Inequality Problems

These are problems that you should try yourself. The ones at the start are easier but so are some of the later ones. If you want to send me solutions, scan them in and email them to me at **eugene.gath@ul.ie**. *I will be happy to make comments and suggestions. I will also email you back my solutions.*

1. Prove that $x^4 - 2x^3 + 3x^2 - 2x + 1 \geq 0$ for all x.

2. Suppose $a, b > 0$. For what non-zero value(s) of x does $\dfrac{a + bx^4}{x^2}$ have the least value?

3. Suppose a, b, c are the sides of a triangle.
Prove that $a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc$.

4. Prove that $2\sqrt{x} \geq 3 - \dfrac{1}{x}$ for $x > 0$.

5. Suppose a rectangular closed box has surface area S and volume V.
(a) Prove that $S \geq 6V^{\frac{2}{3}}$ and find the condition(s) for equality.
(b) Prove that, for closed boxes of given surface area, the one with largest volume is a cube.
(c) Prove that for closed boxes for a given volume, the one with the least surface is a cube.

6.(a) Prove that, of all the triangles with the same perimeter P, the one that encloses the greatest area is equilateral. (b) Prove that, of all the triangles with the same area A, the one with the smallest perimeter is equilateral.

7. Suppose a and b are positive real numbers and that n is a positive integer. Prove that

$$(ab^n)^{\frac{1}{n+1}} \leq \frac{a + nb}{n + 1}$$

and that equality holds if and only if $a = b$.

8. Suppose $a_1, a_2, \ldots, a_n$ are real numbers and that $n > 1$. Prove that

$$\frac{1}{n-1}(a_1 + a_2 + \cdots + a_n)^2 - (a_1^2 + a_2^2 + \cdots + a_n^2) \leq 2a_1a_2.$$

9. Prove that $1^1 \cdot 2^2 \cdot 3^3 \cdots n^n < \left(\dfrac{2n+1}{3}\right)^{\frac{1}{2}n(n+1)}$.

10. Let $x, y \in \mathbb{R}$. Prove that $(x+y+1)^2 + \frac{1}{3} \geq xy$ and find when equality occurs.

11. Let $a, b, c \in \mathbb{R}$. Prove the inequalities: $3(ab+bc+ca) \leq (a+b+c)^2 \leq 3(a^2+b^2+c^2)$.

12. Let $a, b, c > 0$. Prove that $\sqrt{3a+2} + \sqrt{3b+2} + \sqrt{3c+2} \geq 3\sqrt{3}$
and find when equality occurs.

13. Let x, y, z be positive real numbers such that $x+y+z=1$.
Prove that $\dfrac{xy}{z} + \dfrac{yz}{x} + \dfrac{zx}{y} \geq 1$ and find when equality occurs.

14. Let x, y, z be positive real numbers such that $x+y+z=1$.
Prove that $\dfrac{x^2+y^2}{z} + \dfrac{y^2+z^2}{x} + \dfrac{z^2+x^2}{y} \geq 2$ and find when equality occurs.

15. Let x, y, z be positive real numbers such that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$.
Prove that $(x-1)(y-1)(z-1) \geq 8$ and find when equality occurs.

16. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$.
Prove that $\dfrac{1}{a^2+1} + \dfrac{1}{b^2+1} + \dfrac{1}{c^2+1} + \dfrac{1}{d^2+1} \geq 2$ and find when equality occurs.

17. Let $a, b \in \mathbb{R}$ and $a \neq 0$.
Prove that $a^2 + b^2 + \dfrac{1}{a^2} + \dfrac{b}{a} \geq \sqrt{3}$ and find when equality occurs.

18. Suppose that $a, b, c > 0$ and $ab+bc+ca > a+b+c$. Prove that $a+b+c > 3$.

19. Let a, b, c, d be positive real numbers such that $a^2+b^2+c^2+d^2=4$.
Prove that $a+b+c+d \geq ab+bc+cd+da$ and find when equality occurs.

20. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$.
Prove that $\dfrac{1}{a+bc+abc} + \dfrac{1}{b+ca+bca} + \dfrac{1}{c+ab+cab} \geq 1$ and find when equality occurs.

21. Let a, b be positive real numbers.
Prove that $\sqrt[3]{\dfrac{a}{b}} + \sqrt[3]{\dfrac{b}{a}} \leq \sqrt[3]{2(a+b)\left(\dfrac{1}{a} + \dfrac{1}{b}\right)}$.

22. Suppose x, y, z are non-negative real numbers, with $x + y + z = 1$.
Prove that $x^2 + y^2 + z^2 \leq 1$ and find when equality occurs.
Try do this without using Lagrange Multipliers.

23. Suppose x, y are positive real numbers with $x^2 + y^2 > 2$.
Prove that $x^2 + y^3 < x^3 + y^4$.