Abstract.

We discuss nonlinear tipping phenomena for non-autonomous systems using an example of a bistable ecosystem model with environmental changes represented by time-varying parameters [Scheffer et al. Ecosystems 11 2008]. We give simple testable criteria for tipping from the herbivore-dominating equilibrium to the plant-only equilibrium in terms of properties of the corresponding autonomous system with fixed-in-time parameters. Specifically, we use classical bifurcation analysis to identify a codimension-three degenerate Bogdanov-Takens bifurcation: the source of a dangerous subcritical Hopf bifurcation and the organising centre for bifurcation-induced tipping (B-tipping). Furthermore, we introduce basin instability analysis to identify parameter paths along which rate-induced tipping (R-tipping) is guaranteed to occur without crossing any bifurcation. We then produce tipping diagrams for the non-autonomous system in the plane of the magnitude and rate of a parameter shift to reveal tipping-tracking transitions due to canard-like solutions and non-trivial dynamics arising from the interaction between B-tipping and R-tipping. Analysis of non-monotone parameter shifts reveals an intriguing tipping diagram with R-tipping tongues and wiggling tipping-tracking transition curves. In the diagram, we identify “points of no return” where tipping cannot be prevented by the parameter trend reversal and “points of return tipping” where tipping is inadvertently induced by the parameter trend reversal. Our results give new insight into the sensitivity of ecosystems to the magnitudes and rates of environmental change. More generally, a comparison with modified saddle-node and subcritical Hopf normal forms reveals some universal tipping properties for a non-monotone passage through a basin instability boundary and a generic dangerous bifurcation.

Key words. Tipping points, tipping diagrams, ecosystem dynamics, B-tipping, dangerous bifurcation, Bogdanov-Takens bifurcation, R-tipping, non-adiabatic effects of parameter change, basin instability, parameter paths, canards, slow passage through subcritical Hopf bifurcation, points of return, points of no return, points of return tipping.

AMS subject classifications. 37N25, 37B55, 37GXX, 92D40

1. Introduction. Tipping points are strongly nonlinear phenomena which can be described in layman’s terms as large, sudden and often unexpected changes in the state of a system, caused by small and slow changes in the external inputs [1, 2]. The notion of a tipping point was popularised by Gladwell [3] and has since been used in a wide range of applications including climate science [4–6] and ecology [1, 7–12]. Scientists have identified interesting questions in relation to different tipping mechanisms [2, 13], generic early warning signals near a tipping point [14–17], and the possibility of preventing tipping [18–22], that need to be addressed in more rigorous terms. For example, Article 2 of the 1992 United Nations Framework Convention on Climate Change (UNFCCC) pointed out two critical factors: the level and the time frame for changing greenhouse gas concentrations [23], suggesting that there are at least two tipping mechanisms of great importance to the contemporary climate. More generally, tipping phenomena can be classified by a type of instability and analysed in more depth, although this often requires mathematical techniques beyond traditional stability theory [2, 24–26].

Early mathematical models described tipping points as dangerous bifurcations that occur at critical levels of an input parameter [27, 28]. Such bifurcations have a discontinuity in the branch of stable states (attractors) at the bifurcation point, which explains why a system can remain near one stable state up to a critical level,
but is destined to transition to a different state past the critical level [29]. However, tipping points are not just bifurcations. Some systems have critical rates of parameter change, meaning that they are very sensitive to how fast external conditions or inputs change. Such systems can tip to a different state, despite the absence of any classical bifurcation, when the input parameter varies slowly but fast enough [7,22,25,30,31]. Ashwin et al. used the framework of non-autonomous dynamical systems to identify three different tipping mechanisms [2]. Bifurcation-induced tipping (B-tipping) occurs when the changing parameter passes through a critical level or a (dangerous) bifurcation, at which point the stable state loses stability or simply disappears. In other words, B-tipping describes the adiabatic effects of a parameter change. Rate-induced tipping (R-tipping) occurs when the parameter changes faster than some critical rate and the system deviates from the moving stable state sufficiently far to cross some tipping threshold, e.g. the boundary of the domain of attraction. In other words, R-tipping describes the non-adiabatic effects of a parameter change. Noise-induced tipping (N-tipping) occurs when noisy fluctuations drive the system past some tipping threshold. Shi et al. gave an alternative but similar classification of tipping mechanisms based on relative timescales of the input and of the noisy system alone [13]. Additionally, tipping points can be described as either reversible or irreversible, depending on whether or not the system returns to the original stable state in the long term [26]. So far, B-tipping and R-tipping have been discussed in isolation in the literature. However, real-world tipping phenomena will often involve different critical factors and different tipping mechanisms. Motivated by this observation, we use classical bifurcation analysis [32] in conjunction with the concepts of parameter paths and basin instability [2,26] to analyse the effects of the rate of parameter change near the two generic dangerous bifurcations of equilibria: saddle-node and subcritical Hopf bifurcations [29]. In this way, we give new insight into testable criteria for R-tipping and reveal non-trivial phenomena such as multiple critical rates that arise from the interaction between B-tipping and R-tipping.

Ecological models appear to be a perfect test bed for this type of study. B-tipping has been observed and studied extensively in different ecosystems [4,33–37], although the concept of a “global tipping point” in the context of planetary boundaries has recently received some criticism [38]. Ecologists speak of a “regime shift” when the bifurcation is safe or explosive, and of a “critical transition” when the bifurcation is dangerous [1]; we refer to [29] for the classification of bifurcations into safe, explosive and dangerous. Similarly, there is great and rapidly growing interest in R-tipping in the context of ecological dynamics [37,39]. To the best of our knowledge, the first examples of R-tipping were reported in ecosystems [7,10–12,30]. More precisely, R-tipping conceptualises a failure to adapt to changing environments [40] in the sense that the stable state is continuously available but the system is unable to adjust to its changing position when the change happens too fast. This raises the interesting research question of whether tipping phenomena observed in nature are predominantly rate induced. What is more, the related question of whether tipping can be avoided or prevented has recently received much attention in the ecosystem literature [18–21]. Proper mathematical analysis of the interaction between critical levels and critical rates, or between B-tipping and R-tipping, is exactly what is needed to gain more insight into these questions. Lastly, there is a strong need to better understand whether ecosystems are sensitive to the magnitudes of environmental change, the rates of environmental change, or to both. This is of particular importance in view.

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1 In a certain sense, N-tipping can be thought of as a special case of R-tipping.
of a highly variable contemporary climate, intensifying human activity and rapidly
decreasing resources.

The paper is organised as follows. Section 2 introduces the ecological model given
by two non-autonomous ordinary differential equations and discusses the key nonlin-
erarity due to a modified type-III functional response. It also introduces the concepts
of a parameter path and a moving equilibrium. In Sec. 3 we perform classical bifur-
cation analysis of the corresponding autonomous system with fixed in time param-
eters, obtain two-dimensional bifurcation diagrams in the parameter plane of the plant
growth rate and herbivore mortality rate, and uncover a codimension-three degenerate
Bogdanov-Takens bifurcation as the organising centre for B-tipping and the source
of a dangerous subcritical Hopf bifurcation. We give simple criteria for B-tipping in
the non-autonomous system in terms of dangerous bifurcations in the autonomous
system. In Sec. 4 we introduce the concept of basin instability for the correspond-
ing autonomous system to give testable criteria for R-tipping in the non-autonomous
system. We superimpose regions of basin instability on classical bifurcation diagrams
to highlight rate-induced instabilities that cannot be captured by classical bifurca-
tion analysis. We then obtain two-dimensional R-tipping diagrams in the parameter
plane of the rate and magnitude of parameter shift for monotone and non-monotone
parameter shifts, uncover R-tipping tongues with two critical rates, draw parallels
between R-tipping tongues and resonance tongues, and demonstrate that tracking-
tipping transitions correspond to canard-like solutions that, rather surprisingly, track
moving unstable states. In Sec. 5 we describe non-trivial tipping phenomena arising
from the interaction between B-tipping and R-tipping such as tipping diagrams with
multiple critical rates, which we explain in terms of different timescales and bifur-
cation delays. In Sec. 6 we partition the tipping diagrams into “points of tracking”,”
“points of return”, “points of no return” and “points of return tipping” to give new in-
sight into the problem of preventing tipping by a parameter trend reversal. We then
depart from the ecological model and produce tipping diagrams capturing both B-
tipping and R-tipping for modified (tilted) normal forms of the two generic dangerous
bifurcations of equilibria namely saddle-node and subcritical Hopf. By comparison
with the ecological model we show that the tipping diagram from Sec. 5 appears to
be typical for non-monotone parameter shifts that cross a basin instability boundary
and a generic dangerous bifurcation and then turn around. Section 7 summarises our
findings.

2. The Ecosystem Model and its Key Nonlinearity. We consider a simple
ecosystem model, where the time evolution of plant $P \geq 0$ and herbivore $H \geq 0$
biomass concentrations is modelled using two coupled autonomous ordinary differen-
tial equations [7]:

\[
\frac{dP}{dt} = rP - CP^2 - H g(P),
\]

\[
\frac{dH}{dt} = (E e^{-bP} g(P) - m) H,
\]

(2.1)

(2.2)

together with eight parameters listed in Table 1. The first two terms on the right-
hand side (r.h.s.) of Eq. (2.1) describe logistic plant growth from 0 to the carrying
capacity $r/C$. The third term describes grazing with a nonlinear dependence on the
plant biomass $P$. Specifically, the functional response in units inverse day

\[
g(P) = c_{max} \frac{P^2}{P^2 + a^2} e^{-bP},
\]

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Table 1
Description of the system parameters and their values [7].

