MA3059
Combinatorics and Graph Theory

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Abstract

These notes are designed to accompany the textbook "A first course in discrete mathematics" by Ian Anderson. The Chapter and Section numbering, which is not always consecutive here, refers to the corresponding chapters and sections of the book.

Chapter 1

Counting and Binomial Coefficients

1.1 Basic Principles

The Sum Rule: if a first task can be done in n_1 ways, and a second task in n_2 ways, and these tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do *either* task.

Example 1.1.1.

Tonight you could go to any one of 6 pubs or to any one of 5 films, but you will not go to a pub *and* a film. How many choices do you have?

$6 + 5 = 11$ choices. ■

The Product Rule: suppose that you must carry out 2 tasks. (Note difference from Sum Rule situation.) If the first task can be done in n_1 ways, and the second task can be done in n_2 ways *after the first task has* *been done*, and the order of choice matters (i.e., the choice a, b is not the same as the choice b, a), then there are $n_1 n_2$ ways to do *both* tasks.

Example 1.1.2.

In how many ways can one choose 2 students to be class representative and assistant class representative from a class of 14 students?

Can choose class representative in 14 ways, *then* assistant representative in 13 ways. Note that order of choice matters. Total number of choices is $(14)(13) = 182$.

If the problem were instead to choose 2 equal representatives where the order of choice does not matter, then we must use combinations — see §1.3. ■

Both of these Rules extend naturally to situations with more than 2 tasks.

Example 1.1.3. `[cf. Anderson Example 1.12]`

Show that the number of subsets of a finite set S of cardinality $|S|$ is $2^{|S|}$.

Here we are asked: in how many ways can choose a subset of S? Let $S = \{a_1, a_2, \ldots, a_{|S|}\}$. Choosing a subset of S is a procedure where

- 1st task is: do we choose a_1?
- 2nd task is: do we choose a_2?
- $\cdots$

- $|S|^{th}$ task is: do we choose $a_{|S|}$?

Each task can be done in 2 ways after the preceding tasks have been done, and the order matters. By the product rule, we can choose a subset of S in $(2)(2)\cdots(2) = 2^{|S|}$ ways. ∎

For example, if $S = \{a, b, c\}$, then its $2^3 = 8$ subsets are $\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, S$.

In some problems both Rules are needed.

Example 1.1.4.

An organization must choose a president and secretary from 3 Russians and 4 Germans, and these 2 officers must come from different countries. In how many ways can this be done?

Obvious approach: can first choose the president in 7 ways. But then can choose the secretary in 3 or 4 ways, depending on who was already chosen as president ... this method fails.

Instead, begin by using the sum rule:

$$\begin{aligned}
\text{answer} &= (\text{number of ways of choosing president and secretary with R as president}) \\
&\quad + (\text{number of ways of choosing president and secretary with G as president}) \\
&= (3)(4) + (4)(3) \qquad \text{by the product rule} \\
&= 24. \qquad \blacksquare
\end{aligned}$$

Read §1.1.

Exercises:

1. Solve Example 1.1.4 by counting the number of ways that the 2 officers can be chosen without regard to country, then subtracting from this the number of ways they can be chosen from the same country.

2. A class contains 18 mathematics students and 325 computer science students. How many ways are there to choose two representatives, one from mathematics and one from CS? How many ways are there to choose one representative, who can come either from mathematics or from CS? [5850, 343]

3. A *bit string* is a "word" each of whose constituent "letters" is 0 or 1. For example, 00110, 0 and 11 are bit strings of length 5, 1 and 2 respectively. How many bit strings are there of length 8? How many bit strings of length 10 begin and end with a 1? How many bit strings of length n are palindromes (i.e., read the same when reversed, such as 0010100)? [$2^8, 2^8, 2^{n/2}$ when n is even and $2^{(n+1)/2}$ when n is odd]

1.2 Factorials (and Permutations)

An ordered arrangement of r elements of a set S is called an r*-permutation* of S. If $r = |S|$, then we drop the "$r-$" and call this a *permutation* of S.

Example 1.2.1.

$S = \{a, b, c, d\}$. Then b, c, d, a is a permutation of S; b, c is a 2-permutation of S. ■

Theorem 1.2.1. `[Anderson Theorem 1.2]` *The number of r-permutations of a set S with n distinct elements (where $n \geq r$) is*

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1).$$

Proof. Use the product rule. (Recall that this rule can be invoked only when the order of the choices matters.) One can choose the 1st item in n ways, then the 2nd in $n-1$ ways, then the 3rd in $n-2$ ways, ..., then the rth in $n-(r-1) = n-r+1$ ways. Hence the number of ways to choose r items in order is $n(n-1)(n-2)\cdots(n-r+1)$. ∎

Clearly

$$P(n,r) = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)} = \frac{n!}{(n-r)!}.$$

In particular, $P(n,n) = n!$ Note that by definition $0! = 1$.

Alternative notation: $P(n,r) = {}^nP_r = P^n_r$.

Example 1.2.2.

16 players enter a tournament that has a 1st and 2nd prize. The number of ways these prizes can be awarded is $P(16,2) = (16)(15) = 240$. ∎

Example 1.2.3.

6 people (A, B, C, D, E, F) are to be seated at a circular table. How many different circular arrangements are possible, if two arrangements are considered the same when one can be obtained from the other by rotation?

♢

For example, is the same as

but different from

Solution: write each arrangement in a linear way, starting with the person at the top of the diagram and reading clockwise — e.g., the above three examples are $ACEBDF, DFACEB, AFDBEC$. We already saw that the first two of these are the same, and they are also the same as $CEBDFA, EBDFAC, BDFACE$ and $FACEBD$ by a similar argument. Thus one circular arrangement correspond to 6 linear arrangements. But the number of linear arrangements is $P(6,6) = 6!$ Hence the answer is $(6!)/6 = 5! = 120$. ■

Example 1.2.4.

Same as Example 1.2.3, but now in addition A, B, C, D, E, F are 3 married couples with A, B, C female. How many circular arrangements are possible with alternating sexes?

Again we count the number of linear arrangements then divide by 6. (Check that this is still valid.) One can have linear arrangements of the form $MFMFMF$ or $FMFMFM$. In each of these, one can fill the M boxes in $P(3,3) = 3! = 6$ ways, then the F boxes in $3! = 6$ ways. By the product and sum rules, the total number of linear arrangements is $(6)(6) + (6)(6) = 72$. Hence the answer is $72/6 = 12$. ■

Read §1.2. Exercise 1.1.

Exercises:

1. A multiple-choice test contains 10 questions. There are 4 possible answers for each question. In how many ways can a student answer the test if every question is answered? In how many ways can a student answer the test if the student can leave answers blank? $[4^{10}, 5^{10}]$

2. In how many ways can a photographer at a wedding arrange 6 people in a row, including the bride and groom, if (a) the bride must be next to the groom? (b) the bride is not next to the groom? (c) the bride is positioned somewhere to the left of the groom? [(a) 240, (b) 480, (c) 360]

3. How many ways are there to arrange the letters a, b, c and d such that a is not immediately followed by b? [18]

4. In how many ways can n men and n women be arranged in a row with men and women alternating? $[2(n!)^2]$

1.3 Selections (Combinations)

An *unordered* selection of r elements of a set S is an *r-combination* or *r-selection* of S. The number of such selections, where $|S| = n \geq r$, is written $C(n, r)$ or C_r^n or ${}^nC_r$ or $\binom{n}{r}$.

Theorem 1.3.1. `[Anderson Theorem 1.3]` *The number of r-combinations of a set S with n distinct elements (where $n \geq r$) is*

$$C(n, r) = \frac{n!}{r!(n-r)!} \ .$$

Proof. All r-permutations of S can be got by forming all r-combinations of S then all permutations of the r elements in each r-combination. From the product rule, this can be written as

$$P(n, r) = C(n, r) \cdot P(r, r),$$

so

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{r!(n-r)!} \ . \qquad \blacksquare$$

It follows that `[Anderson Theorem 1.4(i)]`

$$C(n, n-r) = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = C(n, r).$$

Example 1.3.1.

A department contains 10 men and 15 women. In how many ways can a committee of 6 members be chosen if it must have more women than men?

$$\begin{aligned}
\text{Number of committees } &= (\text{Number of committees with 6 women}) \\
&\quad + (\text{Number of committees with 5 women}) \\
&\quad\quad + (\text{Number of committees with 4 women}) \\
&= C(15,6) + C(15,5)\cdot C(10,1) + C(15,4)\cdot C(10,2), \\
&\qquad\qquad \text{by the product rule,} \\
&= \frac{15!}{6!9!} + \frac{15!}{5!10!}\cdot(10) + \frac{15!}{4!11!}\cdot\frac{10!}{2!8!} \\
&= \ldots \qquad \blacksquare
\end{aligned}$$

Theorem 1.3.2. `[Anderson Theorem 1.4(ii)]` *(Pascal's Identity) Let n and r be positive integers with $n \geq r$. Then*

$$C(n+1,r) = C(n,r-1) + C(n,r).$$

This identity can be used to construct the entries in Pascal's triangle. One can prove the result using the formula of Theorem 1.3.1 (see Anderson), but the following *combinatorial argument* is more elegant and similar arguments can be used to prove other combinatorial identities in situations where Theorem 1.3.1 is cumbersome to use.

Proof. $C(n+1, r)$ is the number of ways of choosing r objects from $n+1$ objects where the order of choice doesn't matter. Mark one of the $n+1$ objects. The r objects selected from the $n+1$ objects may or may not include the marked object. There are $C(n, r-1)$ ways to do the former and $C(n, r)$ ways to do the latter. Apply the sum rule to finish. ■

Consider an experiment where all n outcomes are equally likely. For example, in a Lotto draw where one must select 5 numbers from $1, 2, \ldots, 49$, there are $n = C(49, 5)$ equally likely outcomes. Then the probability that any one of a group of m particular outcomes occurs is m/n. You need to know this fact to understand some of Anderson's examples and exercises.

Read §1.3. Exercises 1.2–1.7.

1.4 Binomial Coefficients and Pascal's Triangle

Theorem 1.4.1. *(Binomial Theorem) Let x and y be symbols that can take any value. Let n be a positive integer. Then*

$$(x+y)^n = \sum_{r=0}^{n} C(n, r) x^{n-r} y^r.$$

Proof. When $(x+y)^n = (x+y)(x+y)\cdots(x+y)$ is expanded, the terms in the product are each of the form $x^{n-r}y^r$ for $r = 0, 1, \ldots, n$. Fix r. To count the number of terms of the form $x^{n-r}y^r$, observe that to obtain such a term it is necessary to use $n-r$ of the n factors $(x+y)$ to get x^{n-r}. We can choose these $n-r$ factors in $C(n, n-r)$ ways, i.e., the coefficient of $x^{n-r}y^r$ is $C(n, n-r) = C(n, r)$. ■

Theorem 1.4.2. `[Anderson Theorem 1.7]` *Let n be a positive integer. Then*

$$(i) \qquad \sum_{r=0}^{n} C(n, r) = 2^n$$

$$(ii) \qquad \sum_{r=0}^{n} (-1)^r C(n, r) = 0.$$

Proof. (i) Take $x = y = 1$ in the binomial theorem.
(ii) Take $x = 1$ and $y = -1$ in the binomial theorem. ■

Theorem 1.4.3. *(Vandermonde's Identity) Let r, m, n be positive integers with $r \leq m$ and $r \leq n$. Then*

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k) \cdot C(n, k).$$

Proof. Suppose that the set S contains m distinct elements and the set T contains a further n distinct elements. Then $S \cup T$ contains $m+n$ distinct elements and $C(m+n, r)$ is the number of ways one can choose r elements from $S \cup T$.

