THIRTIETH IRISH MATHEMATICAL OLYMPIAD

Saturday, 6 May 2017

First Paper

Solutions

1. Proposed by Stephen Buckley.

   Determine, with proof, the smallest positive multiple of 99 all of whose digits are either 1 or 2.

   Solution

   We call a number *eligible* if its digits are all either 1 or 2. A number is divisible by $99 = (9)(11)$ if and only if it is divisible by both 9 and 11.

   Suppose $N \in \mathbf{N}$ has base-10 expansion $a_n a_{n-1} ... a_2 a_1$. We define three digit-sums (full, odd, even):

$$S(N) := \sum_{1 \le i \le n} a_i\,, \qquad o(N) := \sum_{\substack{1 \le i \le n \\ i \text{ odd}}} a_i\,, \qquad e(N) := \sum_{\substack{1 \le i \le n \\ i \text{ even}}} a_i\,.$$

   As is well known (and easily established), N is divisible by 9 if and only if $S(N)$ is divisible by 9, while N is divisible by 11 if and only if $d(N) := o(N) - e(N)$ is divisible by 11.

   Suppose first that $S(N) = 9$. Then N has at least five digits,

$$3 \le o(N) \le 9 - 2 = 7\,, \qquad 2 \le e(N) \le 9 - 3 = 6\,,$$

   and $o(N), e(N)$ are of opposite parity. Consequently, $0 < |d(N)| \le 5$, and so N cannot be divisible by 11.

   Thus, we must have $S(N) \ge 18$ for eligible N to be divisible by 99. Suppose next that $S(N) = 18$. To minimize eligible N with $S(N) = 18$, we must certainly minimize the number of digits in N. We immediately rule out nine 2s, since then $d(N) = 10 - 8$ is not divisible by 11.

   The next smallest number of digits involves picking eight 2s and two 1s. The smallest such number is the one with 1s in the leading positions, i.e. $N = 1122222222$. Then $o(N) = e(N) = 9$, and N is divisible by 11, and hence by 99. Thus, this is the minimal example with $S(N) = 18$.

   It remains only to consider eligible numbers N with $S(N) \ge 27$. Such numbers have at least 14 digits, so are larger than the one we found above. Thus, the minimal number is indeed 1122222222.

2. Proposed by Finbarr Holland.

Solve the equations

$$a + b + c = 0,\ a^2 + b^2 + c^2 = 1,\ a^3 + b^3 + c^3 = 4abc$$

for a, b, and c.

Solution

As is usual with problems of this kind, we eliminate one of the "unknowns" thereby reducing the number of equations as well.

So, suppose a, b, c satisfy the given equations, and eliminate c, say. Then, from the first, we deduce that

$$\begin{aligned} a^3 + b^3 + c^3 &= a^3 + b^3 - (a+b)^3 \\ &= a^3 + b^3 - (a^3 + 3a^2b + 3ab^2 + b^3) \\ &= -3ab(a+b) \\ &= 3abc. \end{aligned}$$