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
<th>Default value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C &gt; 0$</td>
<td>Competition factor of plants</td>
<td>m$^2$g$^{-1}$d$^{-1}$</td>
<td>0.02</td>
</tr>
<tr>
<td>$a &gt; 0$</td>
<td>Half-saturation constant of functional response</td>
<td>g m$^{-2}$</td>
<td>10</td>
</tr>
<tr>
<td>$b \geq 0$</td>
<td>Exponent determining the reduced quality of food if food biomass is too high</td>
<td>m$^2$g$^{-1}$</td>
<td>0 - 0.04</td>
</tr>
<tr>
<td>$b_c \geq 0$</td>
<td>Exponent determining the predation efficiency of herbivores at high food biomass</td>
<td>m$^2$g$^{-1}$</td>
<td>0 - 0.04</td>
</tr>
<tr>
<td>$E &gt; 0$</td>
<td>Assimilation efficiency of herbivores</td>
<td>dimensionless</td>
<td>0.4</td>
</tr>
<tr>
<td>$c_{max} &gt; 0$</td>
<td>Maximum food intake of herbivores when $b_c = 0$</td>
<td>d$^{-1}$</td>
<td>1</td>
</tr>
<tr>
<td>$m &gt; 0$</td>
<td>Herbivore mortality rate</td>
<td>d$^{-1}$</td>
<td>0 - 0.2</td>
</tr>
<tr>
<td>$r &gt; 0$</td>
<td>Maximum plant growth rate</td>
<td>d$^{-1}$</td>
<td>0 - 2.5</td>
</tr>
</tbody>
</table>

is a modification of the classical monotone and strictly-increasing type-III functional response $c_{max}P^2/(P^2 + a^2)$ [41] with an exponential factor $e^{-b_c P}$ to account for a decline in foraging at high plant biomass. The resulting non-monotone $g(t)$, shown in Fig. 1(a) for different predation efficiency $b_c$, is believed to describe a wide range of terrestrial and aquatic ecosystems; see [7,42] and references therein. For example, rabbits graze more with faster-growing plants as long as the plants are small enough, but avoid overgrown bushes in fear of predators and are unable to graze on plants that have grown too tall. Similarly, in aquatic ecosystems, phytoplankton can be heavily consumed at early life stages by herbivorous zooplankton, but higher-density phytoplankton colonies become less prone to exploration and foraging. Moving on to the herbivore dynamics, the first term on the r.h.s. of Eq.(2.2) describes an increase in herbivore biomass. The increase term consists of three factors: reproduction and grazing $g(P)$, herbivore assimilation efficiency $E$, and exponential decline $e^{-bP}$ due to reduced food quality at high plant biomass. The last term on the r.h.s. of Eq. (2.2) represents herbivore death at the constant rate $m$.

In mathematical terms, the ecosystem model (2.1)–(2.2) with the modified functional response (2.3) is a singular perturbation problem because it has a different number of equilibrium solutions for $b + b_c = 0$ and $0 < b + b_c \ll 1$. To see that, consider the net per-capita herbivore growth shown in in Fig. 1(b):

$$h(P) = \frac{dH/dt}{H} = E \frac{c_{max} P^2}{P^2 + a^2} e^{-(b+b_c)P} - m,$$

whose roots correspond to non-zero herbivore equilibrium concentrations. When $b + b_c = 0$, the net per-capita herbivore growth is a strictly-increasing function of $P$ with a single root $P_3$ [Fig.1(b)]. However, when $0 < b + b_c \ll 1$, the net per-capita herbivore growth has a maximum at the optimal plant biomass

$$P_{opt} \approx \left( \frac{2a^2}{b + b_c} \right)^\frac{1}{2},$$

and can have no roots at all, one double root, or two distinct roots at $P_3 < P_{opt}$ and $P_4 > P_{opt}$ [Fig.1(b)]; see the Appendix for the derivation of $P_{opt}$, $P_3$ and $P_4$. The
additional nonlinearity of \( h(p) \) that arises from a decline in foraging at high plant biomass (\( b_c > 0 \)), from reduced food quality at high plant biomass (\( b > 0 \)), or from a combination of both [Fig. 1(b)], is key to our study. Throughout the paper, we refer to \( b + b_c \) as the nonlinearity parameter, and work with different but fixed in time values of \( b \) and \( b_c \), as indicated in Table 1.


Ecosystems are open systems that are inevitably subject to changing environmental conditions. These include climatic changes and weather anomalies, disease outbreaks, decline in resources or habitat quality, and human activity. In the model, environmental changes can be described by a time-dependent plant growth rate \( r(t) \) and herbivore mortality rate \( m(t) \), which are the input parameters for this study. Specifically, we fix six of the system parameters to the values or ranges given in Table 1, and allow \( r(t) \) or \( m(t) \) to vary smoothly in time from one asymptotic value to another. For example, \( r(t) \) could describe the occurrence of a wet season, owing to a weather anomaly or El Niño Southern Oscillations (ENSO), while \( m(t) \) could describe a disease outbreak among herbivores. This gives the non-autonomous ecosystem model

\[
\frac{dP}{dt} = r(t)P - CP^2 - H g(P),
\]

\[
\frac{dH}{dt} = (E e^{-bP} g(P) - m(t))H,
\]

where the exact time-dependence of \( r(t) \) and \( m(t) \) is specified in Secs. 4 and 6 ahead.

Our analysis of tipping points in the non-autonomous system (2.5)–(2.6) with time-varying environmental conditions is motivated by the need to better understand whether ecosystems are sensitive to the magnitude of environmental change, the rate of environmental change, or to both.

### 2.1.1. Moving Equilibria and Parameter Paths.

An equilibrium or a steady state for the autonomous system (2.1)–(2.2) is a pair

\[ e(r, m) = (P, H), \]

for which \( dP/dt = dH/dt = 0 \). Typically, the position of an equilibrium depends on the input parameters \( r \) and/or \( m \). When the input parameters vary over time, \( e(r, m) \)

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**Figure 1.** (a) The functional response \( g(P) \) with dependence on \( b_c \). (b) The key system nonlinearity: For \( b + b_c > 0 \), the net per-capita herbivore growth \( h(P) = (dH/dt)/H \) has optimal plant biomass \( P_{opt} \) where the growth is maximal, and may change sign twice at \( P_3 \) and \( P_4 \); \( m = 0.1 \).

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changes its position in the \((P, H)\) phase space and we speak of a \textit{moving equilibrium}

\[ e(t) = e(r(t), m(t)), \]

also known as a quasistatic equilibrium [2]. Note that \(e(t)\) is a property of the autonomous system \(\text{(2.1)-(2.2)}\) and the changing environment, but it is not a solution to the non-autonomous system \(\text{(2.5)-(2.6)}.\)

As the input parameters \(r(t)\) and \(m(t)\) evolve smoothly over time, they trace out a continuous \textit{parameter path} in the two-dimensional \((r, m)\) parameter plane. We use the notions of a moving equilibrium and a parameter path to discuss the differences and interaction between B-tipping and R-tipping.

3. B-tipping: Classical Bifurcations. The non-autonomous system \(\text{(2.5)-(2.6)}\) undergoes \textit{B-tipping} when the input parameters pass through a dangerous bifurcation of the corresponding autonomous system \(\text{(2.1)-(2.2)}.\) "Dangerous" means that, in a one-parameter bifurcation diagram like the ones shown in Fig. 4, there is a discontinuity in the attracting set at a bifurcation point [27]. In this section we treat the input parameters \(r\) and \(m\) as fixed in time \textit{bifurcation parameters}, compute bifurcation curves in the \((r, m)\) parameter plane, and uncover different B-tipping mechanisms including the two generic dangerous bifurcations of equilibria namely saddle-node and subcritical Hopf bifurcations. Then, given a parameter path of environmental change, we identify \textit{critical levels} of \(r\) and \(m\) along the path whenever the path crosses a dangerous bifurcation.

3.1. Existence of Equilibrium Solutions. Equilibrium solutions for the autonomous system \(\text{(2.1)-(2.2)}\) are pairs \(e = (P, H)\) of non-negative \(P\) and \(H\), which satisfy the following conditions:

\begin{align*}
\text{(3.1)} & \quad rP - CP^2 - H g(P) = 0, \\
\text{(3.2)} & \quad (E e^{-bP} g(P) - m)H = 0.
\end{align*}

When \(H = 0\), there are two equilibria: a \textit{trivial} equilibrium \(e_1\), and a \textit{plant-only} equilibrium \(e_2\):

\[ e_1 = (0, 0), \quad e_2 = (r/C, 0). \]

When \(H \neq 0\), the equilibrium conditions \(\text{(3.1)-(3.2)}\) become

\begin{align*}
\text{(3.3)} & \quad H = \frac{(r - CP)(P^2 + a^2)}{c_{\text{max}} P e^{-b_c P}}, \\
\text{(3.4)} & \quad h(P) = E c_{\text{max}} \frac{P^2 e^{-(b+b_c)P}}{P^2 + a^2} - m = 0.
\end{align*}

Note that condition \(\text{(3.4)}\), which gives the \(P\)-component of equilibrium solutions, depends on the nonlinearity parameter \(b + b_c\) rather than on \(b\) and \(b_c\) individually, which simplifies the discussion. What is more, the condition and thus the \(P\)-components of the ensuing equilibrium solutions are \(r\)-independent. Independently of \(b + b_c\), the net per-capita herbivore growth \(h(P)\) equals \(-m\) for \(P = 0\). When \(b + b_c = 0\), the herbivore growth \(h(P)\) is strictly increasing and levels off at \(Ec_{\text{max}} - m\) for large \(P\) [Fig. 1(b)]. Thus, if \(Ec_{\text{max}} - m > 0\), Eq. \(\text{(3.4)}\) has one positive root, giving three equilibrium solutions for the system. Most importantly, the maximum number of equilibrium solutions may increase in the presence of the key nonlinearity. When \(b + b_c > 0\), the herbivore growth \(h(P)\) has a global maximum at \(P = P_{\text{opt}} > 0\), and
Figure 2. \(P\) and \(H\) components of the herbivore-dominating equilibrium \(e_3\) and the plant-dominating equilibrium \(e_4\) obtained from (solid curve) numerically solving Eqs. (3.3)–(3.4), and from (dashed curves) first-order asymptotic approximations. Panels (a) and (b) show the dependence on \(m\) for fixed \(r = 1\), and panels (c) and (d) show the dependence on \(r\) for fixed \(m = 0.1\); \(e_1\) and \(e_2\) are included for reference. \(b = b_c = 0.02\), see Table 1 for other parameter values.