The r elements can be chosen from $S \cup T$ in a variety of ways: r elements from S and 0 elements from T, or $r-1$ elements from S and 1 element from T, or $r-2$ elements from S and 2 elements from T, . . . , or 0 elements from S and r elements from T. Applying the sum and product rules to convert this statement into an equation, we get

$$C(m+n,r) = C(m,r)\cdot C(n,0) + C(m,r-1)\cdot C(n,1) + \cdots + C(m,0)\cdot C(n,r),$$

which is the desired identity. ■

A non-combinatorial proof of this Theorem is suggested in Exercise 1.10.

Read §1.4. Exercises 1.8–1.12.

Exercises:

1. Prove Theorem 1.7(i) of Anderson using a combinatorial argument. Hint: recall Example 1.1.3.

2. Give a combinatorial proof that

$$\sum_{k=1}^{n} kC(n,k) = n2^{n-1}.$$

   Hint: count in two ways the number of ways of selecting a committee and its chairperson.

1.5 Selections (Combinations) with Repetitions

In this Section an element of a set may be *chosen more than once*, unlike §1.3 and §1.4.

Theorem 1.5.1. `[Anderson Theorem 1.9]` *The number of r-permutations of a set with n distinct elements, with repetition allowed (i.e., any r-permutation may contain some elements more than once) is n^r.*

Proof. Use the product rule: in each r-permutation one can choose each of the r items in n ways, so one can choose the entire r-permutation in n^r ways. ■

Example 1.5.1.

The 2-permutations of $\{a, b, c\}$ with repetition allowed are $(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)$: that is, $9 = 3^2$ in all as predicted by Theorem 1.5.1. ■

Next we consider *combinations with repetition.*

Example 1.5.2.

In how many ways can one select 4 pieces of fruit from a bowl containing 4 apples, 4 oranges and 4 pears, if the order of selection doesn't matter and only the type of fruit matters (e.g., all apples are indistinguishable from each other)?

Choosing 4 pieces of fruit means for example choosing

$$\begin{array}{c|c|c} a & o & p \\ xx & x & x \end{array} \quad \text{or} \quad \begin{array}{c|c|c} a & o & p \\ & xx & xx \end{array} \quad \text{or} \quad \begin{array}{c|c|c} a & o & p \\ xxxx & & \end{array} \quad \textit{etc.}$$

Based on these diagrams, each choice corresponds to a sequence of 4 xs and 2 bars. That is, in a list of 6 items we have to choose where to put the 4 xs. This can be done in

$$C(6,4) = \frac{6!}{2!4!} = \frac{(6)(5)}{2} = 15 \text{ ways.} \qquad \blacksquare$$

We use the approach of Example 1.5.2 to prove the next result.

Theorem 1.5.2. `[Anderson Theorem 1.10]` *Let n and r be positive integers. The number of r-combinations of a set with n distinct elements, when repetition is allowed, is $C(n+r-1,r)$.*

Proof. Each r-combination corresponds to to a sequence of r xs and $n-1$ bars: $xx||x|x|xxxx|xx|$ — the bars divide the list into n compartments (one for each element of the set) while the xs indicate how often that element is selected in the r-combination. The number of such x-bar sequences is the number of ways that we can choose the positions of the r xs from the $r+(n-1)$ places available. This number is $C(n+r-1,r)$. $\blacksquare$

Example 1.5.3.

In how many ways can you choose a dozen doughnuts from 5 different types when at least one of each type must be chosen?

Start by choosing one of each type — can do this in 1 way. The problem now becomes: choose 7 doughnuts from 5 types with repetition allowed. By Theorem 1.5.2, one can do this in

$$C(5+7-1,7) = C(11,7) = \frac{11!}{7!4!} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2} = 330 \text{ ways.}$$

By the product rule, the answer is $(1)(330) = 330$ ways. ■

Example 1.5.4.

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2 and x_3 are non-negative integers?

Observe that any solution of this equation corresponds to selecting 11 items from a set with 3 elements, so that x_1 items of type 1, x_2 items of type 2 and x_3 items of type 3 are chosen. By Theorem 1.5.2 the answer is

$$C(3+11-1, 11) = C(13, 11) = C(13, 2) = \frac{(13)(12)}{(2)(1)} = 78. \quad ■$$

Read §1.5. Exercises 1.13–1.19.

1.6 A Useful Matrix Inversion

Suppose that $a_n = \sum_{k=0}^{n} C(n,k) b_k$. Can one invert this formula to express b_n in terms of $a_0. \ldots, a_n$?

For $k > n$, define $C(n,k)$ to be 0. (Plausible as there is no way of choosing k items from n without repetition when $k > n$.) Let A be the $(n+1) \times (n+1)$ matrix whose (i,j) entry for $i, j, = 0, \ldots, n$ is $A_{ij} = C(i,j)$.

For example, if A is 5×5, then

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

Theorem 1.6.1. `[Anderson Theorem 1.14]` *Let B be the $(n+1) \times (n+1)$ matrix whose (i, j) entry for $i, j = 0, \ldots, n$ is $B_{ij} = (-1)^{i+j} C(i, j)$. Then $B = A^{-1}$.*

Proof. The (i, j) entry in BA is

$$(BA)_{ij} = \sum_{m=0}^{n} B_{im} A_{mj} = \sum_{m=0}^{n} (-1)^{i+m} C(i, m) C(m, j).$$

If $i < j$, then for each m one of $C(i, m)$ and $C(m, j)$ is zero, so $(BA)_{ij} = 0$. If $i = j$, then for each m, one of

$C(i,m)$ and $C(m,j)$ is zero except when $m = i$, so $(BA)_{ii} = (-1)^{i+i}C(i,i)C(i,i) = 1$. Finally, if $i > j$, then

$$
\begin{aligned}
(BA)_{ij} &= \sum_{m=0}^{n}(-1)^{i+m}C(i,m)C(m,j) \\
&= \sum_{m=j}^{i}(-1)^{i+m}C(i,m)C(m,j) \\
&= \sum_{m=j}^{i}(-1)^{i+m}\frac{i!}{m!(i-m)!}\cdot\frac{m!}{j!(m-j)!} \\
&= (-1)^{i}\frac{i!}{j!(i-j)!}\sum_{m=j}^{i}(-1)^{m}\frac{(i-j)!}{(i-m)!(m-j)!} \\
&= (-1)^{i}C(i,j)\sum_{m=j}^{i}(-1)^{m}C(i-j,i-m).
\end{aligned}
$$

Now set $m = j + r$ so that the sum over r (instead of m) starts at $r = 0$:

$$
\begin{aligned}
(BA)_{ij} &= (-1)^{i}C(i,j)\sum_{r=0}^{i-j}(-1)^{j+r}C(i-j,i-j-r) \\
&= (-1)^{i+j}C(i,j)\sum_{r=0}^{i-j}(-1)^{r}C(i-j,r) \\
&= 0,
\end{aligned}
$$

by Theorem 1.4.2(ii).

We have now verified that $BA = I_{n+1}$, the $(n+1) \times (n+1)$ identity matrix. It follows from linear algebra theory that also $AB = I_{n+1}$, and we are done. ∎

Now the relation $a_n = \sum_{k=0}^{n} C(n,k) b_k$ for all n can be written as

$$\begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = A \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix}.$$

Multiplying on the left by B, we obtain

$$B \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = I_{n+1} \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix}.$$

That is,

$$b_n = \sum_{k=0}^{n} (-1)^{n+k} C(n,k) a_k \quad \text{for all } n. \tag{1.1}$$

We shall use this result later.

1.7 The Pigeonhole Principle

This Section is not covered in Anderson.

Theorem 1.7.1. *(The Pigeonhole Principle) If $k+1$ objects are distributed among k boxes, then at least one box contains more than one object.*

Proof. Suppose the conclusion is false. Then we have k boxes, each containing at most one object, so we have a total of at most k objects. But this contradicts the hypothesis of the theorem. Consequently the conclusion must be true. ■

Example 1.7.1.

If 30 students are in a class, then at least two have a surname that begins with the same letter.

Here object = surname of a student, and box = letter of alphabet. There are 30 objects, 26 boxes, and each object goes into the box corresponding to its first letter. By the pigeonhole principle (since $30 > 26$), some box contains at least two objects, which is what we wanted to show. ■

Example 1.7.2.

If 5 points are chosen at random inside a square of side 2 cm., show that at least 2 points are within a distance $\sqrt{2}$ cm. of each other.

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Here we have 5 points and 4 boxes, so there is a box containing at least two points. ■

Theorem 1.7.2. *(The Generalized Pigeonhole Principle) If N objects are distributed among k boxes, then at least one box contains at least $\ulcorner N/k \urcorner$ objects, where $\ulcorner m \urcorner$ is the smallest integer i that satisfies $i \geq m$.*

Proof. Suppose the conclusion is false. Then the total number of objects is at most

$$k\left(\ulcorner N/k \urcorner - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

a contradiction. ■

Example 1.7.3.

A class of 25 students comprises engineering, science and arts students. Show that

(i) the class has at least 9 engineering, or at least 9 science, or at least 9 arts students;

(ii) the class has at least 3 engineering, or at least 19 science, or at least 5 arts students.

For part (i), objects = students, and the boxes are engineering, science and arts. Put each student into the associated box. Thus 25 students go into 3 boxes. By the GPP, some box has at least $\ulcorner 25/3 \urcorner = 9$ students.

Part (ii) is more subtle. Enlarge the class by adding 16 engineering and 14 arts imaginary students. We now have $25 + 16 + 14 = 55$ students. Similarly to part (i), there are at least $\ulcorner 55/3 \urcorner = 19$ engineering or science or arts students in the enlarged class. Remove the imaginary students. We see that the original class has at least 3 engineering, or at least 19 science, or at least 5 arts students. ■

Ingenious applications of the PP can prove surprising results. For example, the following result can be found in "Discrete mathematics and its applications" (4th edition) by K.H. Rosen:

Theorem 1.7.3. *Let n be a positive integer. Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.*

Exercises:

1. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark. (a) How many socks must he take out to be sure that he has at least two of the same colour? (b) How many must he take out to be sure that he has at least two black socks? [(a) 3 , (b) 14]

2. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

3. How many students, each of whom comes from one of 50 countries, must be enrolled in a university to guarantee that there are at least 100 who come from the same country? [4951]

4. Show that if five integers are selected from the first eight positive integers, then there must be a pair of these whose sum is 9. Is the conclusion still true if four (not five) integers are selected?

5. A computer network consists of six computers. Each computer is linked directly to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers. Can you generalize this conclusion to a network consisting of N computers?

6. Let (x_i, y_i, z_i), for $i = 1, 2, \ldots, 9$, be a set of nine distinct points with integer coordinates in xyz space. Show that the midpoint of at least one pair of these points has integer coordinates. Hint: each of x_i, y_i, z_i is either even or odd.