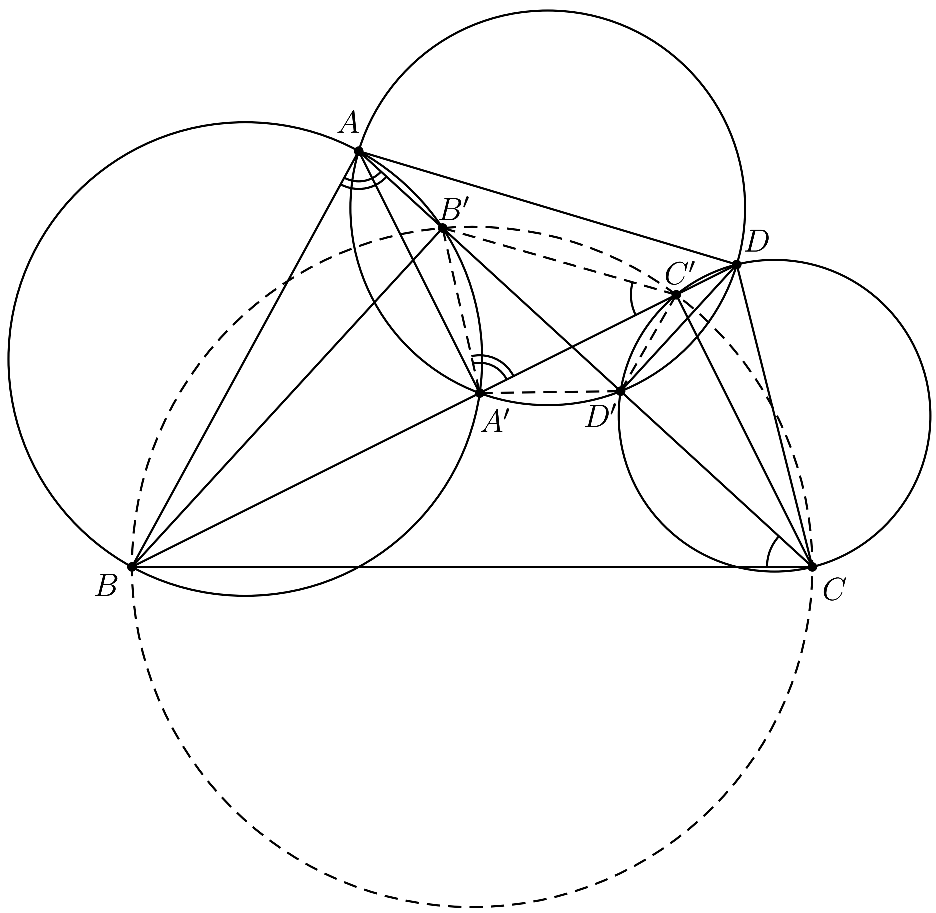
This and the third equation forces $abc = 0$. Hence, one of a, b, c is zero. Say $c = 0$. Then, by the first and second equations, $a = -b$, and $1 = 2a^2$. Thus one solution is $a = \pm 1/\sqrt{2}, b = \mp 1/\sqrt{2}, c = 0$, and any permutation of this triple is a solution. Conversely, every such triple is a solution.

3. Proposed by Jim Leahy.

Four circles are drawn with the sides of the quadrilateral $ABCD$ as diameters. The two circles passing through A meet again at A', the two circles through B at B', the two circles through C at C' and the two circles through D at D'. Suppose that the points A', B', C' and D' are distinct. Prove that the quadrilateral $A'B'C'D'$ is similar to the quadrilateral $ABCD$.

(*Note:* Two quadrilaterals are *similar* if their corresponding angles are equal to each other *and* their corresponding side lengths are in proportion to each other.)

Solution



Join AA', BB', CC' and DD'.

We have $\angle AA'B = 90°$ and $\angle AA'D = 90°$, which implies that the points B, A' and D are collinear. Similarly the points B, C' and D are collinear. It follows that the line BD passes through A' and C'. Similarly, the line AC passes through B' and D'.

Since $AB'A'B$ is cyclic, $\angle BAB' + \angle B'A'B = 180°$. However, since B, A' and C' are collinear we also have $\angle B'A'C' + \angle B'A'B = 180°$. Therefore $\angle B'A'C' = \angle BAB' = \angle BAC$.

Furthermore, we have $\angle BCB' = \angle BC'B'$ since they are in the same segment. Thus, $\angle BCA = \angle B'C'A'$.

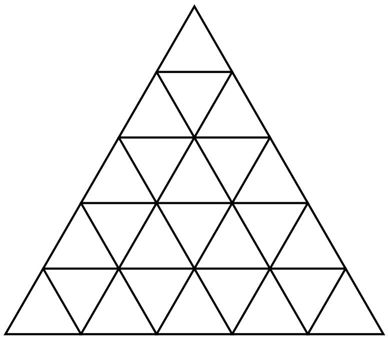
Combining the results of Steps 3 and 4, we deduce that $\triangle A'B'C'$ and $\triangle ABC$ are similar.

