FUNCTIONAL EQUATIONS

Problems

Q1. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
\[ f(x + yf(x)) = f(xf(y)) - x + f(y + f(x)) \]
for all $x$ and $y$ real numbers.

Q2. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
\[ f(f(x) + y) = f(x^2 - y) + 4f(x)y \]
for all $x$ and $y$ real numbers.

Q3. (Romanian TST 2011). Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
\[ 2f(x) = f(x + y) + f(x + 2y) \]
for all $x \in \mathbb{R}$ and $y \geq 0$.

Q4. (IMO 2009 Shortlist) Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
\[ f(xf(x + y)) = f(yf(x)) + x^2, \]
for all $x, y \in \mathbb{R}$.

Q5. (IMO 1993) Determine all functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that
\[ f(xf(y)) = yf(x) \text{ for all } x, y \in \mathbb{R}_+ \]
and as $x \to \infty$, then $f(x) \to 0$.

Q6. (IMO 1999) Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
\[ f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1 \]
for all $x, y \in \mathbb{R}$.

Q7. (IMO 1999) Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
\[ f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x) \]
for all $x, y \in \mathbb{R}$.

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Strategies:

1. (Find Solutions:) Find a class of solutions by inspection, i.e. \( f(x) = c \), a constant, \( f(x) = ax + b \) for some \( a, b \in \mathbb{R} \), \( f(x) = x^2 \), \( a^2 \) etc. If the functional equation contains trigonometric terms, you can consider terms of the form \( f(x) = \sin ax + \cos bx \), etc. Your class of solutions often dictates your strategies.

Once you have found a particular solution \( f_0(x) \), it is sometimes useful to substitute \( f(x) = g(x) + f_0(x) \) to get a new functional equation in \( g(x) \). This may be simpler to manage.

2. (Basic Substitutions:) Apply good substitutions, often to either cancel things out or make terms equal to zero, or to obtain new equations similar to the first, and then solve a system of equations. For example, if the original equation has a summand on the right hand side, try to apply a substitution that makes the same summand occur again but on the left hand side, or in a different place on the right hand side.

3. Try to find \( f(k) \) for some fixed \( k \), usually \( k = 0 \) or \( k = 1 \). As soon as you have found some specific values \( f(k) \), try to input \( k \) in the equation in a different way from before. E.g. if you made \( x = k \) and you got \( f(k) \), you could now make \( y = k \) and get a new simplified equation...

4. Is \( f \) an odd or even function? i.e. \( f(x) = f(-x) \), \( \forall x \in \mathbb{R} \)? \( f(x) = -f(-x) \), \( \forall x \in \mathbb{R} \)?

5. Is \( f \) injective or surjective? If so, can you prove it? If \( f \) is injective, then try to get something like \( f(f(x)) = f( \) some concrete expression ) to find \( f(x) = \) the concrete expression. If \( f \) is surjective, you can replace any \( y \) by \( f(x) \)...  

6. Is \( f \) strictly (or monotone) increasing (or decreasing)?

7. Fixed Points: Is there some thing special about fixed points of \( f \)? i.e. all \( w \in \mathbb{R} \) such that \( f(w) = w \).

8. Something more advanced; Use symmetry of solutions (e.g. in quadratic equations). For example if you are trying to prove \( f(x) = x^2 \), it might be easier to prove \( f(z_1 - z_2) = (z_1 - z_2)^2 \) and then that \( x \in \mathbb{R} \) be written in the form \( x = z_1 - z_2 \) for \( z_1, z_2 \in \text{Range}(f) \).

9. Is the function \( f \) periodic? i.e. is there a positive \( d \) such that \( f(x + d) = f(x) \) for some or all values of \( x \)?

10. Can you reduce the problem to a Cauchy equation? i.e. \( f(x + y) = f(x) + f(y) \), \( \forall x, y \in \mathbb{R} \)? We know that if \( f \) satisfies any of the following conditions, then \( f(x) = cx \) for some constant \( c \):

(a) \( f \) is defined over the rationals
(b) \( f \) is bounded on any closed interval of \( \mathbb{R} \).
(c) \( f \) is monotone increasing, i.e. \( f(x) \leq f(y) \) whenever \( x \leq y \) or monotone decreasing, i.e. \( f(x) \geq f(y) \) whenever \( x \leq y \).

11. At the end, remember to check that your solutions work in your solution write-up!
Q1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that 

$$f(x + yf(x)) = f(xf(y)) - x + f(y + f(x))$$

for all $x$ and $y$ real numbers.

**Hint:** Try a few special values for $x$ and $y$. If we denote the equation $f(x + yf(x)) = f(xf(y)) - x + f(y + f(x))$ by $P(x, y)$, then try $P(0, 0), P(0, 1), P(1, 0), P(1, 1), P(x, 0), P(0, y), P(x, 1)...$ Every time you make a substitution and find a new property, try to set up your next substitutions so that they use that property...

**Solution:** Letting $x = y = 0$, then $f(f(0)) = 0$. Letting $x = y = 1$, then $f(f(1)) = 1$. Letting $x = 1, y = 0$, then $f(1) = f(f(0)) - 1 + f(f(1))$, and since $f(f(0)) = 0$ and $f(f(1)) = 1$, then $f(1) = 0$.