7. During a month with 30 days a baseball team plays at least 1 game per day, but no more than 45 games in total. Show that there must be a period of some number of consecutive days during which the team plays exactly 14 games. Hint: Let a_j be the number of games played on or before the jth day of the month. Show that the sequences $a_1, \ldots, a_{30}$ and $a_1 + 14, \ldots, a_{30} + 14$ are 60 positive integers lying between 1 and 59, so by the PP...

8. An arm wrestler is the champion for a period of 75 hours. The arm wrestler had at least one match per hour, but no more than 125 matches altogether. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches. Hint: imitate the previous exercise.

Chapter 2

Recurrence

2.1 Some Examples

A *recurrence relation* or *difference equation* for a sequence $a_0, a_1, \ldots$ is a formula that gives each a_n in terms of earlier terms in the sequence.

Example 2.1.1.

Consider the recurrence relation

$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n = 2, 3, \ldots$$

where we are also given the *initial conditions* $a_0 = 1, a_1 = 2$.

This enables us to compute $a_2 = a_1 + a_0 = 2 + 1 = 3$, $a_3 = a_2 + a_1 = 3 + 2 = 5$, $a_4 = a_3 + a_2 = 5 + 3 = 8$, etc. ■

A particular sequence is a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Example 2.1.2.

Find a recurrence relation for the number of ways to arrange n distinct objects in a row. Give its solution.

Let a_n be the number of permutations of n objects. (Of course we know a formula for a_n from §1.2 but pretend for the moment that we don't.) A recurrence relation for a_n can be found as follows: there are n choices for the position of the 1st object, then the remaining $n-1$ objects can be arranged in a_{n-1} ways (by definition of a_{n-1}). By the product rule,

$$a_n = (n)(a_{n-1}) \quad \text{for } n = 2, 3, \ldots$$

This is our recurrence relation.

To solve it, observe that by invoking it repeatedly one gets

$$a_n = na_{n-1} = n(n-1)a_{n-2} = n(n-1)(n-2)a_{n-3} = \cdots = n(n-1)(n-2)\cdots(3)(2)a_1,$$

and clearly we have the initial condition $a_1 = 1$, so $a_n = n!$ That is, the sequence $a_n = n!$ for $n = 1, 2, \ldots$ is a solution of the recurrence relation $a_n = na_{n-1}$ with the initial condition $a_1 = 1$. ■

Example 2.1.3. *(Towers of Hanoi)*

This puzzle contains 3 vertical pegs with n discs of different sizes on 1 peg, increasing in size from top to bottom. The object of the puzzle is to move all the discs to one of the other pegs, subject to the following rules:

(i) only one disc can be moved at a time;

(ii) one can never put a bigger disc on a smaller disc.

We shall determine and solve a recurrence relation for H_n, the number of moves needed.

Initially we have n discs on peg 1 and the other 2 pegs are empty. By definition of H_{n-1}, we can move the top $n-1$ discs to peg 2 using H_{n-1} moves. Then use 1 move to transfer the largest disc from peg 1 to peg 3. Now, again use H_{n-1} moves to move the $n-1$ discs from peg 2 to peg 3. We are done! That is,

$$H_n = 2H_{n-1} + 1 \quad \text{for } n = 2, 3, \ldots \tag{2.1}$$

To solve (2.1) we also need an initial condition: clearly $H_1 = 1$. Now

$$\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1, \quad \text{by (2.1)}, \\
&= 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1, \quad \text{by (2.1)}, \\
&= 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&= \cdots \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^n - 1,
\end{aligned}$$

as this series is geometric and can easily be summed. ■

Read §2.1.

2.2 Solving Recurrence Relations

Some recurrence relations can be solved recursively (like the examples of §2.1) but most can't. We now discuss a class of recurrence relations that can be solved systematically using a different approach.

A *linear homogeneous recurrence relation of order k with constant coefficients* has the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \tag{2.2}$$

where the $c_1, \ldots, c_k$ are constants and $c_k \neq 0$.

Fact: such a recurrence relation has a unique solution if in addition k initial conditions are given: $a_1 = C_1, \ldots, a_k = C_k$, where the C_i are known constants.

To solve (2.2), try $a_n = r^n$ where the constant r is yet to be determined. Substituting this into (2.2) we get

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k},$$

i.e.,

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0 \tag{2.3}$$

— the *auxiliary* or *characteristic equation* of (2.2). The solutions of (2.3) are the *characteristic roots* of (2.2).

The case $k = 1$ can easily be solved by recursion — see the Examples of §2.1 — so we first consider the case $k = 2$.

Theorem 2.2.1. `[Anderson Theorem 2.1]` *Consider the second-order recurrence relation* $a_n = c_1 a_{n-1} + c_2 a_{n-2}$*, where the initial values* a_1 *and* a_2 *are given. Suppose that the characteristic equation* $r^2 - c_1 r - c_2 = 0$ *has roots* r_1 *and* r_2*.*

(i) If $r_1 \neq r_2$*, then* $a_n = k_1 r_1^n + k_2 r_2^n$ *for* $n = 1, 2, \ldots$ *where* k_1 *and* k_2 *are chosen so that the initial conditions are satisfied.*

(ii) If $r_1 = r_2$*, then* $a_n = k_1 r_1^n + k_2 n r_1^n$ *for* $n = 1, 2, \ldots$ *where* k_1 *and* k_2 *are chosen so that the initial conditions are satisfied.*

Proof. (i) Choose k_1 and k_2 such that $a_1 = k_1 r_1 + k_2 r_2$ and $a_2 = k_1 r_1^2 + k_2 r_2^2$. That is,

$$k_1 = \frac{a_1 r_2 - a_2}{r_1 r_2 - r_1^2} \quad \text{and} \quad k_2 = \frac{a_1 r_1 - a_2}{r_1 r_2 - r_2^2}$$

(k_1 and k_2 can be chosen according to these formulas since $r_1 \neq r_2$). Thus $a_n = k_1 r_1^n + k_2 r_2^n$ for $n = 1, 2$.

We continue by using induction. Suppose that $a_n = k_1 r_1^n + k_2 r_2^n$ for $n = 1, 2, \ldots, m$, where $m \geq 2$. Then

$$\begin{aligned}
a_{m+1} &= c_1 a_m + c_2 a_{m-1} \\
&= c_1(k_1 r_1^m + k_2 r_2^m) + c_2(k_1 r_1^{m-1} + k_2 r_2^{m-1}), \quad \text{by the inductive hypothesis,} \\
&= k_1 r_1^{m-1}(c_1 r_1 + c_2) + k_2 r_2^{m-1}(c_1 r_2 + c_2) \\
&= k_1 r_1^{m-1}(r_1^2) + k_2 r_2^{m-1}(r_2^2), \quad \text{by definition of } r_1 \text{ and } r_2, \\
&= k_1 r_1^{m+1} + k_2 r_2^{m+1},
\end{aligned}$$

i.e., our formula for $a_1, a_2, \ldots, a_m$ is also valid for a_{m+1}. By the principle of induction, the proof of (i) is complete.

(ii) This is shown in a similar manner. The details are given in Anderson. ■

Example 2.2.1.

Solve the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, where $a_0 = 3$ and $a_1 = 6$.

The fact that the initial term is a_0 instead of a_1 does not affect our theory. The characteristic equation is $r^2 - r - 6 = 0$, with distinct characteristic roots $r_1 = 3$ and $r_2 = -2$. By Theorem 2.2.1 the solution is therefore $a_n = k_1 3^n + k_2(-2)^n$ for some constants k_1 and k_2. We require

$$\begin{cases} 3 = a_0 = \quad k_1 + k_2, \\ 6 = a_1 = 3k_1 - 2k_2, \end{cases}$$

so $k_1 = 12/5, k_2 = 3/5$. Answer:

$$a_n = \frac{1}{5}\Big(12(3)^n + 3(-2)^n\Big) \quad \forall n. \qquad \blacksquare$$

For linear homogeneous recurrence relations of order $k \geq 2$ whose characteristic equations have distinct roots, we have the following analogue of Theorem 2.2.1(i).

Theorem 2.2.2. *Consider the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$. Suppose that its characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$. Then $\{a_n\}$ is a solution of the recurrence relation if and only if $a_n = \sum_{i=1}^{k} k_i r_i^n$ for $n = 0, 1, 2, \ldots$ where $k_1, k_2, \ldots, k_k$ are arbitrary constants.*

This result can be proved by imitating the argument of Theorem 2.2.1.

We now move on to linear *inhomogeneous* recurrence relations of order k with constant coefficients. These have the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \tag{2.4}$$

where $F(n)$ does not depend on the sequence $\{a_n\}$. (Example: $a_n = 2a_{n-1} + 3n$.) The *associated homogeneous recurrence relation* is

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}. \tag{2.5}$$

The solutions of (2.4) and (2.5) are intimately connected.

Theorem 2.2.3. *Let $\{a_n^{(p)}\}_{n=0}^{\infty}$ be one particular solution of the inhomogeneous recurrence relation (2.4). Then the set of solutions to (2.4) — called the "*general solution *of (2.4)" — has the form $\{a_n^{(p)} + a_n^{(h)}\}_{n=0}^{\infty}$, where $\{a_n^{(h)}\}_{n=0}^{\infty}$ is any solution to the associated homogeneous recurrence relation (2.5). Note that $\{a_n^{(h)}\}_{n=0}^{\infty}$ contains k arbitrary constants.*

Example 2.2.2.

$a_n = 2a_{n-1} + 3n$ for $n \geq 1$, with $a_1 = 0$. One can check that a particular solution of the given inhomogeneous recurrence relation is $a_n^{(p)} = -3n - 6$ (we'll discuss later how to find this). The associated homogeneous recurrence relation is $a_n^{(h)} = 2a_{n-1}^{(h)}$, whose solution is $a_n^{(h)} = k2^n$ for arbitrary constant k. Hence the general solution of $a_n = 2a_{n-1} + 3n$ is $a_n = -3n - 6 + k2^n$.

Now recall the initial condition: $0 = a_1 = -3(1) - 6 + (k)(2)$, so $k = 9/2$. Hence $a_n = -3n - 6 + k2^n = -3n - 6 + (9)2^{n-1}$. ∎

How does one find $a_n^{(p)} = -3n-6$ in Example 2.2.2? For certain $F(n)$ one can predict the form of the particular solution $a_n^{(p)}$, up to some constants that can be found explicitly on substituting our "guess" into (2.4).

Theorem 2.2.4. *In the inhomogeneous recurrence relation (2.4), suppose that $F(n) = s^n Q(n)$, where s is some constant and $Q(n)$ is a polynomial in n. Then (2.4) has a particular solution of the form*

(i) $s^n R(n)$ if s is not a characteristic root of the associated homogeneous recurrence relation (2.5);

(ii) $s^n n^m R(n)$ if s is a characteristic root of multiplicity m of (2.5);

in both (i) and (ii), $R(n)$ is a polynomial having the same degree as $Q(n)$.