Similarly, $\triangle A'D'C'$ and $\triangle ADC$ are similar, and therefore the quadrilateral $A'B'C'D'$ is similar to the quadrilateral $ABCD$.

4. Proposed by Mark Flanagan.

An equilateral triangle of integer side length $n \geq 1$ is subdivided into small triangles of unit side length, as illustrated in the figure below for the case $n = 5$. In this diagram, a *subtriangle* is a triangle of any size which is formed by connecting vertices of the small triangles along the grid-lines.

It is desired to colour each vertex of the small triangles either red or blue in such a way that there is no subtriangle with all three of its vertices having the same colour. Let $f(n)$ denote the number of distinct colourings satisfying this condition.



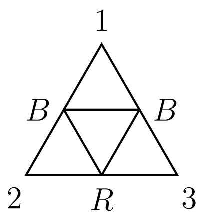
Determine, with proof, $f(n)$ for every $n \geq 1$.

Solution

The answer is $f(1) = 6$, $f(2) = 18$, $f(3) = 36$ and $f(n) = 0$ for $n \geq 4$.

First consider the case $n = 1$. All $2^3 = 8$ possible colourings are valid, except for the two monochromatic colourings. So $f(1) = 6$.

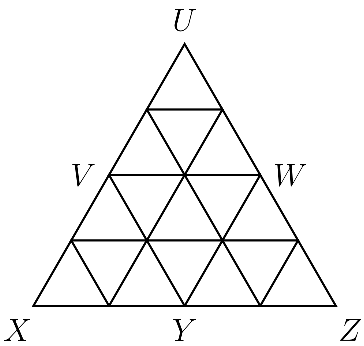
Next consider the case $n = 2$ as shown in the Figure below. The 'inner' triangle cannot be monochromatic, so there are $f(1) = 6$ possible colourings for this inner triangle. Without loss of generality, consider one of these as shown in the Figure below. Point 1 must be Red, and therefore Points 2 and 3 must be either (B, R), (R, B) or (B, B) (3 possibilities). This yields a total of $6 \cdot 3 = 18$, so $f(2) = 18$.



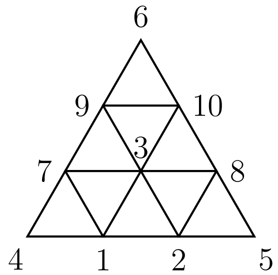
Before considering the next case, we make a simple observation which we formulate as a Lemma.

Lemma 1. There cannot be 3 equally spaced vertices on the base of any subtriangle all of which have the same colour.

Proof. Assume that there are 3 equally spaced vertices X, Y and Z, on the base of a subtriangle all of which have the same colour (say Red). Consider the points formed by the intersections of the lines starting at these vertices and forming $60°$ with the side of the subtriangle. If any of the so-formed three points U, V, W is Red, a triangle with Red vertices is formed. However, if all are Blue, then UVW is a Blue triangle. This proves the Lemma.



Next consider the case $n = 3$ as shown in the Figure below.



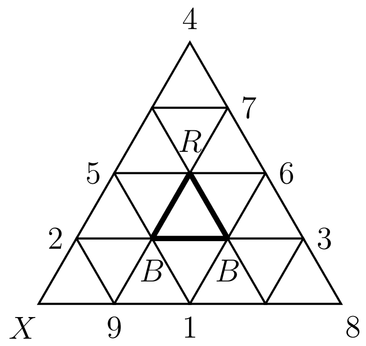
Case 1: First assume that the two points 1 and 2 have the same colour, say Blue. Then Point 3 must be Red, and Lemma 1 also implies that Points 4 and 5 are also Red. This in turn implies that Point 6 is Blue.