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Singular perturbation about \( b + b_c = 0 \) using a stretched variable \( \hat{P} = (b + b_c)P \) gives the \( P \)-component of the plant-dominating equilibrium \( e_4 \):

\[
P_4 = \frac{\ln(E c_{\text{max}}/m)}{b + b_c} - \frac{a^2(b + b_c)}{(\ln(E c_{\text{max}}/m))^2} + O((b + b_c)^2),
\]

and

\[
e_4 = \left( \frac{\ln(E c_{\text{max}}/m)}{b + b_c} + O(b + b_c), \frac{(r - CP_4)(P_4^2 + a^2)}{c_{\text{max}} P_4 e^{-b_c P_4}} \right).
\]

The solid curves in Fig. 2 show the numerically computed components of \( e_3 \) and \( e_4 \), and the dashed curves show the first-order approximations using the \( P \)-formulas \( (3.5) \) and \( (3.7) \) with \( O((b + b_c)^2) = 0 \), and the \( H \)-formula \( (3.3) \). The main advantage of the closed-form approximations is the information about the dependence of the equilibrium positions on the system parameters. In particular, the effect of the nonlinearity parameter \( b + b_c \) can now be discussed in qualitative terms. First of all, R-tipping from the herbivore-dominating equilibrium \( e_3 \) to the plant-only equilibrium \( e_2 \) requires that both equilibria are stable for the same parameter settings (bistability between \( e_2 \) and \( e_3 \)). In the absence of unstable limit cycles, bistability requires one additional equilibrium \( e_4 \) that provides a separatrix between the two stable equilibria \( e_2 \) and \( e_3 \).

It is clear from Eq. \( (3.7) \) that non-monotone herbivore growth due to \( b + b_c > 0 \) is necessary for the additional equilibrium \( e_4 \) to exist.

Figure 2 shows that \( e_3 \) and \( e_4 \) may become degenerate with each other, or each of them may become degenerate with \( e_2 \). These degeneracies are indicative of transcritical and saddle-node bifurcations.

**Degeneracy of \( e_2 \) with \( e_3 \) or \( e_4 \) via transcritical bifurcation.** Equilibrium \( e_3 \) or \( e_4 \) becomes degenerate with \( e_2 \) in a transverse crossing if \( P = r/C \) in Eqs. \( (3.3) \)–\( (3.4) \). Thus, substituting \( P = r/C \) into Eq. \( (3.4) \) defines a curve \( T \) of transcritical bifurcations in the \( (r, m) \) parameter plane

\[
T = \left\{ (r, m) : m = \frac{E c_{\text{max}} e^{-b(b_c) r/C}}{(a C/r)^2 + 1} \right\}.
\]

If \( b + b_c = 0 \), curve \( T \) emerges from the origin and levels off at \( m = E c_{\text{max}} \) for large \( r \) [Fig.3(a)]. Equilibrium \( e_3 \) that bifurcates from \( e_2 \) exists below \( T \). If \( b + b_c > 0 \), curve \( T \) emerges from the origin, has a maximum denoted with \( ST \), and approaches \( m = 0 \) from above for large \( r \) [Fig.3(b)]. Now, \( T \) consists of two different branches separated by \( ST \). Equilibrium \( e_3 \), that bifurcates from \( e_2 \) along the left-hand branch of \( T \), exists below the left-hand branch of \( T \). In contrast, equilibrium \( e_4 \), that bifurcates from \( e_2 \) along the right-hand branch of \( T \), exists above the right-hand branch of \( T \).

**Degeneracy of \( e_3 \) with \( e_4 \) in a saddle-node bifurcation.** Equilibria \( e_3 \) and \( e_4 \) become degenerate in a quadratic tangency when \( r \)-independent Eq. \( (3.4) \) has a non-negative double root, meaning that

\[
h(P) = \frac{dh}{dP} = 0 \quad \text{and} \quad \frac{d^2 h}{dP^2} \neq 0 \quad \text{for} \quad P \geq 0,
\]

and the corresponding \( H \) from Eq. \( (3.3) \) is non-negative, meaning that

\[
r \geq CP.
\]
Conditions (3.10) give the cubic equation for $P$:

\begin{equation}
q(P) = (b + b_c)P^3 + a^2(b + b_c)P - 2a^2 = 0.
\end{equation}

A positive root of $q(P)$ is used in Eq. (3.4) to determine the value of $m$ at which $e_3$ and $e_4$ become degenerate

\begin{equation}
m = \frac{E c_{\text{max}} e^{-(b+b_c)P}}{(a/P)^2 + 1},
\end{equation}

and in Eq. (3.11) to determine the corresponding range of $r$ where $e_3$ and $e_4$ become degenerate. Thus, conditions (3.11)–(3.13) define a half-line of saddle-node bifurcations of equilibria in the $(r, m)$ parameter plane

\begin{equation}
S_c = \left\{ (r, m) : q(P) = 0, r \geq CP \text{ and } m = \frac{E c_{\text{max}} e^{-(b+b_c)P}}{(a/P)^2 + 1} \right\}.
\end{equation}

If $b + b_c = 0$, then $q(P)$ has no roots, meaning that there is no saddle-node bifurcation of equilibria [Fig.3(a)]. If $b + b_c > 0$, $q(P)$ is negative for $P = 0$, monotonically increasing, and positive for $P$ large enough, meaning that Eq. (3.12) has a unique positive root. This root corresponds to a unique saddle-node half-line $S_c$. Equilibria $e_3$ and $e_4$ exist below $S_c$, and become degenerate and disappear along $S_c$. What is more, conditions (3.9) and (3.14) become identical when $r = CP$, meaning that the end of the half-line $S_c$ lies on $T$. Indeed, Fig.3(b) shows that $S_c$ emerges from the special saddle-node-transcritical bifurcation point $ST$, where the three equilibria $e_2$, $e_3$ and $e_4$ become degenerate.

### 3.2. Stability and Bifurcation Analysis.

Linear stability of equilibria can be determined from the eigenvalues of the Jacobian matrix

\begin{equation}
J = \begin{pmatrix}
r - 2CP - Hg'(P) & -g(P) \\
E e^{-bP} (g'(P) - b g(P)) H & Eg(P) e^{-bP} - m
\end{pmatrix}
\end{equation}

where

\[ g'(P) = -g(P) \left( \frac{2P}{P^2 + a^2} - \frac{2}{P} + b_c \right). \]
At the trivial equilibrium \( e_1 = (0,0) \), the Jacobian matrix has eigenvalues \( \lambda_1 = r > 0 \) and \( \lambda_2 = -m < 0 \), meaning that \( e_1 \) is always a saddle. At the plant-only equilibrium \( e_2 = (r/C,0) \), the Jacobian matrix has eigenvalues \( \lambda_1 = -r < 0 \) and \( \lambda_2 = Ce_{\text{max}} e^{-(b+bc)r/C}/((aC/r)^2 + 1) - m \). Hence, \( e_2 \) is a stable node when \( \lambda_2 < 0 \), a saddle when \( \lambda_2 > 0 \), and undergoes a transcritical bifurcation whenever \( \lambda_2 = 0 \); one can verify that \( \lambda_2 = 0 \) gives the transcritical bifurcation condition (3.9). Stability and bifurcations of the herbivore-dominating equilibrium \( e_3 \) and the plant-dominating equilibrium \( e_4 \) are obtained numerically. The Jacobian matrix shows that the stability of \( e_3 \) and \( e_4 \) depends on the nonlinearity parameter \( b + bc \) as well as on \( bc \) alone, meaning that it needs to be discussed with dependence on \( b \) and \( bc \) individually.