Example 2.2.2 again.
In the notation of Theorem 2.2.4, $s = 1$ and $Q(n) = 3n$. The associated homogeneous recurrence relation is $a_n^{(h)} = 2a_{n-1}^{(h)}$, with characteristic polynomial $r - 2 = 0$ and characteristic root $r = 2$. As $s \neq 2$, we use Theorem 2.2.4 (i): a particular solution has the form $1^n(\beta n + \gamma) = \beta n + \gamma$. To find β and γ, substitute $a_n^{(p)} = \beta n + \gamma$ into $a_n^{(p)} = 2a_{n-1}^{(p)} + 3n$: one gets

$$\beta n + \gamma = 2[\beta(n-1) + \gamma] + 3n.$$

Equating the coefficients of n and the constant terms, we find that $\beta = -3$ and $\gamma = -6$. That is, $a_n^{(p)} = -3n - 6$. ■

Read §2.2.

Exercises:

1. Anderson Exercises 2.1–2.3, 2.5–2.9 and 2.11–2.15.

2. What form does a particular solution of the linear inhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = 2^n n^2$ and $F(n) = (n^2+1)3^n$?
Ans: $p_0 n^2 3^n,\ 3^n n^2(p_1 n + p_0), 2^n(p_2 n^2 + p_1 n + p_0),\ 3^n n^2(p_2 n^2 + p_1 n + p_0)$.

3. Find a formula for the sum a_n of the first n positive integers by observing that $a_n = a_{n-1} + n$ for all $n \geq 1$ and $a_1 = 1$, then applying the techniques of this section.

4. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3^n$ with initial condition $a_1 = 5$.

5. Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$.
Ans: $a_n = 2^n(p_0 + p_1 n + n^2 + n^3/6)$.

6. Let a_n be the sum of the first n "triangular numbers", i.e. $a_n = \sum_{i=1}^{n} i(i+1)/2$. Write down a recurrence relation of order 1 for a_n and then apply the techniques of this section to show that $a_n = n(n+1)(n+2)/6$.

2.3 Generating Functions

Given an infinite sequence $\{a_n\}_0^\infty$, its *generating function* is defined to be the power series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

If the sequence $a_0, a_1, \ldots, a_{n_0}$ is finite, make it infinite by setting $a_n = 0$ for all $n > n_0$; then its generating function is a polynomial of degree n_0.

Example 2.3.1.

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1$$

(sum of a geometric series). From calculus it follows that

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{j=0}^{\infty} (j+1) x^j \qquad \text{for } |x| < 1$$

on setting $j = n - 1$. That is, $(1-x)^{-2}$ is the generating function for the sequence $1, 2, 3, \ldots$ ■

Generating functions are useful when solving linear inhomogeneous recurrence relations with constant coefficients. The idea is to find a closed-form formula for the generating function associated with the sequence that is the solution of the recurrence relation, then from this read off the coefficients of $1, x, x^2, \ldots$.

Example 2.3.2.

Solve the recurrence relation $a_n = 2a_{n-1} + 2^n$ for $n \geq 1$, where $a_0 = 2$.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the solution $\{a_n\}$. Then

$$\begin{aligned} G(x) &= a_0 + \sum_{n=1}^{\infty}(2a_{n-1} + 2^n)x^n \\ &= 2 + 2x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} + \sum_{n=1}^{\infty}(2x)^n \\ &= 2 + 2xG(x) + \frac{2x}{1-2x} . \end{aligned}$$

That is,

$$(1-2x)G(x) = 2 + \frac{2x}{1-2x}, \quad \text{i.e.,} \quad G(x) = \frac{2}{1-2x} + \frac{2x}{(1-2x)^2} .$$

This is a closed-form formula for $G(x)$. To finish we need to find its expansion as a power series $\sum_{n=0}^{\infty} a_n x^n$. To find this, expand the second term in $G(x)$ using partial fractions as in calculus; this yields *(exercise)*

$$G(x) = \frac{1}{1-2x} + \frac{1}{(1-2x)^2} .$$

Now a variant of Example 2.3.1 yields *(check)*

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty}(2x)^n + \sum_{n=1}^{\infty} n(2x)^{n-1} \\ &= \sum_{n=0}^{\infty}[(2x)^n + (n+1)2^n x^n] \\ &= \sum_{n=0}^{\infty}(n+2)2^n x^n. \end{aligned}$$

That is, $a_n = (n+2)2^n$ for $n = 0, 1, \ldots$.

Exercise: check that this satisfies the recurrence relation and initial condition. ■

Read §2.3. Exercises 2.4, 2.10.

We shall deal with §2.4 later. Skip §§2.5, 2.6. Parts of Chapters 3,4 and 5 will be covered in the graph theory part of the course, but first we resume our study of Combinatorics in Chapter 6.

Chapter 6

The Inclusion-Exclusion Principle

6.1 The Principle

When we are asked to count the number of ways that task A *or* task B can be done, and some of these ways cause A and B to be done simultaneously, then

$$\begin{aligned}\text{answer} = (&\text{number of ways to do A}) + (\text{number of ways to do B}) \\ &- (\text{number of ways to do A and B simultaneously}).\end{aligned}$$

This is the *inclusion-exclusion principle*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 6.1.1.

How many positive integers between 1 and 99 are divisible either by 4 or by 6?

$$\text{number divisible by } 4 = \frac{96-4}{4} + 1 = 23 + 1 = 24,$$
$$\text{number divisible by } 6 = \frac{96-6}{6} + 1 = 15 + 1 = 16,$$
$$\begin{aligned}\text{number divisible by 4 and 6} &= \text{number divisible by 12 (l.c.m. of 4 and 6)} \\ &= \frac{96-12}{12} + 1 = 7 + 1 = 8.\end{aligned}$$

By the inclusion-exclusion principle, answer $= 24 + 16 - 8 = 32$. ■

2 sets A, B: $\quad |A \cup B| = |A| + |B| - |A \cap B|$.

3 sets A, B, C: $\quad |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

♢

These formulas are typically used to calculate the value of the left-hand side when all values on the right-hand side are given.

Theorem 6.1.1. `[Anderson Theorem 6.1]` *(Inclusion-exclusion for n sets) Let $A_1, A_2, \ldots, A_n$ be finite sets. Then*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|. \tag{6.1}$$

Proof. Let $x \in A_1 \cup A_2 \cup \cdots \cup A_n$. Suppose that in fact x lies in exactly r of these sets, so $1 \le r \le n$. How many times is x counted in the right-hand side of (6.1)? (The answer should be 1 to prove (6.1) correct.)

In $\sum_{i=1}^{n} |A_i|$ the element x is counted $C(r, 1)$ times. In $\sum_{1 \le i < j \le n} |A_i \cap A_j|$ it's counted $C(r, 2)$ times. Continue like this until an intersection of r sets is reached; after this point, there is no further contribution from the right-hand side. Thus the total number of times x has been counted is

$$C(r, 1) - C(r, 2) + \cdots + (-1)^{r+1} C(r, r) = \sum_{k=1}^{r} (-1)^{k+1} C(r, k).$$

But Theorem 1.4.2 says that $\sum_{k=0}^{r} (-1)^k C(r, k) = 0$, i.e., $1 = \sum_{k=1}^{r} (-1)^{k+1} C(r, k)$, as desired. ■

In Anderson's notation, the conclusion of Theorem 6.1.1 is written as

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} N(i) \; - \sum_{1\le i<j\le n} N(i,j) \; + \sum_{1\le i<j<k\le n} N(i,j,k) \\ - \cdots + (-1)^{n+1} N(1,2,\ldots,n). \tag{6.2}$$

Example 6.1.2.

In how many ways can one distribute r distinct objects into 5 distinct boxes yielding at least one empty box?

Let A_i be the set of distributions that leave box i empty for $i = 1, 2, 3, 4, 5$. We want $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|$. By (6.2) we have

$$|A_1 \cup A_2 \cup \cdots \cup A_5| = \sum_{i=1}^{5} N(i) \; - \sum_{1\le i<j\le 5} N(i,j) \; + \sum_{1\le i<j<k\le 5} N(i,j,k) \\ - \cdots + (-1)^{6} N(1,2,3,4,5).$$

By Theorem 1.5.1 (or the product rule) $N(i) = 4^r$, $N(i,j) = 3^r$, $N(i,j,k) = 2^r$, $N(i,j,k,l) = 1$, $N(1,2,3,4,5) = 0$. Hence

$$|A_1 \cup A_2 \cup \cdots \cup A_5| = (5)4^r - C(5,2)3^r + C(5,3)2^r - C(5,4)1 + C(5,5)0. \qquad \blacksquare$$

If we regard A_i as the set of elements having property P_i, then Theorem 6.1.1 counts the number of objects having one or more of n properties. The following alternative form of the inclusion-exclusion principle is useful when one wants the number of objects having *none* of those n properties.

Let A'_i denote the complement of A_i for each i. Then $A'_1 \cap A'_2 \cap \cdots \cap A'_n$ is the set of objects that has none of the properties $P_1, P_2, \ldots, P_n$.

Theorem 6.1.2. `[Anderson Theorem 6.2]` *Let $A_1, A_2, \ldots, A_n$ be finite sets. Let N denote the total number of elements in the set where we are operating (the "universe"). Then Theorem 6.1.1 implies that*

$$\begin{aligned}|A'_1 \cap A'_2 \cap \cdots \cap A'_n| &= N - |A_1 \cup A_2 \cup \cdots \cup A_n| \\ &= N - \sum_{i=1}^{n} N(i) \ + \sum_{1 \le i < j \le n} N(i,j) \ - \sum_{1 \le i < j < k \le n} N(i,j,k) \\ &\qquad + \cdots + (-1)^n N(1, 2 \cdots, n). \end{aligned} \tag{6.3}$$

Example 6.1.3.

In how many ways can one choose a 5-card hand from a 52-card pack when the hand must contain at least one card from each suit?

Here the universe is the set of all 5-card hands, so $N = C(52, 5)$. We let $A_1 = \{\text{hands with no clubs}\}$, $A_2 = \{\text{hands with no spades}\}$, $A_3 = \{\text{hands with no hearts}\}$, $A_4 = \{\text{hands with no diamonds}\}$. As usual define P_i by making A_i the set of 5-card hands that have property P_i. Then we want $|A'_1 \cap A'_2 \cap A'_3 \cap A'_4|$.

Now $N = C(52, 5)$, $N(i) = C(39, 5)$ for each i, $N(i, j) = C(26, 5)$ for each i and j, $N(i, j, k) = C(13, 5)$, and

$N(1,2,3,4) = 0$. From (6.3) we get

$$\begin{aligned}\text{answer} &= C(52,5) - 4\,C(39,5) + \sum_{1\le i<j\le 4} C(26,5) - \sum_{1\le i<j<k\le 4} C(13,5)\\ &= C(52,5) - 4\,C(39,5) + C(4,2)C(26,5) - C(4,3)C(13,5)\\ &= \cdots \qquad \blacksquare\end{aligned}$$

Exercise: check this answer by solving the problem in a different way.

Read §6.1 (Example 6.2 won't make sense until we've done some graph theory later in the course). Exercises 6.1–6.6, 6.9, 6.11.

Now we return briefly to Chapter 2.

2.4 Derangements

A *derangement* is a permutation where no object is left in its original position. For example, 2143 is a derangement of 1234.

Theorem 2.4.1. `[Anderson Example 6.6 and §2.4]` *The number of derangements of a set S with n elements is*

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right].$$

(The expression in square brackets is part of the Maclaurin series for e^x with $x = -1$; thus $D_n \approx e^{-1}n!$ for n large, which has probability implications — see Anderson.)