Looking now at Points 7 and 8, we see that they cannot both have the same colour; they cannot both be Blue as this would form a monochromatic triangle 678, and they cannot both be Red as this would violate Lemma 1 (their midpoint, 3, is Red). So, without loss of generality, Point 7 is Blue and Point 8 is Red. This in turn implies that Points 9 and 10 must be Red and Blue respectively. It can be easily checked that this colouring is valid. Accounting for symmetries leads to 12 possibilities (2 choices for the colour of Points 1 and 2, 2 further choices for the colouring of Points 7 and 8, and 3 possible rotations of the large triangle).

Case 2: The only other possibility is that Points 1 and 2 have different colours, and the same holds for the pairs of Points $(7, 9)$ and $(8, 10)$. Then, we must have that Points 1, 8, 9 are the same colour (say Blue) and that Points 2, 7, 10 are the other colour (Red) – if not, a vertex of the large triangle will not be colourable without completing a monochromatic triangle. The outer triangle 645 can then be coloured in $f(1) = 6$ different ways and the centre vertex of the large triangle can be coloured in 2 ways (Red or Blue). Accounting for all symmetries, this yields 24 possibilities (2 choices for colouring of Points 1, 8, 9, a further 6 possibilities to colour the outer triangle, and 2 possibilities for the colour of the centre of the large triangle).

Adding together all of these distinct colourings from Cases 1 and 2 yields $f(3) = 12 + 24 = 36$.

Next consider the case $n = 4$ as shown in the Figure below. The 'inner' triangle cannot be monochromatic, so without loss of generality, two vertices are coloured Blue and one vertex is coloured Red as shown in the diagram. The two Blue vertices then imply that Point 1 is Red, and we also conclude from Lemma 1 that Points 2 and 3 are Red. This in turn implies that Point 4 is Blue.



Looking now at Points 5 and 6, we see that they cannot both have the same colour; they cannot both be Blue as this would form a monochromatic triangle 456, and they cannot both be Red as this would violate Lemma 1 (their midpoint is Red). So, without loss of generality, Point 5 is Blue and Point 6 is Red. Since Point 3 is also Red, Lemma 1 implies that Points 7 and 8 must both be Blue, which in turn implies that Point 9 is Red.

But now we are stuck, since we cannot colour the point labelled X without forming a monochromatic triangle (either a Blue triangle $X48$ or a Red triangle $X29$). So $f(4) = 0$.

It is easy to see that the diagram for $n > 4$ contains the diagram for $n = 4$, and so $f(4) = 0$ implies $f(n) = 0$ for $n > 4$. This completes the proof.

5. Proposed by Tom Laffey.

The sequence $a = (a_0, a_1, a_2, \ldots)$ is defined by $a_0 = 0$, $a_1 = 2$ and

$$a_{n+2} = 2a_{n+1} + 41a_n \quad \text{for all } n \geq 0.$$

Prove that a_{2016} is divisible by 2017.

Solution 1

The equation $x^2 - 2x - 41 = 0$ has roots $\alpha = 1 + \sqrt{42}$ and $\beta = 1 - \sqrt{42}$; using $a_0 = 0$ and $a_1 = 2$, one obtains

$$a_n = \frac{(\alpha^n - \beta^n)}{\sqrt{42}} \tag{1}$$

for all $n \geq 0$.

Note that if h and k are positive integers, then $\alpha^{h(k-1)} + \alpha^{h(k-2)}\beta^h + \alpha^{h(k-3)}\beta^{2h} + \cdots + \beta^{(k-1)h}$ is an integer. Since $\alpha\beta = -41$, this result follows once we show that $\alpha^m + \beta^m$ is an integer for every positive integer m. Now $\alpha + \beta = 2$ and $(\alpha^{h+1} + \beta^{h+1}) = (\alpha + \beta)(\alpha^h + \beta^h) - \alpha\beta(\alpha^{h-1} + \beta^{h-1})$, so the integrality of $\alpha^m + \beta^m$ follows using induction.

Next $\alpha^8 - \beta^8 = (\alpha - \beta)(\alpha + \beta)(\alpha^2 + \beta^2)(\alpha^4 + \beta^4) = (2\sqrt{42})(2)(2 \cdot 43)(2 \cdot 2017)$.