To showcase different types of dynamics and bifurcations in the autonomous ecosystem (2.1)–(2.2), we plot six examples of one-dimensional bifurcation diagrams in Fig. 4 for two types of parameter paths. In the left column we fix \( r \) and consider a range of \( m \in (0,0.2] \). In the right column we fix \( m \) and consider a range of \( r \in (0,2] \). In addition to the transcritical \( T \) and saddle-node \( S_e \) bifurcations of equilibria identified in the previous section, there are sub- and supercritical Hopf bifurcations \( H_e \). In a supercritical Hopf bifurcation, a stable equilibrium turns unstable and gives rise to a stable limit cycle [Fig. 4(e)–(f)]. Thus, this bifurcation is safe. In a subcritical Hopf bifurcation, an unstable limit cycle shrinks onto a stable equilibrium and the equilibrium becomes unstable [Fig. 4(b)–(c)]. The subcritical Hopf bifurcation is classified as dangerous because it gives rise to a discontinuity in the attracting set that is the branch of stable equilibria. What is more, a limit cycle can connect to the saddle equilibrium \( e_4 \) and disappear in a homoclinic bifurcation \( h \) [Fig. 4(b)–(c) or (e)–(f)].

For more details and background on bifurcation theory, we refer to [32].

To provide a systematic bifurcation analysis, we obtain two-dimensional \((r,m)\) bifurcation diagrams for different but fixed values of \( b \) and \( bc \) [Fig. 5]. We plot codimension-one supercritical bifurcations as solid curves and subcritical bifurcations as dashed curves. Along a solid (dashed) transcritical bifurcation, the bifurcating branch of equilibria is stable (of saddle type); along a solid (dashed) saddle-node bifurcation, a saddle collides with an attractor (repeller); and along solid (dashed) Hopf and homoclinic bifurcations, the bifurcating limit cycle is attracting (repelling). Transcritical and saddle-node bifurcations of equilibria are obtained using conditions (3.9) and (3.14), respectively. Hopf, homoclinic and saddle-node bifurcations of limit cycles are computed using the numerical continuation software AUTO [43]. For each bifurca-
Figure 4. One-parameter bifurcation diagrams showing the position and stability of equilibria and limit cycles. The left column shows the \((P, H, m)\)-space for (a) \(r = 0.5\), \((b, b_c) = (0.025, 0.025)\) (c) \(r = 1\), \((b, b_c) = (0.02, 0.02)\) and (e) \(r = 1.5\), \((b, b_c) = (0.001, 0.005)\). The right column shows the \((P, H, r)\)-space for (b) \(m = 0.115\), \((b, b_c) = (0.025, 0.025)\) (d) \(m = 0.1\), \((b, b_c) = (0.02, 0.02)\) and (f) \(m = 0.25\), \((b, b_c) = (0.001, 0.005)\). Solid branches indicate stable solutions, dashed branches indicate unstable solutions. Projections onto the \((m, P)\) and \((r, P)\) planes are shown in grey. For the labelling of different bifurcations see Table 2.

In the absence of the key nonlinearity, that is when \(b + b_c = 0\) and \(g(P)\) is the classical type-III functional response, there are just two bifurcation curves: curve \(T\) of supercritical transcritical bifurcations, and curve \(H_c\) of supercritical Hopf bifurcations.
Figure 5. Examples of four qualitatively different (r,m)-bifurcation diagrams obtained for different but fixed (b,bc) = (a) (0, 0), (b) (0.001, 0.005), (c) (0.005, 0.01), (d) (0.025, 0.025). Supercritical (subcritical) bifurcations are plotted as solid (dashed) curves. For the labelling of different bifurcations see Table 2.

[Fig. 5(a)]. These two curves do not interact, and partition the (r,m) parameter plane into three distinct regions with qualitatively different dynamics [Fig. 6, 1–3]. In particular, He gives rise to a stable limit cycle in region 3, which represents stable but oscillatory coexistence between plants and herbivores. These simple dynamics change drastically in the presence of the key nonlinearity.

When b + bc becomes small but non-zero, a number of qualitative changes take place in the bifurcation diagram as expected from the singular perturbation nature of the problem. Specifically, there are three additional co-dimension one bifurcation curves, and four special codimension-two bifurcation points [Fig. 5(b)]. Understanding the new bifurcation diagram is reminiscent of assembling a jigsaw-puzzle. Firstly, a half-line Se of saddle-node bifurcations of equilibria appears. Se emerges from the saddle-node-transcritical bifurcation point ST on T, where T changes from super- to subcritical. Secondly, He is no longer unbounded at both ends, but emerges from the Bogdanov-Takens bifurcation point BT on Se, where Se changes from super- to subcritical. There are two possible types of Bogdanov-Takens bifurcation, and BT is type-I according to the classification in [32, Sec.8.4]. It is known from the unfolding of a Bogdanov-Takens bifurcation that an additional bifurcation curve, namely the curve of homoclinic bifurcations h, must emerge from BT. Along h, the limit cycle originating from He becomes a connecting orbit to the saddle equilibrium e3 and...
Figure 6. Examples of qualitatively different \((P,H)\)-phase portraits for the autonomous system (2.1)-(2.2) showing (filled circles) stable equilibria, (open circles) unstable equilibria, (thick curves) limit cycles and stable/unstable invariant manifolds of saddle equilibria, and (thin curves) examples of typical trajectories. Note the stable limit cycle in regions 3 and 6, the unstable limit cycle in region 7, and two limit cycles in region 8. See Table 3 for parameter values.

Figures 5(c), 6. Examples of qualitatively different \((P,H)\)-phase portraits for the autonomous system (2.1)-(2.2) showing (filled circles) stable equilibria, (open circles) unstable equilibria, (thick curves) limit cycles and stable/unstable invariant manifolds of saddle equilibria, and (thin curves) examples of typical trajectories. Note the stable limit cycle in regions 3 and 6, the unstable limit cycle in region 7, and two limit cycles in region 8. See Table 3 for parameter values.

355 disappears [Fig. 6 h]. Thirdly, there is a generalised Hopf (or Bautin) bifurcation point 356 \(GH\) on \(H_e\), where \(H_e\) changes from super- to subcritical [32, Sec.8.3]. It is known from 357 the unfolding of a generalised Hopf bifurcation that an additional bifurcation curve, 358 namely the curve of saddle-node of limit cycles \(S_{lc}\), must emerge from \(GH\). Along \(S_{lc}\), 359 two limit cycles of which one is stable and the other repelling collide and disappear. 360 Finally, \(S_{lc}\) terminates on \(h\) at a resonant homoclinic bifurcation point \(h_{res}\), where \(h\) 361 changes from super- to subcritical. This new bifurcation structure has five additional 362 regions 4–8 with qualitatively different dynamics; for regions 7–8 see the inset in 363 Fig. 5(c). We would like to point out the appearance of adjacent regions 5 and 7 364 with bistability between the plant-only equilibrium \(e_2\) and the herbivore-dominating 365 equilibrium \(e_3\).

When the combination of \(b\) and \(b_c\) is increased further, bifurcation points \(GH\)

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Table 3
Parameter values chosen for phase portraits in Fig. 6

<table>
<thead>
<tr>
<th>Phase Portrait</th>
<th>r</th>
<th>m</th>
<th>b</th>
<th>b_c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.14</td>
<td>0.025</td>
<td>0.025</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.05</td>
<td>0.025</td>
<td>0.025</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>0.23</td>
<td>0.001</td>
<td>0.005</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.125</td>
<td>0.025</td>
<td>0.025</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.075</td>
<td>0.025</td>
<td>0.025</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0.21</td>
<td>0.005</td>
<td>0.01</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.12</td>
<td>0.025</td>
<td>0.025</td>
</tr>
<tr>
<td>8</td>
<td>1.5</td>
<td>0.18025</td>
<td>0.005</td>
<td>0.01</td>
</tr>
<tr>
<td>h</td>
<td>1.5</td>
<td>0.2684</td>
<td>0.001</td>
<td>0.005</td>
</tr>
</tbody>
</table>

...and $h_{res}$ approach $BT_I$ [Fig. 5(c)]. In the process, region 3 with stable self-sustained oscillations disappears, while bistable region 5 becomes noticeably larger. Then, there are special combinations of $b$ and $b_c$, where $GH$ and $h_{res}$ collide simultaneously with $BT_I$ and disappear in a codimension-three degenerate Bogdanov-Takens bifurcation (not shown in the figure). This collision eliminates $S_{el}$ together with the supercritical part of $H_e$ and with regions 6 and 8. What is more, the Bogdanov-Takens bifurcation point changes to type II. The difference from $BT_I$ is that $H_e$ and $h$ emerging from $BT_{II}$ swap their relative positions and become subcritical [Fig. 5(d)].

Past the special combination of $b$ and $b_c$, there are four bifurcation curves, including the two dangerous bifurcations of equilibria that are of interest for B-tipping: the (solid) half-line $S_e$ of supercritical saddle-node bifurcations, and the (dashed) curve $H_e$ of subcritical Hopf bifurcations. Additionally, there are two special bifurcation points, one of which is the type-II Bogdanov-Takens bifurcation point $BT_{II}$. $H_e$ gives rise to a repelling limit cycle in region 7, which becomes a connecting orbit to the saddle equilibrium $e_4$ and disappears in a homoclinic bifurcation along $h$. Finally, a substantial part of the diagram is occupied by adjacent bistable regions 5 and 7. In these two regions, the plant-only equilibrium $e_2$ and the herbivore-dominating equilibrium $e_3$ are both stable, which is of interest for R-tipping from $e_3$ to $e_2$ studied in the next section.