Proof. The universe is the set of all permutations of S, so $N = n!$ For each i, let A_i be the set of permutations of S that have property P_i: the ith object is *not* moved. Then $D_n = |A_1' \cap A_2' \cap \cdots \cap A_n'|$.

From (6.3) we have

$$|A_1' \cap A_2' \cap \cdots \cap A_n'| = N - \sum_{i=1}^{n} N(i) + \sum_{1 \le i < j \le n} N(i,j) - \sum_{1 \le i < j < k \le n} N(i,j,k) + \cdots + (-1)^n N(1,2,\ldots,n).$$

Now $N(i)$ is the number of permutations of $n-1$ objects (since the ith object is fixed), so $N(i) = (n-1)!$

for all i. Similarly $N(i,j) = (n-2)!\,$, $N(i,j,k) = (n-3)!\,$, etc. Hence

$$\begin{aligned}
|A_1' \cap A_2' \cap \cdots \cap A_n'| &= n! - \sum_{i=1}^{n} (n-1)! \; + \sum_{1 \le i < j \le n} (n-2)! \; - \sum_{1 \le i < j < k \le n} (n-3)! \\
&\qquad + \cdots + (-1)^n 0! \\
&= n! - n(n-1)! \; + C(n,2)(n-2)! \; - C(n,3)(n-3)! \\
&\qquad + \cdots + (-1)^n 0! \\
&= n! - n! + \frac{n(n-1)}{2!}(n-2)! - \frac{n(n-1)(n-2)}{3!}(n-3)! \\
&\qquad + \cdots + (-1)^n \\
&= n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right]. \qquad \blacksquare
\end{aligned}$$

Read §2.4. Note in particular that the formula for D_n given in Theorem 2.4.1 can be elegantly derived using the inversion formula (1.1). Exercises 6.7, 6.8, 6.10.

Skip §§6.2, 6.3.

6.4 Scrabble

This short section comprises a single example: at the beginning of a Scrabble game, how many different possibilities are there for the set of 7 letter tiles that a player chooses?

Read Section 6.4.

Chapter 3

Introduction to Graphs

3.1 The Concept of a Graph

Warning: Unlike most branches of mathematics, definitions in graph theory vary from book to book. We'll follow Anderson's terminology.

A *graph* $G = (V, E)$ is a non-empty set of *nodes* or *vertices* V and a (possibly empty) list of *edges* E whose elements are pairs of vertices.

Example 3.1.1.

$G = (V, E)$, where $V = \{a, b, c, d, e\}$ and $E = \{a, a\}, \{a, b\}, \{a, c\}, \{a, c\}, \{b, c\}, \{b, d\}$. While this formal definition of G is the most precise way of describing the graph, it's easier to understand what it means if we represent G by the following diagram:

♢

You can position a, b, c, d in any way you like and the edges merely have to join the correct pairs of nodes — they do not have to be straight lines. ■

Two nodes a, b are *adjacent* if they are joined by an edge $e = \{a, b\}$. We then say that e is *incident* to a and b. An edge of the form $\{a, a\}$ is called a *loop*. If two or more edges join the same pair of nodes, they are called *multiple* or *parallel* edges. Example 3.1.1 contains two multiple edges.

If a graph has neither loops nor multiple edges, we say it is *simple*. The London Underground map is not a simple graph, as it has multiple edges (but no loops).

The *degree* of a vertex v, written $d(v)$, is the number of edges incident to v (where each loop is counted twice). In Example 3.1.1,

If a graph is of the type used by chemists to describe molecules, where nodes represent atoms and edges represent bonds, then the degree of a node is the same as its valency.

A node of degree 1 is called a *pendant* node.

Theorem 3.1.1. `[Anderson Theorem 3.1]` *(the handshaking lemma) In any graph, the sum of the degrees of all the nodes equals twice the number of edges.*

Proof. Adding the degrees of all nodes means counting each edge twice (once for each of the nodes it joins). ∎

Corollary 3.1.1. `[Anderson Exercise 3.1]` *Every graph has an even number of nodes of odd degree.*

Proof. Let $G = (V, E)$ be a graph. Let $A(B)$ be the sum of the degrees of the nodes of even(odd) degree. By Theorem 3.1.1, $2|E| = A + B$. Since A is a sum of even numbers, A is even. Also $2|E|$ is even. Thus $B = 2|E| - A$ is even. This implies that B must be the sum of an even number of odd numbers, which is the required result. ∎

Example 3.1.2.

Can you construct a graph with 12 edges such that 2 nodes have degree 3 and the rest have degree 4?

Let x denote the number of nodes of degree 4. By Theorem 3.1.1, $2(12) = 2(3) + 4x$, which gives $x = 9/2$. This is impossible, so no such graph exists. ∎

Example 3.1.3.

During a party several people may shake hands. Consider the graph with nodes = people and two people joined by an edge if they shook hands. The degree of each node is the number of people with whom that person shook hands. By Corollary 3.1.1, when the party ends an even number of people have shaken hands an odd number of times. ∎

Example 3.1.4.

A house has only one door. Prove that some room in the house must have an odd number of doors.

Consider graph with nodes = rooms (and an extra node representing the exterior world). Join 2 nodes by an edge only if there is a door linking the corresponding 2 rooms. By Corollary 3.1.1 there is an even number of rooms having an odd number of doors. But the exterior of the house is one of these as the house has only one door. Thus the even number isn't 0, it must be 2 or 4 or 6 etc. Hence there's at least one such room inside the house. ∎

Like all branches of mathematics, there are standard examples of graphs that are frequently useful. The first useful family that we meet are the *complete* graphs K_n, where $n \geq 1$; each K_n is a simple graph with n vertices where all pairs of vertices are adjacent.

$\diamondsuit$

If $G = (V, E)$ is a graph, then $G' = (V', E')$ is a *subgraph* of G if $V' \subset V$, $E' \subset E$ and edges in E' are incident only to nodes of V'. Put another way, G' can be got from G by erasing some nodes and/or edges but leaving no edge without a node at both ends. For example, K_3 is a subgraph of K_4.

The notation (p, q)-graph means a graph with p nodes and q edges.

Read §3.1.

Exercises:

1. There used to be 26 teams in the U.S. National Football League with 13 teams in each of two divisions. A League guideline said that each team's 14-game schedule should include exactly 11 games against teams in its own division and 3 games against teams in the other division. By considering part of a graph model of this scheduling problem, show that this guideline could not be satisfied by all the teams!

2. For each of the following, describe a graph model and then answer the question.

   (a) Must the number of people at a party who know an odd number of other people be even?

   (b) Must the number of people at a party who do not know an odd number of other people be even?

   (c) Must the number of people ever born who had (have) an odd number of brothers and sisters be even?

   (d) Must the number of families in Kerry with an odd number of children be even?

3. A graph has 12 edges and 6 nodes, each of which has degree 2 or 5. How many nodes are there of each degree?

4. Either find a graph that models the following or show that none exists: Each of 102 students will be assigned the use of 1 of 35 computers, and each of the 35 computers will be used by exactly 1 or 3 students.

5. (a) Is it possible to have a group of 11 people, each of whom knows exactly 3 others of the group? If it is possible, draw a graph illustrating the situation. If it's impossible, explain why.

   (b) Same as part (a), except 11 is replaced by 8.

6. Show that K_n has $n(n-1)/2$ edges for $n = 1, 2, \ldots$

7. Suppose that there are 7 committees with each pair of committees having a common member and each person is on two committees. How many people are there? Hint: use the previous exercise.

3.2 Paths in Graphs

When applied problems are modelled using graphs, the solution of the problem often involves moving around the graph following the edges from node to node.

Let x_0 and x_k be two nodes in a graph G. A *walk* from x_0 (the *initial vertex*) to x_k (the *final vertex*) is a sequence of edges of the form $(x_0, x_1), (x_1, x_2), (x_2, x_3), \cdots, (x_{k-1}, x_k)$. This walk has *length k* (the number of edges in the walk).

If all the edges are distinct, the walk is called a *trail*. If all the vertices (except possibly x_0 and x_k) are distinct then the walk is called a *path*. If no vertex is repeated in a walk, then of course no edge is repeated, so every path is a trail. A path for which $x_0 = x_k$ is called a *cycle*.

Example 3.2.1.

$a - b - c - a - d$ is a trail of length 4. It is not a path. $a - b - c - a$ is a cycle of length 3. ■

♢

We shall use the notation C_n to denote a cycle of length n, which contains n nodes. Also, let P_n denote a path of length $n-1$ whose initial and terminal nodes are not the same (so P_n contains n nodes).

$\diamondsuit$

A graph is *connected* if there is a path joining each pair of distinct vertices. Otherwise the graph is *disconnected*. A disconnected graph can be written as a union of connected subgraphs called *components* where different components have no vertex in common. Two nodes are in the same component of a graph if and only if there is a path joining them.

Example 3.2.2.

(i) the graph of Example 3.2.1 is connected.

(ii) a disconnected graph with 2 components:

$\diamondsuit$

Read §3.2. Exercise 3.3.

3.3 Trees

A *tree* is a connected simple graph with no cycles.

$\diamondsuit$

Trees are used to model structures in such diverse areas as computer science, chemistry, geology, botany and psychology.

The requirement that a tree be connected forces it to have sufficiently many edges, while the requirement that it have no cycles constrains it from having too many edges. These opposing tendencies are neatly balanced in a tree, as the next theorem shows.

Theorem 3.3.1. `[Anderson Theorem 3.4]` *Let T be a simple graph. Suppose it has p nodes. Then the following statements are equivalent:*

(i). T is connected and has no cycles (i.e., T is a tree)

(ii). T has no cycles and has $p-1$ edges

(iii). T is connected and has $p-1$ edges.

Proof. We shall show that $(i) \Leftrightarrow (ii)$ and $(i) \Leftrightarrow (iii)$. (Our argument is not identical to Anderson's.)

$(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$: Let T be a tree with p nodes. We must show that T has $p-1$ edges. Use strong induction on p. The case $p = 1$ is obvious. Assume that a tree with p nodes has $p-1$ edges for $p = 1, 2, \ldots, k$. Suppose that the tree T has $k+1$ vertices. Remove an edge from T. This yields two trees T_1 and T_2 with k_1 and k_2 vertices respectively, where $k_1 + k_2 = k + 1$. By the inductive hypothesis, T_1 has $k_1 - 1$ edges and T_2 has $k_2 - 1$ edges. Hence T has

$$(k_1 - 1) + (k_2 - 1) + 1 = k_1 + k_2 - 1 = k \text{ edges},$$

as desired. By the principle of induction, $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ for all $p \geq 1$.

$(ii) \Rightarrow (i)$: Let T be a simple graph with p nodes, $p-1$ edges and no cycles. We must show that T is connected. If T is disconnected, then it contains at least two components. Add an edge joining a pair of nodes from two separate components. This cannot yield a cycle. If (considering all nodes and edges) the new graph thereby obtained is not connected, add another edge joining two separate components. Continue adding one edge at a time until a connected graph T' is obtained. Then T' is simple and has no cycles; it is a tree. As it has p nodes, we know from the first part of the proof that T' must therefore contain $p-1$ edges. But this is impossible since we started with $p-1$ edges then added one or more edges. Thus our guess that T was disconnected must have been wrong, i.e., T is connected.

