1 Introduction

Let’s consider a laser which consist of an optical resonator filled with an active medium (excited atoms, electrons or molecules).

An active medium is necessary for light amplification, and we will consider a semiconductor material connected to a battery. More specifically, the external electrical current from the battery transfers or excites electrons from the lower energy band (called the valence band) into the highest energy band (called the conduction band) [1]. Electrons in the semiconductor conduction band will be called carriers. An incident photon can create its identical twins by forcing excited electrons to descend to the lower-energy valence band via stimulated emission. The external electrical current continuously excites the electrons from the valence band back into the conduction band so that they can produce more photons. This is how light is amplified. The type of the active medium or, more precisely, the spacing between the energy bands, determines the frequency band for amplification, that is the frequency range of incident photons that are likely to cause stimulated emission.

An optical resonator is used for two reasons. Firstly, it enhances the amplification process via forcing the light to make several passages through the active medium before it escapes to the outside world. Secondly, it imposes a particular resonant frequency or frequencies of light oscillation. The simplest example of an optical resonator is a set of two parallel semi-transparent mirrors (also called a Fabry-Perot resonator). Such an optical resonator imposes infinitely many resonant frequencies of oscillations. These resonant oscillations are called optical modes. They are clearly distinguishable spatial light profiles such as standing waves. Given some initial energy in each optical mode but no active medium, the amplitude of each mode will decay exponentially to zero because the mirrors are semi-transparent. However, in the presence of the active medium, those optical modes whose frequencies fall within the active-medium amplification-band will be amplified. If the amplification of an optical mode is stronger than its losses, the mode can exhibit sustained oscillations giving
rise to an intense and highly monochromatic laser beam. Typically, only one or a few optical modes receive enough gain to lase.

The total electric field inside the laser is a real-valued dynamical variable that depends on space and time. However, it is very convenient to separate the spatial and temporal dependence and write the electric field as a sum of products of the space-dependent optical modes $u_j(r)$ and time-dependent complex-valued mode amplitudes $B_j(t)$

$$E(r,t) = \frac{1}{2} \sum_j \left[ u_j(r) B_j(t) + \bar{u}_j(r) \bar{B}_j(t) \right]. \tag{1}$$

The summation index $j$ denotes possible modes of the passive or transparent optical resonator. A resonator is passive or transparent when the probabilities of stimulated emission and absorption for the light within the resonator are equal, meaning the light propagates without any losses or amplification. Furthermore, it is useful to rewrite $B_j(t)$ in terms of the slowly-varying complex-valued amplitude $A_j(t)$ and the fast-varying phase $\nu_j t$, where $\nu_j$ will be chosen later on

$$B_j(t) = A_j(t) e^{-i\nu_j t}, \tag{2}$$

Furthermore, we rewrite the complex-valued $A_j(t)$ in terms of the real-valued amplitude $|A_j(t)|$ and phase $\psi_j(t)$

$$A_j(t) = |A_j(t)| e^{-i\psi_j(t)}. \tag{3}$$

The optical frequencies $\nu_j \approx 10^{14} \text{ s}^{-1}$ are high, and the terms $e^{-i\nu_j t}$ are fast-varying. The terms $E(t)$, $\psi(t)$ vary at the rates $\sim 10^{10} \text{ s}^{-1}$, meaning they are slowly-varying compared to $e^{-i\nu_j t}$. The lasing frequency $\nu_{\text{las},j}$ of the $j$-th optical mode is calculated as the time derivative of the total phase of the $j$-th component of the electric field $E(t,r)$

$$\nu_{\text{las},j} = \nu_j + \frac{d\psi_j(t)}{dt}. \tag{4}$$

Furthermore, the mode intensity (units W/m$^2$) can be calculated as

$$I_j(t) = \frac{1}{2} c \epsilon_0 n_b^2 |E_j(t)|^2 \tag{5}$$

where $c$ is the speed of light, $\epsilon_0$ is the permittivity of free space, and $n_b$ is the refractive index of passive (transparent) semiconductor material.