To conclude the bifurcation analysis, we quantify in Fig. 7 “the special combinations of $b$ and $b_c$” that give rise to a codimension-three degenerate Bogdanov-Takens bifurcation. In the $(b, b_c)$ parameter plane, these “special combinations” lie on a curve which separates the regions with type-I and type-II Bogdanov-Takens bifurcation points. In other words, Fig. 7 shows the projection of a codimension-three bifurcation curve from the four-dimensional $(r, m, b, b_c)$ parameter space onto the $(b, b_c)$ parameter plane. Points labelled (a)-(d) in Fig. 7 refer to the values of $b$ and $b_c$ chosen for the $(r, m)$ bifurcation diagrams in Fig. 5. The asterisk indicates the values of $b$ and $b_c$ used in Ref. [7].

3.3. Summary of B-tipping: Simple Criteria and Robustness. In summary, classical bifurcation analysis for fixed in time input parameters describes quasistatic or adiabatic effects of changing the plant growth rate $r$, the herbivore mortality rate $m$, and the decline in herbivore growth at high plant biomass quantified by the nonlinearity parameter $b + b_c$. When $b + b_c = 0$, we do not expect any B-tipping owing to the lack of dangerous bifurcations. However, when $b + b_c > 0$, meaning that there is a decline in herbivore growth at high plant biomass, a number of different B-tipping...
mechanisms appear in the ecosystem model. The most dominant are the two generic
dangerous bifurcations of equilibria, namely supercritical saddle-node and subcritical
Hopf bifurcations. Additionally, but less important for this study, there is a sub-
critical transcritical bifurcation [Fig. 4(b), (d) and (f)], a supercritical saddle-node
bifurcation of limit cycles [e.g. see the inset in Fig. 5(c)], and a supercritical homo-
clinic bifurcation [Fig. 4(e)–(f)]. Thus, to identify B-tipping in the non-autonomous
system (2.5)–(2.6), it is sufficient to consider an \((r,m)\) bifurcation diagram for the
autonomous system (2.1)–(2.2) with a prescribed parameter path.

**Testable criterion for B-tipping.** If a parameter path in a \((r,m)\) bifurcation
diagram crosses one of the dangerous bifurcations for the autonomous system (2.1)–
(2.2), then there is a time-varying external input \((r(t), m(t))\) that traces out this path
and gives rise to B-tipping in the non-autonomous system (2.5)–(2.6).

Figure 8 shows an example of such a path, denoted with \(\Delta m\) in panel (a), together
with the dynamics of the non-autonomous system (2.5)–(2.6), where \(m(t)\) drifts slowly
along the path [panel (b)]. If the system starts near the stable equilibrium \(e_3\) at the
lower end \(p_1\) of the path and \(m(t)\) increases over time, then the non-autonomous
system tracks the moving stable equilibrium \(e_3(t)\) up to the point of the dangerous
bifurcation \(S_c\) [Figure 8(b)]. As \(m(t)\) passes through the bifurcation, which defines the
critical level of \(m\), the system undergoes a sudden and abrupt transition to the other
stable equilibrium \(e_2(t)\). Such instability is also described as a **dynamic** or **adiabatic**
bifurcation [44,45]. To discuss the robustness of B-tipping, we can invoke the notions
of genericity and co-dimension of a bifurcation [32, Sec.2.5]. Specifically, B-tipping
due to generic co-dimension one bifurcations is unaffected by small perturbations to
the system or to the parameter path. We refer to [46] for a more general and precise
definition of B-tipping, and for rigorous criteria for B-tipping.

**4. Irreversible R-Tipping: Beyond Classical Bifurcations.** In this sec-
tion we go beyond the traditional bifurcation theory and adiabatic effects of parameter
change, and consider genuine non-autonomous instabilities and non-trivial non-
adiabatic effects of parameter change. These instabilities arise solely from the time
variation of the input parameters $r$ and $m$, cannot be captured by classical bifurcation analysis, and thus require an alternative approach. Specifically, we ask: Are there parameter paths in the $(r,m)$ bifurcation diagram that do not cross any bifurcations of the stable equilibrium $e_3$ but give rise to tipping? The answer is yes. This was demonstrated in [7] and is examined here in more depth. Consider the $(r,m)$ bifurcation diagram with a parameter path $\Delta_r$ that does not cross any bifurcations in Fig. 9(a). If the non-autonomous system starts at the stable equilibrium $e_3$ near the lower end $p_1$ of the path, and $r(t)$ increases slowly enough along the path, then the non-autonomous system is able to adapt to the changing environment and track the moving stable equilibrium $e_3(t)$ [blue trajectory in Fig. 9(b)]. However, if $r(t)$ increases slowly but faster than some critical rate, the non-autonomous system fails to adapt to the changing environment and undergoes a critical transition from $e_3(t)$
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to the other stable equilibrium $e_2(t)$ [red trajectory in Fig. 9(b)]. This happens even though $e_2(t)$ is continuously available and never loses stability along the path. Such non-autonomous instability is called irreversible R-tipping [2]. Note that there can be transient phenomena of reversible R-tipping that are not studied here, where the system fails to track the moving stable state, suddenly moves to a different state, but in the long term returns to and follows the original stable state [25, 26].

4.1. The Vicious Cycle. Intuitively, R-tipping is the result of a vicious cycle that could potentially tip the system to a different state if the input parameters vary too fast. In the ecosystem model, the vicious cycle arises from the key nonlinearity identified in Sec. 2, that is from non-monotone herbivore growth (2.4) that changes sign from positive to negative at high plant biomass $P = P_4$ [see Fig.1(b)].

The effect can be understood as follows. Consider a stable herbivore population with a lower than optimal plant biomass $P_3$ for some $r = r_-$. Then, consider a smooth increase in the plant growth rate from $r_-$ to $r_+$. The increase results in faster growing plants, and moves the stable equilibrium to a larger herbivore population with the same plant biomass $P_3$. If $r(t)$ increases slowly enough, herbivores manage to graze and grow fast enough so that the larger herbivore population is able to maintain the same plant biomass at larger $r = r_+$. However, if $r(t)$ increases too fast, herbivores may be unable to keep up and prevent the plant biomass from growing past its optimal value $P_{opt}$. This, in turn, triggers the vicious cycle: past the optimal plant biomass, the larger the plant biomass gets, the less the herbivores graze and grow, allowing the plant biomass to grow even larger. The ultimate result is negative net herbivore growth causing a sudden collapse of the herbivore population. This is accompanied by a sudden increase in the plant biomass to $P_4$. Even though there is no classical bifurcation along the parameter path between $r_-$ and $r_+$, the rate of change of $r(t)$ alone prevents the system from adapting to the modified stable equilibrium. In the proceeding sections, we perform a systematic mathematical analysis of the vicious cycle mechanism that gives rise to irreversible R-tipping as shown in Fig. 9(b).

4.2. Simple Criteria for Irreversible R-tipping. It turns out that, similarly to B-tipping, much can be understood about irreversible R-tipping in the non-autonomous system (2.5)–(2.6) from certain properties of the corresponding autonomous system (2.1)–(2.2) [46]. The difference is that R-tipping is related to global, rather than local, properties of the stable state (an attractor). Specifically, we need the following ingredients to give testable criteria for irreversible R-tipping:

(i) A stable base equilibrium $e(p)$ whose position depends on the input parameter(s) $p$. For example, stable equilibrium $e_3$ for the ecosystem model can be given in terms of

$$p = (r, m),$$

by the asymptotic expansion formula (3.6).

(ii) Bistability or multistability - at least one additional attractor $a$ that coexists with $e$ for the same setting of the input parameters. For example, there is bistability between $e_3$ and $e_2$ in the $(r, m)$ parameter regions 5 and 7.

(iii) A continuous parameter path $\Delta$ in the space of the input parameters, that does not cross any dangerous bifurcations of the base equilibrium $e$. For example, the horizontal path $\Delta_r$ in the $(r, m)$ bifurcation diagram from Fig. 10(a) does not cross any bifurcations.

(iv) The basin of attraction of the base equilibrium, defined as the set of all initial states $(P_0, H_0)$ whose trajectories converge to the stable base equilibrium $e(p)$
in time

\[ B(e, p) = \{ (P_0, H_0) \in \mathbb{R}^2 : (P(t), H(t)) \to e(p), \ t \to +\infty \} , \]

together with the evolution of \( B(e, p) \) along the chosen parameter path. For example, Fig. 10(b)–(d) shows the (blue shaded) basin of attraction of \( e_3 \) for three different settings of \( r \) along the parameter path \( \Delta_r \). The boundary between the basins of attraction of \( e_3 \) and \( e_2 \) is given by the stable invariant manifold of the saddle equilibrium \( e_4 \).

(v) **Basin instability on a path.** Let \( B(e, p) \) denote the basin of attraction of \( e(p) \) together with its basin boundary. Then, we say that the stable base equilibrium \( e(p) \) is basin unstable on a parameter path \( \Delta \) if there are two points on the path, \( p_1, p_2 \in \Delta \), such that \( e(p_1) \) is outside the basin of attraction of \( e(p_2) \):

\[ e(p_1) \notin B(e, p_2). \]

For example, consider \( \Delta_r \) from Fig. 10(a) and pick \( r_- \). The stable equilibrium \( e_3(r_-) \) is contained within the basin of attraction of \( e_3(r) \) for all \( r_- < r < r_+ \), lies on the basin boundary of \( e_3(r_+) \) [Fig. 10(c)], and is outside the basin of attraction of \( e_3(r) \) for all \( r_+ < r < r_+ \) [Fig. 10(d)]. Thus, \( e_3 \) is basin unstable on the path \( \Delta_r \) because \( e_3(r_1) \notin B(e_3, r_2) \) for \( r_1 = r_- \) and any \( r_2 \in (r_+, r_+) \).

