \frac{1}{2}$ and $f(x) = a(x^2 + x - 1) - \frac{1}{2}$, for any $a \neq 0$.

We check that both of these work.

For the first function, $f(\frac{1}{2} - x) = f(\frac{1}{2} + x)$. Also, $f(0) = \frac{1}{2} - \frac{a}{2} = f(1)$ and $f(-1) = \frac{1}{2} + \frac{a}{2}$. So $f(f(1)) = f(f(0)) = f(f(-1))$ as required.

For the second function, $f(-\frac{1}{2} - x) = f(-\frac{1}{2} + x)$. Also, $f(0) = -\frac{a}{2} - \frac{1}{2} = f(-1)$ and $f(1) = \frac{a}{2} - \frac{1}{2}$. So $f(f(1)) = f(f(0)) = f(f(-1))$ as required.

In summary, the solutions are $a(x^2 - \frac{1}{2})$, $\quad a(x^2 - x - 1) + \frac{1}{2}$, $\quad a(x^2 + x - 1) - \frac{1}{2}$, for any $a \neq 0$.

4. Proposed by Stephen Buckley.

We say that a rectangle with side lengths a and b *fits inside* a rectangle with side lengths c and d if either ($a \leq c$ and $b \leq d$) or ($a \leq d$ and $b \leq c$). For instance, a rectangle with side lengths 1 and 5 fits inside another rectangle with side lengths 1 and 5, and also fits inside a rectangle with side lengths 6 and 2.

Suppose S is a set of 2019 rectangles, all with integer side lengths between 1 and 2018 inclusive. Show that there are three rectangles A, B, and C in S such that A fits inside B, and B fits inside C.

Solution 1

We write $R \leq R'$ if R fits inside R'. (Note that $\leq$ is a preorder: it is reflexive and transitive. It might not be a partial order because two different rectangles might have matching widths and lengths.)

We call an n-element subset C of S an *n-chain* if its elements can be listed in "increasing order", i.e. in the form

$$R_1 \leq R_2 \leq \cdots \leq R_n \,.$$

An n-chain is allowed to contain congruent rectangles; for example, all n rectangles $R_1, R_2, \ldots, R_n$ may be congruent to each other. We call $A \subset S$ an *anti-chain* if it contains no 2-chains. In particular, no two rectangles are congruent in an anti-chain. For each rectangle R in S, we select a chain C_R with R as a maximal element, and which has maximal cardinality among chains having R as a maximal element. Let $f(R)$ be the size of C_R.

Claim 1: The subset $f^{-1}(n)$ is an anti-chain for all $n \in \mathbf{N}$.

To prove this claim, it suffices to show that if $f(R) = f(R')$ for distinct $R, R' \in S$, then $\{R, R'\}$ is an anti-chain. But this is easy since if $f(R) = f(R')$ and if $R' \leq R$, then $S_{R'} \cup \{R\}$ is a chain with maximal element R and size $f(R) + 1$, contradicting the definition of $f(R)$.

Claim 2: An anti-chain A of S has at most 1009 elements.

Let A be an anti-chain with n elements: $A = \{R_1, ..., R_n\}$, where the width and length of R_i are w_i and l_i, respectively, with $w_i \leq l_i$.

We assume that the rectangles are ordered so that $w_i \leq w_j$ for $i \leq j$. If $w_i = w_j$ for some $i < j$, then it is clear that $\{R_i, R_j\}$ is a chain, contradicting our anti-chain assumption. Similarly, we deduce that l_i must be strictly decreasing in i. Thus, we have numbers

$$w_1 < w_2 < \cdots < w_n \leq l_n < \cdots < l_2 < l_1 \,. \tag{4}$$

There are at least $2n - 1$ distinct numbers above, and all are between 1 and 2018, so $2n - 1 \leq 2018$ and we deduce Claim 2.

Combining Claim 1 and Claim 2, we see that $f^{-1}(1) \cup f^{-1}(2)$ has at most 2018 elements. Since this is less than the cardinality of S, we must have $f(R) \geq 3$ for some $R \in S$, as required.