### 2 Single-Mode Laser Rate Equations

In the framework of semiclassical laser theory one can derive the laser rate equations. The calculations start from classical Maxwell’s equations and quantum-mechanical Schrödinger equation, and are rather complicated. We will skip the intermediate steps and present the final result. Also, we will focus on the case when only a single optical mode can be amplified, we will speak of single-mode theory.
2.1 The laser field equation

The equation of motion describing the complex-valued amplitude of a single-mode electric field is

\[
\frac{dB}{dt} = \left[ -\frac{1}{2} \gamma_c + \Gamma \frac{c}{n_b} g(\bar{N}) - i \left( \Omega_{tr} - \Gamma \frac{\nu}{n_b} \delta n(\bar{N}) \right) \right] B, \quad (6)
\]

where \( \gamma_c \) is the optical resonator decay rate (units \( s^{-1} \)) which accounts for the mirror and intracavity losses, \( \Omega_{tr} \) is the resonant frequency of the passive or transparent resonator and \( \Gamma \) quantifies the spatial overlap of the optical mode with the semiconductor active medium and is called the confinement factor (dimensionless). The additional variable \( \bar{N} \), which is the average carrier density (units \( m^{-3} \)), appears in two functions on the right-hand side. The local gain of the active medium (units \( m^{-1} \))

\[
g(\bar{N}) = g_{th} + \xi (\bar{N} - \bar{N}_{th}), \quad (7)
\]

quantifies the amount of light produced \((g(\bar{N}) > 0)\) or absorbed \((g(\bar{N}) < 0)\) per unit length. The parameter \( \xi \) (units \( m^2 \)) is the gain coefficient. The local change in the refractive index of the active medium

\[
\delta n(\bar{N}) = -\frac{c}{\nu} \alpha \xi (\bar{N} - \bar{N}_{tr}) \quad (8)
\]
describes the change in the refractive index relative to the refractive index of a passive or transparent medium. The linewidth enhancement factor \( \alpha \) quantifies how strongly \( \delta n \) depends on \( \bar{N} \).

There are two important levels of the carrier density \( \bar{N} \) which appear in the formulas for \( g(\bar{N}) \) and \( \delta(\bar{N}) \). The threshold carrier density \( \bar{N}_{th} \) is the carrier density for which the gain and the loss terms in Eq. (6) are equal

\[
\Gamma \frac{c}{n_b} g(\bar{N}_{th}) = \Gamma \frac{c}{n_b} g_{th} = \frac{1}{2} \gamma_c \quad \Rightarrow \quad g_{th} \equiv g(\bar{N}_{th}) = \frac{\gamma_c n_b}{2c\Gamma}.
\]

The transparency carrier density \( \bar{N}_{tr} \) is the carrier density of the passive or transparent (zero gain) material

\[
g(\bar{N}_{tr}) = 0 \quad \Rightarrow \quad \bar{N}_{th} = \bar{N}_{tr} + \frac{n_b \gamma_c}{2c\xi \Gamma}.
\]

While \( \bar{N}_{tr} \) is an inherent property of the active medium material, \( \bar{N}_{th} \) depends on the optical resonator design/parameters.

The resonant frequencies of the laser resonator

\[
\Omega_j(\bar{N}) = j \frac{\pi c}{n_r(\bar{N}) L}, \quad (9)
\]

are given in terms of the resonator length \( L \) and the carrier-dependent refractive index

\[
n_r(\bar{N}) = n_b + \Gamma \delta n(\bar{N}) = n_b - \Gamma \frac{c}{\nu} \alpha \xi (\bar{N} - \bar{N}_{tr}), \quad (10)
\]
where $n_b$ is the refractive index of the passive or transparent active medium. In the single-mode theory we consider just one fixed $j$. To make a connection with Eq. (6) we note that

$$\Omega_tr \equiv \Omega(\bar{N}_{tr}) = j \frac{\pi c}{n_b L},$$

and expand $\Omega(\bar{N})$ about $\bar{N} = \bar{N}_{tr}$, which is equivalent to expanding about $\delta n = 0$:

$$\Omega(\bar{N}) = j \frac{\pi c}{n_r(N) L} = j \frac{\pi c}{L(n_b + \Gamma \delta n)} = j \frac{\pi c}{L} \left( \frac{1}{n_b} - \frac{\Gamma}{n^2_b} \delta n + \frac{2\Gamma^2}{n^3_b} (\delta n)^2 + \mathcal{O}((\delta n)^3) \right)$$

$$= j \frac{\pi c}{n_b L} + j \frac{\pi c}{n_b^2 L} \alpha \frac{\Gamma c \xi}{\nu} (\bar{N} - \bar{N}_{tr}) + \mathcal{O}((\delta n)^2)$$

$$= \Omega_{tr} + \frac{\Omega_{tr}}{\nu} \Gamma \frac{c \xi}{n_b} (\bar{N} - \bar{N}_{tr}) + \mathcal{O}((\delta n)^2).$$

Making realistic assumptions

$$\delta n << n_b \quad \text{and} \quad \Omega_{tr} \approx \nu,$$

we can approximate $\Omega(\bar{N})$ as a linear function of $\bar{N}$:

$$\Omega(\bar{N}) = \Omega_{tr} + \frac{\Omega_{tr}}{\nu} \Gamma \frac{c \xi}{n_b} (\bar{N} - \bar{N}_{tr}). \quad (11)$$

Substitute Eqs. (7) and (8) into Eq. (6), use the formula for $g_{th}$ given below Eq. (8), use Eq. (11) to define

$$\Omega_{th} \equiv \Omega(\bar{N}_{th}) = \Omega_{tr} + \Gamma \frac{c}{n_b} \alpha \xi (\bar{N}_{th} - \bar{N}_{tr}), \quad (12)$$

and arrive at

$$\frac{dB}{dt} = \left[ -i \Omega_{th} + \Gamma \frac{c}{n_b} \xi (\bar{N} - \bar{N}_{th})(1 - i\alpha) \right] B. \quad (13)$$

Next, use Eq. (2) to get the rate equation for the slowly-varying amplitude

$$\frac{dA}{dt} = \left[ -i(\Omega_{th} - \nu) + \Gamma \frac{c}{n_b} \xi (\bar{N} - \bar{N}_{th})(1 - i\alpha) \right] A. \quad (14)$$

### 2.2 The carrier equation

The equation for the carrier density can be derived using the density matrix operator and taking the expectation values. The evolution of the average carrier density can be written as

$$\frac{d\bar{N}}{dt} = J - \gamma_N \bar{N} - \frac{\epsilon_0 n_b e}{\hbar \nu} g(\bar{N}) |A|^2. \quad (15)$$
or
\[
\frac{d\bar{N}}{dt} = J - \gamma_N \bar{N} - \frac{\epsilon_0 n_b c}{\hbar \nu} \left[ g_{th} + \xi (\bar{N} - \bar{N}_{th}) \right] |A|^2.
\]

where \( J \) is the carrier pump rate due to electric current and \( \gamma_N \) is the carrier decay rate due to non-radiative recombinations of electrons. The threshold pump rate is defined as
\[
J_{th} = \gamma_N \bar{N}_{th}.
\]

## 2.3 Laser parameter values

Typical values for the laser parameters are \( \alpha = 0 - 6, n_b = 3.4, \gamma_c = 5 \times 10^{11} \text{ s}^{-1}, \gamma_N = 2 \times 10^9 \text{ s}^{-1}, \bar{N}_{tr} = 2 \times 10^{24} \text{ m}^{-3}, \xi = 10^{-19} \text{ m}^2, \Gamma = 0.1. \)