$(iii) \Rightarrow (i)$: Let T be a connected simple graph with p nodes and $p-1$ edges. We must show that T has no cycles. If T has a cycle, then we can remove one edge of that cycle without disconnecting the graph. If the new graph still has a cycle, again remove an edge from a cycle. Continue removing one edge from each cycle until no cycles remain. The graph T'' that we now have is connected, simple, and has p nodes and no cycles; it is a tree. We know from the first part of the proof that T'' must therefore contain $p-1$ edges. But this is impossible since we started with $p-1$ edges then removed one or more edges. Thus our guess that T contained a cycle must have been wrong, i.e., T has no cycle. ∎

Example 3.3.1.

Consider the chemical molecules (both are trees) for butane and isobutane:

♢

Both have the molecular formula C_4H_{10}, but they have slightly differing chemical behaviour. This is because, regarding the above molecular diagrams as graphs, these graphs are "fundamentally different" or *nonisomorphic*. ■

Two graphs are *isomorphic* if there exists a one-to-one correspondence (a "pairing off") between the vertices in the two graphs such that the number of edges joining each pair of nodes in the first graph is the same as the number of edges joining the corresponding pair of nodes in the second graph.

Intuitively, two graphs are isomorphic if you can redraw one to look like the other.

Example 3.3.2.

The graphs G_1 and G_2

are isomorphic since we can redraw G_1 as

Now the pairing-off of vertices is clear: a with p, b with m, c with o, and d with n. ■

♢

Returning to Example 3.3.1, butane has 2 nodes of degree 4 that are each adjacent to 3 nodes of degree 1, but isobutane has 3 nodes of degree 4 with this property. Any isomorphism would have to make those 2 butane nodes correspond (in a one-to-one and onto way) with the 3 nodes in isobutane. This is impossible, so no isomorphism exists.

Read §3.3. Exercise 3.2.

Exercises:

1. `[Anderson Theorem 3.3]` Show that if T is a tree with at least two nodes, then T has at least two pendant vertices. Hint: suppose that T has p nodes, of which m are pendant nodes. Use Theorems 3.1.1 and 3.3.1.

2. Construct all nonisomorphic simple graphs with 3 vertices. *[There are 4.]*

3. Construct all nonisomorphic simple graphs with 4 vertices and at most 3 edges. *[There are 7.]*

3.4 Spanning Trees

Example 3.4.1.

Consider a network of computers in five cities (see diagram) where various pairs of cities are linked directly by broadband computer lines. Each such line has a monthly rental cost indicated. Which lines should one rent in order to minimize the total cost, while ensuring that all computers can communicate with each other?

♢

The solution is given by a subgraph H that includes all nodes, is connected, and whose edges have total minimal cost. This subgraph H cannot contain any cycle, since this would mean that an unnecessary line has been rented. Thus H is a tree. ∎

Given a graph G, if a subgraph H of G contains all vertices of G and is a tree, we say that H is a *spanning tree* of G.

Example 3.4.2.

Let G be

It has four spanning trees:

Thus spanning trees are not in general unique. ∎

♢

In Example 3.4.1 the edges of the graph have non-negative *weights*, and we want a spanning tree the sum of whose weights is as small as possible. This is called a *minimal spanning tree*.

Given a weighted connected graph, there are two standard algorithms for finding minimal spanning trees.

Kruskal's algorithm (1956): successively choose edges of minimal weight without forming any cycles until this can no longer be done.

Prim's algorithm (1957): starting with any edge of minimal weight, form a tree that grows by adding edges of minimal weight (without forming any cycles) until this can no longer be done.

At every stage in Prim's algorithm one is required to have a tree, but in Kruskal's algorithm this may not be the case.

Example 3.4.3.

(Example 3.4.1 solved by Kruskal)

In the algorithm choices can be resolved arbitrarily. ■

♢

Example 3.4.4.

(Example 3.4.1 solved by Prim)

In the algorithm choices can be resolved arbitrarily. ■

◊

Note that the problem does not have a unique solution, but every minimal spanning tree does have the same weight, i.e., the total rental cost is the same for each possible solution.

Kruskal's and Prim's algorithms are examples of *greedy algorithms*. These are algorithms that at each iteration make the choice that yields the greatest short-term gain, without worrying about the long-term effect of that choice. Greedy algorithms yield satisfactory results for some discrete problems (e.g., finding minimal spanning trees), but for others they can produce severely nonoptimal solutions.

Theorem 3.4.1. *Let G be a weighted connected graph. Then Prim's algorithm yields a minimal spanning tree for G.*

Proof. Suppose G has n vertices. As long as the tree constructed by Prim's algorithm contains fewer than n nodes, at least one of these nodes is adjacent to a node not in the tree because G is connected. Thus the algorithm will not terminate until it includes all nodes of G, i.e., it does construct a spanning tree of G.

Let $T_i = (e_1, e_2, \ldots, e_i)$ be the spanning tree constructed after i iterations of Prim's algorithm, where $i = 1, 2, \ldots, n-1$. (We are describing the tree in terms of its edges e_j.) Let S be a minimal spanning tree of G that is chosen to have as many edges as possible in common with T_{n-1}, the final tree produced by Prim's algorithm. If $T_{n-1} = S$ we are done, so suppose that $T_{n-1} \neq S$. We shall show that this is impossible.

Let the first edge chosen by Prim that is not in S be $e_k = \{a, b\}$, where $a \in T_{k-1}$ and $b \in T_k$ (if $k = 1$, we set $T_0 = \{a\}$). Since S is a spanning tree of G, there is a path P in S from a to b. This path must contain at least one edge e^* that is incident both to a vertex in T_{k-1} and a vertex not in T_{k-1}. Clearly $e^* \neq e_k$. If the weight $wt(e^*) < wt(e_k)$, then Prim would have chosen e^* over e_k at the kth iteration (there is no risk of a cycle being formed, since $e_1, e_2, \ldots, e_{k-1}, e^*$ all lie in the tree S). Consequently we have $wt(e^*) \geq wt(e_k)$.

Modify S by replacing e^* by e_k, thereby obtaining a new subgraph S' of G. Now S contains all nodes of G so S' must contain all nodes of G. Furthermore, if S' contains a cycle C, then $e_k \in C$ (as otherwise C is a subset of the tree S, which is impossible), which implies that there is a path P' from a to b in $S' \setminus \{e_k\} = S \setminus \{e^*\}$; but this implies that $P' \neq P$, so $P' \cup P \subset S$ will contain a cycle which is impossible as S is a tree. We conclude that S' cannot contain a cycle. Also, S is a spanning tree so it has n nodes; then by Theorem 3.3.1(ii) it has $n-1$ edges; hence by construction S' has $n-1$ edges; now by Theorem 3.3.1, since S' has no cycles and n nodes, it is a tree. To summarize, we now know that S' is a spanning tree of G.

As $wt(e^*) \geq wt(e_k)$, we have $wt(S) \geq wt(S')$, which implies that S' also is a minimal spanning tree of G. By its construction, S' has one more edge in common with T_{n-1} than S had. But this contradicts the choice of S. Thus our supposition that $T_{n-1} \neq S$ must have been wrong, i.e., $T_{n-1} = S$, so T_{n-1} is a minimal spanning tree of G. ■

Read §3.4. Exercises 3.4, 3.5, 3.7–3.9.

Exercises:

1. Let G be a graph. Prove that G is connected if and only if G has a spanning tree. Hint: see the last paragraph of §3.4 in Anderson.

2. Do you believe the proof in Anderson that Kruskal's algorithm yields a minimal spanning tree? Can our

proof for Prim's algorithm be modified to prove the same result for Kruskal's algorithm?

3.5 Bipartite Graphs

A graph $G = (V, E)$ is *bipartite* if its vertices can be divided into two disjoint sets B and W such that all edges in the graph are incident to a node from B and a node from W. We call $V = B \cup W$ a *bipartition* of V. For all positive integers m and n, the *complete bipartite graph* $K_{m,n}$ is the simple bipartite graph where $V = B \cup W$ with $|B| = m, |W| = n$, and all possible edges run from B to W.
$\diamondsuit$

Theorem 3.5.1. `[Anderson Theorem 3.5]` *A connected graph is bipartite if and only if it has no cycle of odd length.*

Proof. If the graph $G(V,E)$ is bipartite with bipartition $V = B \cup W$, then every cycle has the form $b - w - b - w - \cdots$ or $w - b - w - b - \cdots$, where $b \in B$ and $w \in W$. Every cycle either starts and ends in B or starts and ends in W. It follows that no cycle has odd length, as this would forbid its initial and final nodes to both lie in B or both lie in W.

Now suppose that $G = (V,E)$ has no cycle of odd length. Choose a vertex x in G. Let y be any other vertex in G. Observe that the lengths of all walks from x to y must have the same parity (odd/even), as otherwise the union of an odd and an even walk would yield a walk of odd length that began and ended at x; this could be split into a union of cycles, at least one of which must have odd length (impossible!) since the sum of their lengths is odd. Consequently the following definition makes sense: let $B(W)$ be the set of $y \in V$ such that the length of every path from x to y is even(odd).

Our previous arguments show that $V = B \cup W$ with B and W disjoint. No edge e can be incident to a node $v_1 \in B$ and a node $v_2 \in B$, as that would imply a walk (equal to the union of e and a path from x to v_1 and a path from x to v_2) of odd length that returned to its starting point, which by the argument above leads to a contradiction. Similarly no edge can be incident to two nodes in W. That is, G is bipartite. ■

Read §3.5. Exercise 3.6.

Exercise: Let G be a simple (p,q)-graph with $q > p^2/4$. Show that G cannot be bipartite.

3.6 Planarity

A graph is *planar* if it can be drawn on a flat surface without any edges crossing. When it is drawn in this way, we call it a *plane* graph.

Example 3.6.1.

K_4 is planar because we can draw it as

which is plane. ■

◊

Any plane graph divides the page into one or more *regions*, including an infinite region.

Example 3.6.2.

(i)

(ii)

— 1 region only. ■

◇

Theorem 3.6.1. `[Anderson Theorem 3.7]` *(Euler's formula) Let G be a connected plane (p, q)-graph. If G divides the plane into r regions, then $r = q - p + 2$.*

Proof. We use induction on q. If $q = 0$, then $p = 1$ (since G is connected) and $r = 1$, so $q - p + 2 = 1 = r$.

Assume that the theorem is true for $q = k$, where $k \geq 0$. We want to deduce its truth for $q = k + 1$. Thus suppose that G is a connected plane $(p, k + 1)$-graph that divides the plane into r regions. If G is a tree, then $r = 1$ and by Theorem 3.3.1 we have $p = k + 2$, so $q - p + 2 = k + 1 - (k + 2) + 2 = 1 = r$, as desired. Thus we can assume that G is not a tree. This implies that it has a cycle. Delete an edge from that cycle, obtaining a connected (why?) graph G' with k edges. Furthermore, G' divides the plane into $r - 1$ regions. By the inductive hypothesis applied to G',

$$k - p + 2 = r - 1, \quad \text{i.e.,} \quad (k + 1) - p + 2 = r.$$

That is, the inductive hypothesis has been verified for $q = k + 1$.

By the principle of induction, the result holds true for all $q \geq 0$. ■

Theorem 3.6.1 implies that, given a planar graph (such as K_4), then every plane drawing of this graph will have the same number of regions (e.g., K_4 must have $6 - 4 + 2 = 4$ regions).