Solution 2

We represent a rectangle by the ordered pair (x, y) of its side lengths, where $x \leq y$. We shall write $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$. When the x_i are positive, this means that a rectangle with side lengths x_1 and y_1 fits inside a rectangle with side lengths x_2 and y_2. A set of ordered pairs will be called a *chain* if its elements can be listed as $R_1, R_2, \ldots, R_n$ such that $R_i \leq R_j$ whenever $i \leq j$.

Let $T(m, n)$ be the set of all pairs (x, y) of integers that satisfy $m \leq x \leq y \leq n$. Of course, $T(m, n)$ is empty if $m > n$ and it contains exactly one element, namely (m, m), if $m = n$. More generally, if $d = n - m \geq 0$, the number of elements in $T(m, n)$ is equal to the triangular number $t_{d+1} = \sum_{i=1}^{d+1} i = \frac{1}{2}(d+1)(d+2)$. To see this, observe that, for $i = 1, 2, \ldots, d+1$, the pair $(x, m+i-1)$ is in $T(m, n)$ exactly for i values of x, namely $x = m, m+1, \ldots, m+i-1$.

Because we do not identify congruent rectangles that are different, some pairs (x, y) may appear more than once in the set S mentioned in the problem. We will describe S by selecting a subset S_0 of $T(1, 2018)$ and attaching to each of its elements a multiplicity. Multiplicities are positive integers and they represent the number of rectangles in S that have the given side lengths. More formally, S is represented by $S_0 \subset T(1, 2018)$ together with a map $\mu : S_0 \to \mathbb{Z}_+$ that takes values in the positive integers. We will then write $S = (S_0, \mu)$. The number of elements in S is equal to $\sum_{R \in S_0} \mu(R)$. A chain in S is a chain in S_0, but when we calculate its length, we take multiplicities into account. For example, if S_0 contains just one element but the multiplicity of it is three, then S contains a chain of length three, consisting of three congruent triangles, the side lengths of which give the element of S_0.

We are going to prove the following slightly more general statement by induction on $d = n - m \geq 1$ for odd d.

Claim. *If $S = (S_0, \mu)$ where S_0 is a subset of $T(m, n)$, $n - m$ is odd and $\mu : S_0 \to \mathbb{Z}_+$ is a map such that $\sum_{R \in S_0} \mu(R) \geq n - m + 2$, then S contains a chain with at least three elements.*

Note that the validity of this claim depends on $d = n - m$ only and not on the individual values of m and n, because $(x_1 - k, y_1 - k) \leq (x_2 - k, y_2 - k)$ is equivalent to $(x_1, y_1) \leq (x_2, y_2)$ for any integer k.

Let $E(m, n)$ be the subset of $T(m, n)$ that consists of those pairs (x, y) for which $x = m$ or $y = n$. If $d = n - m = 1$ we have $E(m, n) = T(m, n)$. Note that $E(m, n)$ is a chain for any $d = n - m \geq 1$, because

$$(m, m) \leq (m, m+1) \leq \ldots \leq (m, n-1) \leq (m, n) \leq (m+1, n) \leq \ldots \leq (n, n)\,.$$

A crucial observation for the proof is that for $d \geq 2$ the set $T(m, n)$ is the disjoint union of $E(m, n)$ and $T(m+1, n-1)$.

In the inductive step we use the claim for $d - 2$ when we prove the claim for d. Because we deal with odd d only, it suffices to consider $d = 1$ in the base case of the induction.

If $d = 1$, the set $T(m,n) = T(m,m+1)$ contains three elements, namely $(m,m) \leq (m,m+1) \leq (m+1,m+1)$, so it is a chain. Therefore, any possible set S in this case is a chain as well.

For the inductive step, we assume that $S' = (S'_0, \mu')$ contains a three-element chain if it contains at least $n' - m' + 2$ elements, $S'_0 \subset T(m', n')$ and $0 < n' - m' = n - m - 2$.