Letting $x = 1, y$ any real number, then $0 = f(1) = f(f(y)) - 1 + f(y)$, so $f(y) = 1 - f(f(y))$. In particular, $f(0) = 1$. Letting $x$ be any real number, $y = 0$, then $f(x) = f(x) - x + f(f(x))$ so $f(f(x)) = x$.

So from the previous relation we get $f(x) = 1 - f(f(x)) = 1 - x$.

**Note:** Once we obtained the equation $f(y) = 1 - f(f(y))$, we can substitute $a = f(y)$ to get directly $a = 1 - f(a)$ so $f(a) = 1 - a$. However, we haven’t proven that this equation holds for all real numbers $a$, but only for the case when $a$ is an output of the function $f$ (as $a = f(y)$).

If any number $a \in \mathbb{R}$ is an output of $f$, (meaning that $a = f(t)$ for some $t$ real) then we say that $f$ is a surjective function. If we managed to prove that $f$ is surjective, then the problem is finished because then the equation $f(a) = 1 - a$ holds for all real numbers $a$.

For example, $f(f(x)) = x$ implies that $f$ is surjective because every $x$ is an output for $f$.

Another useful property of functions is injectivity: a function $f$ is injective if different inputs yield different outputs: For injective functions,

$$f(x_1) = f(x_2)$$

is only possible if $x_1 = x_2$.

For example, $f(f(x)) = x$ implies that $f$ is injective because $f(x_1) = f(x_2)$ would imply $x_1 = f(f(x_1)) = f(f(x_2)) = x_2$.

**Important Note:** The solution is not complete once you have found a formula for $f$, like $f(x) = 1 - x$. You still need to substitute the formula in the original equation and check if $P(x, y)$ holds true for this value of $f$:

$$f(x + yf(x)) = f(xf(y)) - x + f(y + f(x)) \iff$$

$$1 - (x + y(1 - x)) = 1 - x(1 - y) - x + 1 - (y + 1 - x)$$

which you can check by simplifying.

Q2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that 

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x$ and $y$ real numbers.
**Hint:** Note that for a carefully chosen value of $y$, (depending on $x$ and $f(x)$, the terms $f(f(x) + y)$ and $f(x^2 - y)$ cancel each other. This will lead you to two possible values for $f(x)$. Then show that either one or the other holds for $f(x)$.

**Solution:** Letting $y = \frac{f(x) - x^2}{2}$ gives $f(f(x) + y) = f(x^2 - y)$ hence $4f(x)\frac{f(x) - x^2}{2} = 0$. Hence for every $x$ we have that

1. either $f(x) = 0$
2. or $f(x) = x^2$.

The problem is that for some values of $x$, rule (1) could hold, while for other values of $x$, rule (2) could hold.

Assume that there exists $x \neq 0$ such that $f(x) \neq 0$. Then $f(x) = x^2$ and hence for every $y$,

$$f(x^2 + y) = f(x^2 - y) + 4x^2y.$$  

Since $f(x^2 - y) = (x^2 - y)^2$ or 0, it follows that $f(x^2 + y) = (x^2 - y)^2 + 4x^2y = (x^2 + y)^2$ or $4x^2y$. But we know $f(x^2 + y) = (x^2 + y)^2$ or 0 so the only compatible choices are $f(x^2 - y) = (x^2 - y)^2$ and $f(x^2 + y) = (x^2 + y)^2$. Since every real number $t$ can be written as $x^2 + y$ for some $y$, we have proven that in this case $f(t) = t^2$ for all real numbers.

In fact in this way we have also checked that $f(t) = t^2$ does satisfy the equation

$$f(f(x) + y) = f(x^2 - y) + 4f(xy)$$

Q3. (Romanian TST 2011). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$2f(x) = f(x + y) + f(x + 2y),$$

for all $x \in \mathbb{R}$ and $y \geq 0$.

**Hint:** Try to find $f(0)$ and you’ll get $2f(0) = f(0) + f(0)$ which is a hint that $f(0)$ could be any constant $c$ you want. Since $f(0)$ doesn’t matter, substitute $g(x) = f(x) - f(0)$ and you’ll get

$$2g(x) = g(x + y) + g(x + 2y)$$ and $g(0) = 0$.

**Solution:** Set $x = 0$ to get $2g(0) = g(0) + g(2y)$. But $g(0) = 0$ so $g(2y) = -g(y)$. Replace $y$ with $2y$ so $g(4y) = -g(2y) = g(y)$.

In the original equation for $g$ replace $x$ with $y$ to get $2g(y) = g(2y) + g(3y)$. Using $g(2y) = -g(y)$ we get $g(3y) = 3g(y)$.

In the original equation for $g$ replace $x$ with $2y$ to get $2g(2y) = g(3y) + g(4y)$. Using $g(2y) = -g(y)$ and $g(3y) = 3g(y)$ and $g(4y) = g(y)$ we get $g(y) = 0$. Hence $f(x) = c$ are the only solutions.

Q4-6. See Adrian Tang’s 2010 IMO Summer Training (but beware some small typos/computation errors in the text).