Now $2016 = 8 \times 252$ and

$$\begin{aligned} a_{2016} &= \frac{(\alpha^{8\times 252} - \beta^{8\times 252})}{\sqrt{42}} \\ &= \frac{(\alpha^8 - \beta^8)}{\sqrt{42}} \times \frac{(\alpha^{8\times 252} - \beta^{8\times 252})}{(\alpha^8 - \beta^8)}. \end{aligned}$$

The second factor is an integer, on applying the result above with $k = 252$, $h = 8$, and the first factor is an integral multiple of 2017. Hence 2017 divides a_{2016}, as claimed.

Solution 2

We start in a similar manner to Solution 1 to obtain the formula (1). The binomial theorem then gives

$$a_{2016} = \frac{1}{\sqrt{42}}\left(\left(1 + \sqrt{42}\right)^{2016} - \left(1 - \sqrt{42}\right)^{2016}\right) = 2\sum_{k=0}^{1007} \binom{2016}{2k+1} 42^k.$$

The key observation now is that for any prime number p and $1 \leq m < p$ we have

$$(p-1)(p-2)\cdots(p-m) \equiv (-1)(-2)\cdots(-m) \equiv (-1)^m m! \pmod{p}$$

and because $\gcd(p, m!) = 1$ this implies that

$$\binom{p-1}{m} = \frac{(p-1)(p-2)\cdots(p-m)}{m!} \equiv (-1)^m \pmod{p}.$$

Because 2017 is a prime number, we see now that

$$\binom{2016}{2k+1} \equiv -1 \pmod{2017}$$

for all $0 \le k \le 1007$. Hence,

$$a_{2016} \equiv -2\sum_{k=0}^{1007} 42^k \equiv -\frac{2}{41}(42^{1008} - 1) \pmod{2017}$$

and so a_{2016} is divisible by 2017 iff $42^{1008} \equiv 1 \pmod{2017}$. This could be checked by a tedious calculation, but can also be seen with the aid of Legendre symbols and quadratic reciprocity as follows. By Euler's criterion, we have $42^{1008} \equiv \left(\frac{42}{2017}\right) \pmod{2017}$. Next, we compute the Legendre symbol $\left(\frac{42}{2017}\right)$ via

$$\begin{aligned}
\left(\frac{42}{2017}\right) &= \left(\frac{2}{2017}\right)\left(\frac{3}{2017}\right)\left(\frac{7}{2017}\right) \\
&= \left(\frac{3}{2017}\right)\left(\frac{7}{2017}\right) \\
&= \left(\frac{2017}{3}\right)\left(\frac{2017}{7}\right) \\
&= \left(\frac{1}{3}\right)\left(\frac{1}{7}\right) = 1,
\end{aligned}$$

where in the second line we have used the fact that $\left(\frac{2}{2017}\right) = 1$ since $2017 \equiv 1 \pmod{8}$, and the third line follows from the fact that $2017 \equiv 1 \pmod{4}$. This gives the desired result that $42^{1008} \equiv 1 \pmod{2017}$.

Note that all this works so smoothly because $2017 = 2016 + 1$ is a prime and 2016 is divisible by $3, 7$ and 8.

Solution 3

We start in a similar manner to Solution 1 to obtain the formula (1). Note that $p = 2017$ is prime. Next we show that 42 is a quadratic residue mod 2017 (either by observing that $119^2 \equiv 42 \pmod{2017}$, by using the argument in Solution 2, or otherwise). Finally, letting $r^2 \equiv 42 \pmod{2017}$, we interpret the formula (1) over the field of integers mod 2017 to obtain

$$a_{p-1} \equiv \frac{(1+r)^{p-1} - (1-r)^{p-1}}{r} \pmod{2017}.$$

By Fermat's little theorem, p divides $((1+r)^{p-1} - 1) - ((1-r)^{p-1} - 1)$, and so p divides ra_{p-1}. Thus p divides a_{p-1}, i.e., 2017 divides a_{2016}.