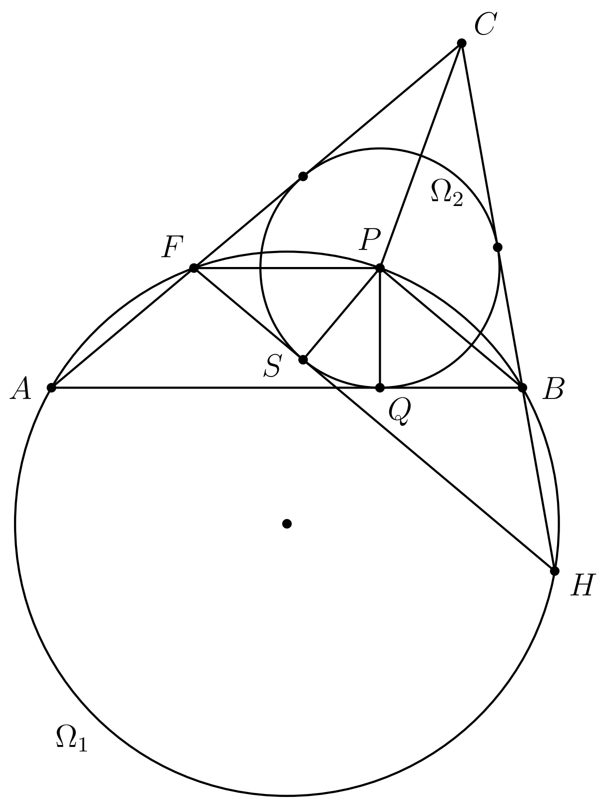
Suppose $S = (S_0, \mu)$ contains at least $n - m + 2$ elements and $S_0 \subset T(m,n)$. We consider $S_0 \cap E(m,n)$ and $S_0 \cap T(m+1, n-1)$. Because $E(m,n)$ is a chain, $S \cap E(m,n)$, which is $S_0 \cap E(m,n)$ with the multiplicities given by μ, is a chain as well. If this set with multiplicities contains at least three elements, the proof is finished. Otherwise, $S \cap T(m+1, n-1)$, i.e. $S_0 \cap T(m+1, n-1)$ equipped with the multiplicities given by μ, contains at least $n - m + 2 - 2 = (n-1) - (m+1) + 2$ elements, hence it contains a chain with at least three elements by the inductive assumption. This proves the claim.

The statement of the problem follows now because $d = n - m = 2018 - 1 = 2017$ for $T(1, 2018)$ and S is supposed to contain $2019 = n - m + 2$ elements.

5. Proposed by Jim Leahy.

Points A, B and P lie on the circumference of a circle Ω_1 such that $\angle APB$ is an obtuse angle. Let Q be the foot of the perpendicular from P on AB. A second circle Ω_2 is drawn with centre P and radius PQ. The tangents from A and B to Ω_2 intersect Ω_1 at F and H respectively. Prove that FH is tangent to Ω_2.

Solution



Step 1: Let the tangents from A and B to Ω_2 intersect at C, and let S be the foot of the perpendicular from P on FH (not necessarily on the circumference of Ω_2, yet). Join P to F, B, and C.

Step 2: Now P is the incentre of triangle ABC, so $\angle ABP = \angle CBP$. Since $FPBA$ is a cyclic quadrilateral, $\angle ABP = \angle CFP$, and therefore $\angle CFP = \angle CBP$.

Step 3: Since $\angle FCP = \angle BCP$ and CP is common, the triangles FCP and BCP are congruent and so $|PF| = |PB|$.

Step 4: Since $FPBH$ is a cyclic quadrilateral, $\angle PFH = \angle CBP = \angle ABP$. This equality may be restated as $\angle PFS = \angle QBP$.

Step 5: Since $\angle PFS = \angle QBP$ (from Step 4), $\angle BQP = \angle FSP$ $(= 90°)$, and $|PF| = |PB|$ (from Step 3), the triangles FPS and BPQ are congruent. This implies that $|PS| = |PQ|$.

Step 6: It follows that S lies on the circumference of Ω_2. Since PS is perpendicular to FH, FH is tangent to Ω_2.