**Testable criterion for irreversible R-tipping.** If a stable equilibrium \( e(p) \) of the autonomous system (2.1)–(2.2) is basin unstable on a parameter path \( \Delta \), meaning that there are \( p_1, p_2 \in \Delta \) such that \( e(p_1) \notin B(e, p_2) \), then there is a time-varying external input \( p(t) = (r(t), m(t)) \) that traces out the path from \( p_1 \) to \( p_2 \) and gives irreversible R-tipping from \( e(t) \) in the non-autonomous system (2.5)–(2.6).

More generally, basin instability is necessary and sufficient to observe irreversible R-tipping in one-dimensional systems [46], and sufficient but not necessary to observe irreversible R-tipping in higher-dimensional systems [26, 47, 48].

The criterion above can be understood intuitively using the example from Fig. 10. Suppose the system starts in the basin of attraction and near the stable equilibrium \( e_3 \) at \( r = r_- \), and undergoes a monotone parameter shift from \( r_- \) to any \( r \in (r_+, r_+) \). We choose a shift from \( r_- \) to \( r_+ \), and consider two extreme scenarios. If \( r(t) \) varies sufficiently slowly, meaning that the speed of the moving equilibrium \( |\dot{e}(t)| \) is much slower than the natural timescales of \( H(t) \) and \( P(t) \), the non-autonomous system is guaranteed to closely track or adiabatically follow the moving stable equilibrium \( e_3(t) \) along the path [26, 46]. However, the dynamics are different when \( r(t) \) shifts smoothly but abruptly from \( r_- \) to \( r_+ \) at some point in time, remains almost constant otherwise, and the speed \( |\dot{e}(t)| \) during the abrupt shift is much faster than the natural timescales of \( H(t) \) and \( P(t) \). Initially, the system approaches \( e_3(r_-) \) because \( r(t) \) is almost constant. Then comes the abrupt shift from \( r_- \) to \( r_+ \), the stable equilibrium \( e(t) \) changes its position, but the system is too slow to respond. Thus, just after the shift, the system is still at its earlier position near \( e_3(r_-) \), which now lies outside the basin of attraction of \( e_3(r_+) \) and inside the basin of attraction of \( e_2(r_+) \) [Fig. 10(d)]. As \( r(t) \) remains almost constant from now on, the system approaches the other stable equilibrium \( e_2(r_+) \). These two qualitatively different scenarios indicate that there is an intermediate critical rate of change of \( r(t) \), above which the system R-tips from \( e_3 \) to \( e_2 \). We refer to [26, 48] for more general and precise definitions of basin instability, for rigorous statements of the sufficient criteria for irreversible R-tipping, and for extension of these ideas to *threshold instability* that captures both reversible and irreversible R-tipping.
4.3. Beyond Traditional Bifurcation Diagrams: Basin Instability. To examine the robustness of irreversible R-tipping from e3, we need to examine the persistence of its basin instability on different parameter paths in the (r, m) bifurcation diagram. To this end, we fix \( b = b_c = 0.025 \) close to the values used in Ref. [7] point (d) in Fig. 7, and explore different parameter paths within adjacent regions 5 and 7 from Fig. 5(d). Recall that \( e_3 \) remains stable and does not bifurcate within or across the boundary \( h \) between those two regions. Specifically, we choose four different points \( p_1 \) within region 5, and mark them with a black dot on different panels in Fig. 11. For each \( p_1 \), we identify the set of points \( p_2 \) within regions 5 and 7 such that \( e_3(p_1) \notin B(e_3, p_2) \), and shade this set in grey in Fig. 11. We call this the region of basin instability

\[
BI(e_3, p_1) = \{ p_2 \in \mathbb{R}^d : e_3(p_1) \notin B(e_3, p_2) \}.
\]

In other words, stable equilibrium \( e_3 \) is basin unstable on any parameter path that stays within regions 5 and 7 and connects \( p_1 \) to some \( p_2 \in BI(e_3, p_1) \). The analysis of \( BI(e_3, p_1) \) unveils a robust region of basin instability that can occupy almost the entirety of regions 5 and 7, and is in line with the intuitive vicious cycle discussion.
Figure 11. (Shading) Region of basin instability $BI(e_3,p_1)$ in the $(r,m)$ bifurcation diagram for stable equilibrium $e_3$ along continuous parameter paths from $p_1$ consists of all points $p_2$ such that $e_3(p_1) \not\in B(e_3(p_2))$ (or $e_3(p_1) \in B(e_2(p_2))$) as defined by Eq. (4.1). $p_1$ is chosen to be at $(r,m) = (a) (0.5,0.12)$, (b) $(0.75,0.075)$, (c) $(1.25,0.11)$ and (d) $(1.25,0.025)$. $b = \beta_c = 0.025$.

from Section 4.1. R-tipping is expected for a variety of parameter paths, even for small-magnitude shifts in $r$. Basin instability is easily achieved for increasing $r$, less easy to achieve for increasing $m$, and appears impossible to achieve for decreasing $r$ alone [Fig. 11(a)–(d)]. Moving $p_1$ to a different position in the $(r,m)$ diagram clearly modifies the region of basin instability in different ways. For example, starting at lower values of $m$ gives a small section of basin instability for shifts in $m$ alone [Fig. 11(b)]. Overall, the region of basin instability persists upon moving $p_1$ to near the upper [Fig. 11(c)] or the lower [Fig. 11(d)] boundary of region 5.

In addition to dangerous magnitudes of environmental change, the ecosystem model appears to be particularly sensitive to how fast the plant growth rate $r$ increases over time. The basin instability analysis quantifies this effect in terms of different starting points $p_1$ in the $(r,m)$ diagram, and shows that sensitivity to the rate of environmental change becomes greatly enhanced at higher herbivore mortality rates $m$. Beyond the specific ecosystem model, this approach can be used to analyse tipping phenomena in any nonlinear system. Superimposing the region of basin instability on a classical bifurcation diagram gives basic information on rate-induced instabilities that cannot be captured by classical bifurcation analysis. What is more, the basic information can be augmented in different ways. For example, one can prescribe the form of $A(t) = (r(t),m(t))$ and the shape of the parameter path (e.g. a straight line path)
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between $p_1$ and $p_2$) and use colour-scale instead of plane gray to indicate different critical rates for each point $p_2$ within $B(e_3, p_1)$. Another possibility is to fix $p_1$ and $p_2$ and analyse critical rates with dependence on the shape of parameter path between $p_1$ and $p_2$. This type of analysis is left for future work.

4.4. Tipping Diagrams for Parameter Shifts. Now, we consider irreversible R-tipping from $e_3(t)$ in the non-autonomous system (2.5)–(2.6). Guided by the basin instability analysis performed in the previous section, we focus on shifts in the plant growth rate $r$. Specifically, we analyse the system response to two shapes of $r(t)$, each of which is parameterized by its magnitude $\Delta_r$ and rate $\varepsilon$. Firstly, we consider a monotone shift

\[ r(t) = r_- + \frac{\Delta_r}{2} \left( \tanh(\varepsilon t) + 1 \right), \]

from $r_-$ to $r_+ = r_- + \Delta_r$ at the rate $\varepsilon$ in units inverse day, and with $\dot{r}_{\text{max}} = \varepsilon \Delta_r / 2$ in units inverse day squared \([c^{\uparrow}] \text{ in Fig. 12(a)}\]. Secondly, we consider a non-monotone shift

\[ r(t) = r_- + \Delta_r \tanh(\varepsilon t), \]

from $r_-$ to $r_+ = r_- + \Delta_r$ and then back to $r_-$ at the rate $\varepsilon$ and $\dot{r}_{\text{max}} = \varepsilon \Delta_r / 2$ \([c^{\downarrow}] \text{ in Fig. 12}\). Such a setting enables parametric study in the form of two-dimensional $(\Delta_r, \varepsilon)$ or $(\Delta_r, \dot{r}_{\text{max}})$ tipping diagrams to identify critical rates $\varepsilon_c$ at which the system switches between tracking and irreversible R-tipping.

In all instances, the non-autonomous system (2.5)–(2.6) is initialised at the moving stable equilibrium

\[ (P(t_0), H(t_0)) = e_3(t_0), \]

at some initial time $t_0$ such that

\[ r(t_0) = r_- + \delta, \]

is $\delta$-close to $r_-$, and $\dot{r}(t_0) \ll \dot{r}_{\text{max}}$ to ensure that the speed of the moving equilibrium $|\dot{e}_3(t_0)|$ is sufficiently small. (Note that initiating the system at $r_0 = r_-$ would require $t_0 = -\infty$.) Henceforth, we set $\delta = \Delta_r / 10^3$. This gives

\[ t_0(\varepsilon) = \frac{1}{\varepsilon} \tanh^{-1} (-0.998) \approx -\frac{3.453}{\varepsilon}, \]
with $\dot{r}(t_0) \approx 0.004 \dot{r}_{max}$ for the monotone input (4.2), and
$$t_0(\varepsilon) = \frac{1}{\varepsilon} \text{sech}^{-1} (10^{-3}) \approx \frac{7.6}{\varepsilon},$$
with $\dot{r}(t_0) \approx 0.002 \dot{r}_{max}$ for the non-monotone input (4.3).