Define the *degree of a region* to be the number of edges that form the boundary of the region, where any edge that can be approached from both sides within the region is counted twice.

Intuitively, if a (p, q)-graph is planar then q cannot be too large relative to p. Various theorems make a more precise statement; we present the simplest result of this type.

Theorem 3.6.2. `[Anderson Exercise 3.13(a)]` *Let G be a simple connected plane (p, q)-graph with $p \geq 3$. Then $q \leq 3p - 6$.*

Proof. Since G is simple, every region is bounded by at least 3 edges ($p \geq 3$ implies this is true for the infinite region). Thus adding the degrees of all the regions gives a total of at least $3r$. But this sum counts every edge exactly twice (this is why we have that convention about counting edges twice that protrude into regions), so $2q \geq 3r$. Theorem 3.6.1 says that $r = q - p + 2$. Hence

$$2q \geq 3(q - p + 2) = 3q - 3p + 6, \qquad \text{i.e.,} \qquad 3p - 6 \geq q. \qquad \blacksquare$$

Theorem 3.6.3. `[Anderson Theorem 3.9]` *K_n is planar for $n \leq 4$ and non-planar for $n \geq 5$.*

Proof. For $n \leq 4$ we can draw a plane representation of K_n. Consider K_5: it is simple with $p = 5$ and $q = C(5, 2) = 10$. But $3p - 6 = 15 - 6 = 9 \not\geq 10 = q$, so K_5 cannot be planar by Theorem 3.6.2. If $n > 5$, then K_5 is a subgraph of K_n, so the non-planarity of K_5 implies that K_n is also non-planar. $\blacksquare$

One can show via related arguments based on the minimum number of edges forming the boundary of regions that $K_{3,3}$ and the Petersen graphs are non-planar `[Anderson Theorem 3.10 and Example 3.13]`.

A remarkable and deep theorem of Kuratowski from 1930 states that K_5 and $K_{3,3}$ are basically the only non-planar graphs, in the sense that every non-planar graph must contain a subgraph that is essentially K_5 or $K_{3,3}$. See Anderson.

Historical note: the *4-colour problem.* When a map of countries is drawn on a flat surface, it is usually coloured so that any two countries having a common border receive different colours. In general, what is the minimum number of colours needed? Certainly 3 is not always enough, e.g.,

— here 4 colours are needed.

$\diamondsuit$

The problem can be rephrased in graph theoretical terms: nodes = countries and edges join adjacent countries. Then the requirement is that adjacent nodes must have different colours. In 1890 Heawood proved 5 colours suffice. Four colours always seemed to work in practice, but no proof was found. It became a celebrated unsolved problem. Finally in 1976 Appel and Haken at the University of Illinois in Urbana-Champaign proved it using a computer to check cases — the programme took 50 days to run! The local post office franked outgoing letters with the slogan "four colours suffice."

Read §3.6. Exercises 3.10–3.13, 3.15, 3.16.

Skip §3.7.

Chapter 4

Travelling Round a Graph

4.1 Hamiltonian Graphs

William Rowan Hamilton (1805-1865) is Ireland's most famous mathematician.

A *hamiltonian cycle* is a cycle that passes exactly once through every node of a graph. A graph is called *hamiltonian* if it contains a hamiltonian cycle.

Example 4.1.1.

hamiltonian cycle: $x - y - z - w - v - u - x$ ∎

♢

Example 4.1.2.

In a group of people, each person has some friends in the group. Can they be seated at a round table in such a way that each person has a friend on each side?

Construct the graph $G = (V, E)$ where $V = \{\text{people}\}$ and an edge joins two nodes if the corresponding people are friends. Then solving the problem means finding a hamiltonian cycle in G. ■

Unfortunately there is no efficient and reliable method for determining if a given graph has a hamiltonian cycle.

Sometimes one can show that a given graph does not have a hamiltonian cycle by attempting to construct one and arriving at an impossibility. To help us do this, note the following 3 rules that apply when constructing hamiltonian cycles. They are based on the observation that a hamiltonian cycle must use exactly 2 edges at every node.

Rule 1. If $d(x) = 2$, then both edges incident at x must be part of any hamiltonian cycle.

Rule 2. Once the hamiltonian cycle we are constructing has passed through a node x, all other unused edges incident at x can be deleted.

Rule 3. No subcycle (i.e., a cycle not containing all nodes) can be formed when constructing a hamiltonian cycle.

Example 4.1.3.

Consider the graph

♢

By Rule 1, $b - a - c$ and $p - n - j$ and $h - m - o$ are part of any hamiltonian cycle. At node b, either $\{b, d\}$ or $\{b, e\}$ is part of the proposed hamiltonian cycle. We must take each case separately.

Case (i): include $\{b, d\}$ in hamiltonian cycle. By Rule 2, delete $\{b, e\}$. By Rule 1, include $c - e - i$ in hamiltonian cycle. By Rule 2 delete $\{c, f\}$. By Rule 1, include $g - f - j$. By Rule 2 delete $\{j, l\}$. By Rule 1 include $i - l - p$. By Rule 2 delete $\{i, k\}$. By Rule 1 include $h - k - o$. But now we have violated Rule 3, as $h - k - o - m - h$ is a subcycle.

Case (ii): include $\{b, e\}$ in hamiltonian cycle, etc... *(Exercise)* This leads to 3 edges used at j, an impossibility.

We conclude that the graph has no hamiltonian cycle. ■

Anderson `[Example 4.3]` uses a similar technique to show that the Petersen graph has no hamiltonian cycle. Note that it is helpful to use Rules 1 and 2 to force you to include or delete edges. Otherwise you have to consider cases, which makes the work much longer.

Theorems that guarantee the existence of hamiltonian cycles generally have the form "if G has enough edges, then G has a hamiltonian path/cycle ." See Anderson Theorem 4.2 for instance.

Read §4.1. Exercises 4.1, 4.6, 4.7.

Exercise: Can you find a quicker way of doing Example 4.1.3 that avoids considering cases?

4.2 Planarity and Hamiltonian Graphs

Let G be a graph. If we know a hamiltonian cycle H for G, then the following algorithm gives a straightforward way of determining whether or not G is planar.

Algorithm: draw G with H forming the perimeter of G. Construct a new graph K whose vertices are called $e_1, e_2, \ldots, e_r$; these are the edges of $G \setminus H$. In K two vertices are joined by an edge if and only if the corresponding edges cross in our drawing of G. Then G is planar if and only if K is bipartite.

Example 4.2.1.

Is G, the following graph, planar?

$\diamondsuit$

To apply the algorithm we must first find a hamiltonian cycle in G, then redraw G with that cycle forming the perimeter of G. Now $H = a - b - c - d - e - f - g - h - a$ is such a cycle. A redrawing of G with H as perimeter is

$\diamondsuit$

Hence K is

$\diamondsuit$

But by Theorem 3.5.1, K is not bipartite since it contains a cycle of odd length: $ae - gb - ac - be - dg - ae$. The algorithm now concludes that G is non-planar. ■

If this algorithm were to show that a given graph is planar, it also gives a recipe for drawing a plane representation of that graph: with $K = B \cup W$, draw the B-edges inside H and the W-edges outside H. See Anderson Example 4.4 for a demonstration.

Read §4.2. Exercise 4.4.

Skip §4.3.

4.4 Gray Codes

A *Gray code* is a binary code where each number from 1 to n, where $n = 2^r$, is coded as an r-bit sequence with the property that for each consecutive pair of numbers $i, i+1$ (and $n, 1$) the corresponding binary sequences are the same in all but one position. Such codes have their origins in electro-mechanical devices that measure how far a rotating wheel has turned while seeking to minimize any errors in measurement; see Anderson.

For example, the numbers 1, 2, 3, 4 (so $n = 4 = 2^2$, i.e., $r = 2$) have a Gray code $1 \equiv 00, 2 \equiv 01, 3 \equiv 11, 4 \equiv 10$. An alternative Gray code would be $1 \equiv 01, 2 \equiv 00$, $3 \equiv 10, 4 \equiv 11$.

If we have a Gray code for $1, \ldots, 2^r$, then there is a simple way of constructing a Gray code for $1, \ldots, 2^{r+1}$. Observe first that $2^{r+1} = (2)(2^r)$, so the new code has to service exactly twice as many numbers as the old one.

Code $1, \ldots, 2^r$ by appending a 0 to the end of each old code for these numbers, then code $1 + 2^r, \ldots, 2^{r+1}$ by using the old code again but taken in sequence from right to left with a 1 appended in each case.

For example, starting from our code $1 \equiv 00, 2 \equiv 01, 3 \equiv 11, 4 \equiv 10$, we have the following code for $1, \ldots, 8$: $1 \equiv 000, 2 \equiv 010, 3 \equiv 110, 4 \equiv 100$, (now we turn around and head back while appending 1s) $5 \equiv 101, 6 \equiv 111, 7 \equiv 011, 8 \equiv 001$. Check that this is a Gray code!

We could continue in the same way, extending the code for $1, \ldots, 8$ to a Gray code for $1, \ldots, 16$, then $1, \ldots, 32$, etc.

Read §4.4. Exercise 4.5.

4.5 Eulerian Graphs

Graph theory began when in 1736 Leonhard Euler published a paper solving a puzzle that had perplexed the citizens of Königsberg in Prussia (now Kaliningrad in Russia), where he lived. The river divided the city into 2 banks and 2 islands, flowing under 7 bridges as it did so (cf. Cork). For a map see Anderson p.44. Was it possible to take a walk that crossed each bridge exactly once and finished where it started?

This puzzle can be rephrased as a problem in graph theory. Use a node to denote each bank and each island, and edges correspond to the bridges joining these nodes. Then the puzzle asks: is there a walk that traverses each edge exactly once and returns to where it started?

An *eulerian circuit* is a closed trail (i.e., no edges repeated and finishes where it started) that contains every edge of a graph. A graph that contains an eulerian circuit is called an *eulerian graph*.

Unlike hamiltonian cycles, there is an easily-checked necessary and sufficient condition for a graph to be eulerian.

Theorem 4.5.1. `[Anderson Theorem 4.4]` *Let G be a connected graph. Then G has an eulerian circuit if and only if all its nodes have even degree.*

Proof. (only if) Assume that G has an eulerian circuit. For any circuit, every time we traverse a node we use exactly two edges incident to that node. Because the circuit is eulerian, it will eventually use every edge at the node exactly once. As these edges are used in pairs, in total there must be an even number of edges incident to the node, i.e., the node has even degree.

(if) Now assume that all nodes of G have even degree. We show how to construct an eulerian circuit.

Algorithm:

1. [initial node] Choose any node a in G.
2. [initial circuit] Construct a circuit C of the form $a - b - c - \cdots - a$ by choosing any edge from a to b (say), then any new edge from b to c (say), etc., until no new edges can be added. This must yield a circuit because all nodes have even degree, so any node we enter we can leave, with the exception of node a since we started there.
3. [subcircuit loop] Repeat Steps 3a-3c until all edges of G have been used.

3a. [new initial node] Choose a node x (say) in C that is adjacent to some unused edge of G. If there is no such node, then C must be eulerian because G is connected and the algorithm stops.

3b. [construct subcircuit S] Construct a circuit S of the form $x - \cdots - x$ as in Step 2, using only edges not previously used.