## 2.4 Nondimensionalisation and Symmetry

To facilitate the numerical computations and presentations of the results, we introduce the normalized variables and time
\[
E = \frac{A}{|A_0|}, \quad N = \frac{(\bar{N} - \bar{N}_{th})}{\bar{N}_{th}}, \quad \tau = \gamma_N t,
\]

where \( |A_0| = \sqrt{2 \hbar \nu \Gamma \gamma_N N_{th} / (\epsilon_0 n_b^2 \gamma_c)} \) is the amplitude of a free-running laser at twice the threshold (\( J = 2J_{th} \)). After nondimensionalisation \([3]\), Equations (14) and (16) can be rewritten in the dimensionless form
\[
\dot{E} = i \Delta E + \beta \gamma (1 - i \alpha) N E, \quad \dot{N} = \Lambda - N - (1 + \beta N) |E|^2,
\]

where the dot denotes a derivative with respect to \( \tau \), and normalized parameters are defined as
\[
\Delta = \frac{\nu - \Omega_{th}}{\gamma_N}, \quad \beta = 1 + \frac{2c \Gamma \xi}{n_b \gamma_c} \bar{N}_{tr} = \frac{2c \Gamma \xi}{n_b \gamma_c} \bar{N}_{th}, \quad \gamma = \frac{\gamma_c}{2 \gamma_N}, \quad \Lambda = \frac{J}{\gamma_N \bar{N}_{th}} - 1 = \frac{J}{J_{th}} - 1.
\]

Typical values for \( \Lambda \) are between 0.5 and 10.

Physically, Eqs. (17)–(18) describe a nonlinear, self-sustained oscillator. What is more, the oscillator is non-isochronous, meaning that its frequency depends of the amplitude. To see this, note that a disturbance in \( E \) will cause a change in \( N \) in accordance with Eq. (18). A change in \( N \) will change the lasing frequency in accordance with Eq. (22) below.

Mathematically, Eqs. (17)–(18) define a three-dimensional dynamical system with with \( S^1 \) symmetry. This means that the equations are invariant or the right-hand side (the vector field) is equivariant under the transformation \( E \rightarrow E e^{i a} \) for \( a \in [0, 2\pi) \); see \([4]\) and Sec.7.4.
in [6]. The $S^1$ symmetry allows reduction of Eqs. (17)–(18) to a two-dimensional dynamical system. Expressing the complex field as

$$E = |E|e^{-i\psi} = \sqrt{S}e^{-i\psi},$$

Eqs. (17)–(18) can be rewritten as

$$\dot{S} = 2\beta\gamma NS, \quad (19)$$
$$\dot{N} = \Lambda - N - (1 + \beta N)S. \quad (20)$$

plus the phase equation

$$\dot{\psi} = -\Delta + \alpha\beta\gamma N, \quad (21)$$

which decouples from the equations for $I$ and $N$ but is necessary for calculating the frequency of the laser defined in Eq. (4):

$$\nu_{\text{las}} = \Omega_{th} + \alpha\beta\gamma c N = \Omega_{th} + \alpha \frac{c}{n_b} \Gamma \xi (\bar{N} - \bar{N}_{th}). \quad (22)$$

**Problem I:** Calculate positions of equilibria of Eqs. (17)–(18) in the $(\Re[E], \Im[E], N)$ phase space as a function of $\Lambda$ for fixed $\alpha, \beta, \gamma$. Determine the stability type of equilibria and bifurcation points. Refer to [5, 6] for bifurcation and stability theory.

**Problem II:** Calculate positions of equilibria of Eqs. (19)–(20) in the $(S, N)$ phase space as a function of $\Lambda$ for fixed $\alpha, \beta, \gamma$. Determine the stability type of equilibria and bifurcation points. Refer to [5, 6] for bifurcation and stability theory.

### 3 Coupled Lasers

When two lasers are coupled either (i) side-to-side at a short distance or (ii) face-to-face with a common coupling mirror the coupled-laser system can be described by the following equations:

$$\dot{E}_1 = i\Delta_1 E_1 + \beta\gamma(1 - i\alpha)N_1 E_1 + iCE_2, \quad (23)$$
$$\dot{E}_2 = i\Delta_2 E_2 + \beta\gamma(1 - i\alpha)N_2 E_2 + iCE_1, \quad (24)$$
$$\dot{N}_1 = \Lambda_1 - N_1 - (1 + \beta N_1)|E_1|^2 \quad (25)$$
$$\dot{N}_2 = \Lambda_2 - N_2 - (1 + \beta N_2)|E_2|^2. \quad (26)$$

where $C$ is the real-valued coupling strength [10, 11, 12, 13]
References