### 4.4.1. Monotone Shifts Across Basin Instability Boundary: Single Critical Rate

Figure 13 sheds more light on the R-tipping from Fig. 9. Firstly, the stable equilibrium $e_3$ can be basin unstable upon increasing $r$ from $p_1$ along the path $\Delta_r$ [Fig. 13(a)]. The boundary of the region of basin instability occurs at $\Delta_r \approx 0.32$.

Secondly, there is a unique critical rate $\varepsilon_c$ which defines the transition between tracking and R-tipping. When $\varepsilon = \varepsilon_c$, the non-autonomous system neither tracks $e_3(t)$ nor R-tips to $e_2(t)$ but, rather surprisingly, follows the moving unstable equilibrium $e_4(t)$ [Fig. 13(b)]. This behaviour is akin to so-called canard trajectories that follow unstable slow manifolds in slow-fast systems [49, 50]. It is interesting to note that
FIGURE 15. (a) Example of a parameter path \( \Delta_r \) in the \((r,m)\)-bifurcation diagram, that crosses the boundary of (shading) the region of basin instability \( BI(e_3,p_1) \) for \( p_1 = (0.5,0.12) \) and the homoclinic bifurcation \( h \) separating regions 5 and 7. (b) The non-autonomous system (2.5)-(2.6) with monotone parameter shift (4.2) from \( p_1 \) along \( \Delta_r \) (blue) tracks the moving stable equilibrium \( e_3(t) \) when \( \varepsilon = \varepsilon_{c1} = 0.0175 < \varepsilon_c \), (green) rather surprisingly tracks the moving repelling limit cycle from region 7 when \( \varepsilon = 0.020342768468207 \approx \varepsilon_c \), and (red) R-tips to \( e_2(t) \) when \( \varepsilon = \varepsilon_{c2} = 0.021 > \varepsilon_c \). The moving equilibria are obtained for \( \varepsilon = \varepsilon_c \). \( b = b_c = 0.025 \).

FIGURE 16. Trajectories of the non-autonomous system (2.5)-(2.6) with non-monotone parameter shifts (4.3) from \( p_1 = (1.0,0.075) \) along a path \( \Delta_r \) with fixed \( m = 0.075 \) and time-varying \( r > 1 \). (a) The system (blue) tracks the moving stable equilibrium \( e_3(t) \) when \( \varepsilon = \varepsilon_{c1} = 0.1 < \varepsilon_{c1} \), (green) rather surprisingly tracks the moving unstable equilibrium \( e_4(t) \) when \( \varepsilon = 0.166491526823788 \approx \varepsilon_{c1} \), and (red) R-tips to \( e_2(t) \) when \( \varepsilon = \varepsilon_{c1}^+ = 0.2 > \varepsilon_{c1} \). (b) Upon further increase in the rate, the system (green) again rather surprisingly tracks the moving unstable equilibrium \( e_4(t) \) when \( \varepsilon = 1.049396269470948 \approx \varepsilon_{c2} \), and (blue) switches back to tracking \( e_3(t) \) when \( \varepsilon = \varepsilon_{c2}^+ = 1.5 > \varepsilon_{c2} \). The moving equilibria are obtained for (a) \( \varepsilon = \varepsilon_{c1} \) and (b) \( \varepsilon = \varepsilon_{c2} \). \( b = b_c = 0.025 \).

Critical-rate canard trajectories can follow different unstable states, depending on the basin boundary for the future-limit autonomous system. For example, Fig. 15(a) shows a parameter path \( \Delta_r \) that starts at \( p_1 \) in region 5 and extends past the subcritical homoclinic bifurcation \( h \) to region 7. Along this path, equilibrium \( e_3 \) is smoothly stable, but its basin boundary changes from the stable invariant manifold of saddle \( e_4 \) (region 5) to a repelling limit cycle (region 7) [Fig. 6]. Given a monotone shift from \( p_1 \) along \( \Delta_r \) and across \( h \), there is R-tipping due to basin instability. The difference from Fig. 13(b) is the (green) critical-rate canard trajectory which now follows the moving unstable limit cycle [Fig. 15(b)].
A systematic analysis of R-tipping for monotone shifts (4.2) from $p_1$ along the path $\Delta_r$ from Fig. 13(a) gives the $(\Delta_r, \varepsilon)$ and $(\Delta_r, \dot{r}_{\text{max}})$ tipping diagrams [Fig. 14]. In the diagrams, the tracking-tipping transitions occur along the curve $c^1$. This curve divides the tipping diagram into separate regions of (white) tracking and (pink) irreversible R-tipping [Fig. 14]. The entire R-tipping region appears to be located past the basin instability boundary $BI$. As $\Delta_r$ decreases, the $c^1$ curve is asymptotic to $BI$ from the right. This suggests that basin instability is both sufficient and necessary for irreversible R-tipping in the ecosystem model. In general, this need not be the case in higher than one dimension, where basin instability is guaranteed to be sufficient but not necessary for R-tipping; see [47, 48] for examples of irreversible R-tipping in two dimensions in the absence of basin instability. What is more, as $\Delta_r$ increases, the $c^1$ curve appears to level off at $\dot{r}_{\text{max}} \approx 0.045$. In other words, R-tipping in the ecosystem model requires sufficiently large $\dot{r}_{\text{max}}$, rather than $\varepsilon$, independently of $\Delta_r$. Thus, one can give simple approximate conditions for irreversible R-tipping along this path in terms of $\Delta_r$ exceeding the basin instability boundary $BI$ and $\dot{r}_{\text{max}}$ exceeding the critical value $\approx 0.045$. Finally, we say that this R-tipping is unique, meaning that there is a unique critical rate $\varepsilon_c$ for every fixed magnitude $\Delta_r$ that exceeds the basin instability boundary $BI$.

4.4.2. Non-monotone Shifts Across Basin Instability Boundary: Two Critical Rates. Now, consider system (2.5)–(2.6) with non-monotone $r(t)$ tracing out the path $\Delta_r$ in Fig. 13(a) from $p_1$ at $r_- = 0.75$ to $r_- + \Delta_r$ and then back to $p_1$. The six solutions for different values of $\varepsilon$ shown in Fig. 16(a)–(b) highlight the main difference from the monotone shift: two different critical rates for the same $\Delta_r$. Specifically, the system tracks $\varepsilon_3(t)$ below the first critical rate $\varepsilon < \varepsilon_{c1}$, then switches from tracking to irreversible R-tipping when $\varepsilon = \varepsilon_{c1}$. R-tips for a range of rates $\varepsilon_{c1} < \varepsilon < \varepsilon_{c2}$, then switches back from irreversible R-tipping to tracking when $\varepsilon = \varepsilon_{c2}$, and continues to track $\varepsilon_3(t)$ for $\varepsilon > \varepsilon_{c2}$. In the $(\Delta_r, \varepsilon)$ and $(\Delta_r, \dot{r}_{\text{max}})$ tipping diagrams shown in Fig. 17, tracking-tipping transitions occur along the curve $c^1$. This curve divides the tipping diagram into two separate regions of (white) tracking and (pink) irreversible R-tipping. The region of irreversible R-tipping is located past
the basin instability boundary $BI$, and is tongue-shaped. The $R$-tipping tongue is reminiscent of a resonance tongue [51] in the sense that the system exhibits a strongly enhanced response to external inputs with optimal timing. This tongue shape can be understood in terms of relative timescales. At high $\varepsilon$, the natural timescales of $H(t)$ and $P(t)$ are slower than $e_3(t)$. Thus, the system is unable to respond to a short impulse $r(t)$. As $\varepsilon$ is decreased, the natural timescales of $H(t)$ and $P(t)$ get closer to $e_3(t)$, the system starts to react to the input and R-tips due to basin instability. This transition is marked by the higher critical rate. As $\varepsilon$ is decreased further, the natural timescales of $H(t)$ and $P(t)$ become comparable to $e_3(t)$, giving rise to a strongly enhanced response in the form of the tipping tongue. As $\varepsilon$ is decreased even further, the natural timescales of $H(t)$ and $P(t)$ become faster than $e_3(t)$, and the system starts to closely track $e_3(t)$. This transition is marked by the lower critical rate. In summary, for a fixed $\Delta_r$ past the basin instability boundary $BI$, the $\varepsilon$-interval of irreversible R-tipping can be bounded by two critical rates, $\varepsilon_{c1}$ from below and $\varepsilon_{c2}$ from above [Fig. 17(a)–(b)]. We describe this as non-unique R-tipping.

5. Interaction Between B-tipping and R-tipping and Multiple Critical Rates. So far, we have discussed B-tipping and R-tipping in isolation. At the same time, we recognise that real-world tipping phenomena will often involve both mechanisms, although the ensuing nonlinear dynamics is less well understood. This section discusses three different types of interplay between critical levels and critical rates, or between B-tipping and R-tipping, and reveals intriguing tipping diagrams.