3c. [enlarge C] Enlarge C to $C + S$ by following the edges of C as far as x, then going around S back to x, then continuing with C back to a. From its description, this algorithm uses all edges of G and yields a circuit. That is, it constructs an eulerian circuit. ∎

Example 4.5.1.

The following graph is connected and all its nodes have even degree, so by Theorem 4.5.1 it is eulerian.

♢

We have now constructed an eulerian circuit. ■

An *eulerian trail* is a trail that contains every edge of a graph but is not closed. A non-eulerian graph that contains an eulerian trail is called a *semi-eulerian graph*.

For semi-eulerian graphs we have the following analogue of Theorem 4.5.1:

Theorem 4.5.2. `[Anderson Theorem 4.5]` *Let G be a connected graph that has no eulerian circuit. Then G has an eulerian trail if and only if it has precisely two vertices of odd degree.*

Proof. (only if) Suppose that G has an eulerian trail $a - \cdots - b$ (so $a \neq b$). Form a new graph G' by adding the edge $\{b, a\}$ to G. Then $a - \cdots - b - a$ is an eulerian circuit of G'. By Theorem 4.5.1, all nodes of G' have even degree. Deleting $\{a, b\}$ from G' to get G, we see that in G vertices a and b have odd degree and all others have even degree, i.e., G has 2 nodes of odd degree.

(if) Assume G has exactly 2 nodes of odd degree; call them x and y. Form a new graph G^* by adding the edge $\{x, y\}$ to G. Then all nodes of G^* have even degree and G^* is clearly connected. By Theorem 4.5.1, we can construct an eulerian circuit of G^* of the form $x - y - \cdots - x$. Deleting the first edge of this circuit yields an eulerian trail $y - \cdots - x$ in G. ■

Note that if G has exactly 2 nodes a, b of odd degree, its eulerian trail must start at a and end at b (or vice versa). This can be helpful in constructing an eulerian trail. Alternatively one can use the ideas from the "if" part of the above proof: add the edge $\{a, b\}$ to G, then construct an eulerian circuit of the form $a - b - \cdots - a$ using the algorithm of Theorem 4.5.1, finally deleting $\{a, b\}$ from this eulerian circuit.

Example 4.5.2.

Can you draw the following figure without lifting pen from paper, without drawing any line twice, and finishing where you started? That is, is this graph eulerian?

♢

What if we remove the restriction that one must finish where one started? Now we are asking if the graph is semi-eulerian.

$\blacksquare$

$\diamondsuit$

Example 4.5.3.

Returning to the bridges of Königsberg problem, what is the analogous situation for the bridges of Cork?

■

♢

Read §4.5 except "An upper bound for the TSP." Exercises 4.2, 4.12, 4.13.

Exercises:

1. For which values of n does K_n have an eulerian circuit?

2. A road inspector must travel a network of roads to inspect them. She wishes to travel along each road exactly once, returning eventually to her starting point. Under which condition on the road network can this be done?

3. A postman must travel a connected network of roads, delivering post to houses on both sides of each road. He wishes to drive along each road exactly twice (once in each direction). Show that this can always be done.

4. A standard set of dominoes has one piece for each (unordered) pair of integers from 0 to 6 inclusive; that is, the 28 pieces are (0,0), (0,1), . . . , (0,6), (1,1), (1,2), . . . , (1,6), (2,2), (2,3), . . . , (2,6), (3,3), (3,4), . . . , (6,6). Here a piece such as (1,0) is not listed because it has already appeared as (0,1).

   Can you arrange the 28 dominoes in a closed pattern (like a large circle), so that each matches with its neighbour in the usual way? (This means that the adjacent halves of every two adjacent dominoes are the same.) Solve this problem by expressing it as a problem in graph theory then applying a theorem about eulerian circuits. Hint: consider the graph with nodes $0, 1, 2, \ldots, 6$ and dominoes as edges — for example the domino (3,5) is the edge incident to nodes 3 and 5.

If all dominoes with a 6 on them are removed from the set, can you arrange the remaining dominoes in a closed pattern, so that each matches with its neighbour?

Can you state a general theorem about arranging dominoes with numbers $0, 1, \ldots, n$ on them into a closed pattern?

4.6 Eulerian Digraphs

A *directed graph* or *digraph* is a graph $G = (V, E)$ where each edge has a direction. The direction is usually indicated by an arrow on the edge.

$\diamondsuit$

Let $x \in V$. The *indegree* of x, written $d^-(x)$, is the number of directed edges in E of the form (y, x), where y can be any vertex. The *outdegree* of x, $d^+(x)$, is the number of edges of the form (x, y). (The indegree of x is the number of arrows entering x, and the outdegree is the number of arrows leaving x.) In the above diagram,

In digraphs, a *directed eulerian circuit* is an eulerian circuit with the extra requirement that it must follow the directions of the arrows (like driving down one-way streets). Digraphs have the following analogue of Theorem 4.5.1, which is proved in much the same way:

Theorem 4.6.1. *Let $G = (V,E)$ be a digraph. Then G has a directed eulerian circuit if and only if G is connected and $d^-(x) = d^+(x)$ for all $x \in V$.*

Read §4.6. Exercises 4.10, 4.11.

Exercises

1. Let $A = \{2, 3, 4, 6, 9, 18, 36\}$ be the set of nodes of a directed graph D, and let (x, y) be a directed edge in D if and only if x divides y. Draw D. Characterize the in-degree of a node x in D, and characterize the out-degree of x.

2. A directed graph D has 6 nodes and 7 edges in which each node has out-degree equal to either 1 or 2. Determine the number of nodes in D of each out-degree, and draw such a directed graph.

3. State and prove a theorem for digraphs that is the analogue of Theorem 4.5.2.

Chapter 5

Partitions and Colouring

5.4 Vertex Colourings

We assign a colour to each node of a graph, subject to the single rule that adjacent nodes must have different colours. (Recall the 4-colour problem in §3.6.) If a graph has n nodes, then it certainly be coloured using n colours (assign a different colour to each node), but most graphs need fewer colours.

The *chromatic number* of a graph G, written $\chi(G)$, is the minimum number of colours needed to colour G.

Example 5.4.1.

(i)

$\diamondsuit$

(ii)

◊

The subgraph consisting of a, c, e, g and the edges joining them is K_4. This needs four colours as each vertex is adjacent to all of the other 3. Hence G needs at least 4 colours, i.e., $\chi(G) \geq 4$. In fact, once we have coloured the K_4 subgraph with 4 colours, it's easy to colour b, d, f, h without using another colour. As G has been coloured using 4 colours, $\chi(G) \leq 4$. We conclude that $\chi(G) = 4$. ■

Welsh-Powell colouring algorithm (different from Anderson's algorithm)

Let S be set of uncoloured nodes of the graph $G = (V, E)$ at each stage of the algorithm. Let i be the colour counter.

1. Set $S = V,\ i = 1$.
2. List nodes of S in order of decreasing degree (may not be unique).
3. Assign colour i to 1st node on list, then in sequence to each node on list not adjacent to one already coloured i.
4. Update S by removing from it those nodes coloured in Step 3.
5. If S is empty, stop; otherwise, add 1 to i and go to Step 2. ■

This algorithm may use more colours than necessary; if it uses say 5 colours, you can only conclude from this that $\chi(G) \leq 5$, *not* that $\chi(G) = 5$. In practice it generally uses a number of colours quite close to or equal to $\chi(G)$.

Example 5.4.2.

$\diamondsuit$

Step 1: $S = \{a, b, c, d, e, f, g\}$, $i = 1$.

Step 2: $S = \{c, d, b, e, g, a, f\}$.

Step 3: Colour nodes c, e with colour 1.

Step 4: $S = \{d, b, g, a, f\}$.

$\diamondsuit$

We have used 3 colours, so $\chi(G) \leq 3$. But G contains a subgraph K_3 (the nodes a, b, c and the edges joining them) so it needs at least 3 colours. We conclude that $\chi(G) = 3$. ■

When a vertex x of degree $d(x)$ is being assigned a colour by the algorithm, the worst scenario is when all the neighbours of x have already been assigned different colours. Thus we can be certain that we can colour x if we have $1 + d(x)$ colours available. This argument proves that the Welsh-Powell algorithm will use at most $\Delta + 1$ colours, where Δ is the maximum degree of any vertex in the graph. This result is analogous to Theorem 5.10 in Anderson.

Vertex colouring has many applications, e.g., to exam scheduling, to the storage of chemicals, and to minimizing the number of rooms needed to house a group of people in a hotel where certain pairs of individuals refuse to share a room. All of these are conflict problems. To model them by a graph, a good rule-of-thumb is to use the edges of the graph to represent the conflicts.

Read §5.4. Exercises 5.12–5.15.

Exercises

1. Find the chromatic number of each graph in Anderson Exercise 4.2.

2. A convention of mathematicians will have rooms available in six hotels. There are 108 mathematicians and because of personality conflicts various pairs of mathematicians must be put in different hotels. The organizers wonder whether six hotels will suffice to separate all conflicts. Model this conflict problem with a graph and restate the problem in terms of vertex colouring.

3. Let G be a graph whose vertices represent final exams in various courses. Two vertices are joined by an edge only if the corresponding exams cannot be scheduled simultaneously (e.g., 1st Science Mathematics and 1st Science Chemistry, because some students take both subjects). What interpretation can be given to a colouring of G? To the chromatic number of G?

4. (a) A set of solar experiments are to be made at observatories. Each experiment begins on a given day of the year and ends on a given day (each experiment is repeated for several years). An observatory can only perform one experiment at a time. What is the minimum number of observatories required to perform a given set of experiments annually? Model this problem as a graph colouring problem.

   (b) Suppose expt. A runs from Sept. 2 to Jan. 3, expt. B from Oct. 15 to March 10, C from Nov. 20 to Feb. 17, D from Jan. 23 to May 30, E from April 4 to July 28, F from April 30 to July 28, G from June 24 to Sept. 30. Find the minimum number of observatories needed.

5. The following 6 committees wish to schedule meetings. Find the least number of meeting times needed to avoid simultaneous meetings for committees with a common member.
   C1 = {O'Brien, Lee, Barry, Takata, James},
   C2 = {Lee, Madison, O'Regan, Healy}, C3 = {O'Regan, Robinson, Lucas, Wong},
   C4 = {Lucas, Albert, Lee, Davies}, C5 = {Twomey, Oulsnam, Bergman},
   C6 = {Oulsnam, Green, Barry}

6. Schedule the final exams for M115, M116, M185, M195, CS101, CS102, CS273, and CS473, using the

fewest number of time slots, if there are no students taking both M115 and CS473, both M116 and CS473, both M195 and CS101, both M195 and CS102, both M115 and M116, both M115 and M185, and both M185 and M195, but there are students in every other combination of courses.

7. How many different channels are needed for 6 t.v. stations numbered $1, 2, \ldots, 6$, if no two stations within 150 miles of each other can operate on the same channel, and the distances in miles between stations are as follows:

	1	2	3	4	5	6
1	–	85	175	200	50	100
2		–	125	175	100	160
3			–	100	200	250
4				–	210	220
5					–	100

8. A zoo wants to set up natural habitats in which to exhibit its animals. Unfortunately, some of the animals will eat some of the others if given the opportunity. How can a graph model and a colouring be used to determine the number of different habitats needed and the placement of the animals in these habitats?