5.1. Monotone Shifts Across Basin Instability Boundary and Dangerous Bifurcation. Figure 15 showed a monotone parameter shift from $p_1$ along the parameter path $\Delta_r$ across the subcritical homoclinic bifurcation $h$ in order to demonstrate the effects of a qualitative change in the basin boundary along the path. To study the interaction between B-tipping and R-tipping, we extend this parameter path away from $p_1$ and past the dangerous (subcritical) Hopf bifurcation [Fig. 19(a)]. The ensuing $(\Delta_r, \varepsilon)$ and $(\Delta_r, \dot{r}_{\text{max}})$ tipping diagrams for monotone shifts (4.2) from $p_1$ along the extended path are shown in Fig. 18. The tipping-tracking transition curve $c^\uparrow$ consists of two distinct parts, which correspond to two different tipping mecha-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure18.png}
\caption{Tipping diagrams in the (a) $(\Delta_r, \varepsilon)$- and (b) $(\Delta_r, \dot{r}_{\text{max}})$-parameter plane for monotone shifts (4.2) from $p_1 = (0.5, 0.12)$ along the parameter path $\Delta_r$ from Fig. 19(a). The tipping-tracking transition curve $c^\uparrow$ separates the diagram into regions of (white) tracking and (pink) tipping. BI indicates the boundary of the basin instability region $BI(e_3(p_1))$, $h$ indicates the homoclinic bifurcation, and $H_e$ indicates the (dangerous) subcritical Hopf bifurcation of $e_3$. $b = b_c = 0.025$. The critical rate $\varepsilon_c$ corresponds to the green canard-like trajectory in Fig. 15(b).}
\end{figure}
The moving equilibria are obtained for (b) switches from tipping to tracking when \( \varepsilon \) decreased, there is an abrupt transition near \( \dot{r} \approx 0.0016560 \) from \( p_1 = (0.5, 0.12) \) along \( \Delta r \) (b) tips from \( e_3(t) \) to \( e_2(t) \) below the first critical rate \( \varepsilon < \varepsilon_{c_1} \), then switches from tipping to tracking when \( \varepsilon \approx 0.0148304 \) (c) switches back to tipping when \( \varepsilon \approx 0.7005963 \approx \varepsilon_{c_2} \), tips for a range of rates \( \varepsilon_{c_1} < \varepsilon < \varepsilon_{c_2} \), (d) switches again from tipping to tracking when \( \varepsilon \approx 0.025 \) (e) \( \varepsilon = \varepsilon_{c_3} \). The three different critical rates \( \varepsilon_{c_1}, \varepsilon_{c_2} \) and \( \varepsilon_{c_3} \) are indicated in the tipping diagram in Fig.20(a).

We would like to point out that the slow passage through a Hopf bifurcation gives rise to a bifurcation delay that does not vanish even if the rate of parameter change tends to zero \([52–54]\). Whereas the bifurcation delay has no effect on the tipping diagram for monotone shifts \((4.2)\), it is expected to manifest itself for non-monotone shifts \((4.3)\) that are considered in the next section.
5.2. Non-monotone Shifts Across Basin Instability Boundary and Dangerous Bifurcation. Monotone parameter shifts across a basin instability boundary and a dangerous bifurcation give rise to an intuitive tipping diagram with two distinct regimes: B-tipping for low rates and unique R-tipping for higher rates. Now, we consider non-monotone shifts along the same path $\Delta_r$ from Fig. 19(a). Specifically, $r(t)$ increases from $p_1$, passes through the basin instability boundary $BI$ and through the dangerous (subcritical) Hopf bifurcation $H_e$, but then turns around and tends back to $p_1$. This turning around allows the system to avoid tipping even if it goes past the critical level for B-tipping namely past the dangerous bifurcation $H_e$. What is more, the bifurcation delay gives the system additional time to turn back before tipping occurs. The nine solutions for different values of $\varepsilon$ shown in Fig. 19(b)-(d) highlight the main difference from the monotone shift: three different critical rates for the same $\Delta_r$. The system tips from $e_3(t)$ to $e_2(t)$ below the first critical rate $\varepsilon < \varepsilon_{c1}$, then switches from tipping to tracking when $\varepsilon = \varepsilon_{c1}$ [Fig. 19(b)], tracks $e_3(t)$ for a range of rates $\varepsilon_{c1} < \varepsilon < \varepsilon_{c2}$, switches back to tipping when $\varepsilon = \varepsilon_{c2}$ [Fig. 19(c)], tips for a range of rates $\varepsilon_{c2} < \varepsilon < \varepsilon_{c3}$, and switches again from tipping to tracking when $\varepsilon = \varepsilon_{c3}$ [Fig. 19(d)].

The ensuing $(\Delta_r, \varepsilon)$ and $(\Delta_r, \dot{r}_{max})$ tipping diagrams for non-monotone shifts \(^{(4.3)}\) from $p_1$ are shown in Fig. 20. Although the separation between different tipping mechanisms is less obvious now, the tipping-tracking transition curve $c^+$ still consists of two different parts that can be associated with the two different tipping mechanisms. At high $\varepsilon$ and between $BI$ and $H_e$, we replicate the distinctive tongue-shaped tipping region from Fig. 17. Thus, we attribute this part of the tipping diagram to irreversible R-tipping. As $\varepsilon$ is decreased, we observe two new features. Firstly, the curve $c^+$ forms a deep tracking wedge whose tip delineates the change from R-tipping to B-tipping. Secondly, as $\varepsilon \to 0$, the curve $c^+$ approaches the critical level $H_e$ for B-tipping. However, this approach is ‘slow’, which is in stark contrast to Fig. 17. The new features can be explained in terms of relative timescales and a bifurcation delay. As $\varepsilon$ decreases below the tipping tongue, the natural timescales of $H(t)$ and $P(t)$ start to exceed the timescale of $e_3(t)$, meaning that the system becomes more able to follow

\[ H(t), P(t) \to e_3(t) \]
the moving stable equilibrium $e_3(t)$. On the one hand, we start to lose R-tipping. On the other hand, the system acquires some characteristics of a slow passage through a Hopf bifurcation. In particular, there is a bifurcation delay that allows the system to spend a noticeable time past the critical level $H_e$ before B-tipping actually occurs. As a result, the tracking-tipping transition due to $H_e$ is shifted to a much larger $\Delta r$. Hence the deep tracking wedge in $c_r$ separates the diagram into regions of (white) tracking and (pink) tipping. The inset shows the wiggling part of of $c_r$, $b = b_c = 0.025$.

5.3. Non-monotone Shifts Across Dangerous Bifurcation. The third type of interaction arises during a non-monotone passage through a dangerous bifurcation, and is more of an interplay between critical levels and critical rates rather than between B-tipping and R-tipping. To be more specific, we consider a parameter path that crosses a dangerous bifurcation with vanishing region of basin instability. The path $\Delta_m$ through a subcritical Hopf bifurcation $H_e$ from Fig. 21(a) is an example of such a path. The difference from the first two types of interaction is that no pure R-tipping is observed along this path. Nonetheless, the system response is expected to depend on the rate $\varepsilon$. For example, the system may avoid tipping despite going past the dangerous bifurcation if it turns around fast enough [20,21]. Thus, in addition to the critical level, we also expect critical rate(s).

\[\text{This manuscript is for review purposes only.}\]
We fix \( r = 1 \), consider non-monotone shifts in the herbivore death rate along \( \Delta_m \):

\[
m(t) = 0.12 + \Delta_m \sinh(\varepsilon t),
\]

and initiate the non-autonomous system (2.5)–(2.6) at the stable equilibrium \( e_3(t_0) \) at time \( t_0(\varepsilon) = \text{sech}^{-1}\left(10^{-3}\right) / \varepsilon \approx -7.6 / \varepsilon \). The resulting tipping-tracking transition curve \( c^\uparrow \) in the \((\Delta_m, \varepsilon)\) tipping diagram shows a complicated rate dependence and is far from trivial [Fig. 21(b)]. Owing to the absence of basin instability and R-tipping, it is expected that \( \Delta_m \) has to exceed the critical level \( H_e \) for tipping to occur. What is less obvious is the presence of multiple critical rates. Past \( H_e \), there is a range of shift magnitudes \( \Delta_m \) with a unique critical rate. However, for larger \( \Delta_m \), the curve \( c^\uparrow \) has a ‘bump’ that gives rise to three critical rates for a fixed \( \Delta_m \). One can think of this ‘bump’ as a remnant of the R-tipping tongue found for paths \( \Delta_m \) starting at lower values of \( m \). Most interestingly, there is an interval of \( \Delta_m \) where the winding part of \( c^\uparrow \) gives rise to several critical rates for the same \( \Delta_m \) [inset in Fig. 21(b)].

6. Points of Return, Points of No Return, Points of Return Tipping.

Tipping is often defined as a large, sudden and possibly unexpected change in the state of the system, caused by a slow or small change in the external input (e.g. environmental conditions). Although “sudden” and “unexpected” suggest that foreseeing and preventing tipping may be difficult, it should in general be possible [19]. In this section, we are guided by the question: Given a monotone parameter shift that gives tipping, when can tipping be prevented by a parameter-shift reversal? Certain aspects of this question have been explored in the context of B-tipping near a saddle-node bifurcation. For example, Hughes et al. [19] speak of “living dangerously on borrowed time” to describe a window of opportunity for ecosystems to return to safer conditions before an otherwise inevitable tipping occurs. Biggs et al. [18] ask whether early-warning indicators for tipping provide sufficient warning to modify the ecosystem’s management and avert undesired regime shifts by “turning back from the brink”. Gandhi et al. [49, 55] consider non-monotone parameter shifts through the (global) saddle-node on a limit cycle bifurcation to identify a new resonance mechanism in the context of spatially localised (vegetation) patterns. Ritchie et al. [21] model systems near a saddle-node bifurcation and analyse relations between the time and amplitude of a saddle-node crossing to avoid B-tipping [21]. Most recently, Alkhayuon et al. [56] investigate “avoided” B-tipping and R-tipping near a subcritical Hopf bifurcation in the box model of the Atlantic Meridional Overturning Circulation (AMOC) in the context of collapse of the AMOC and climate change mitigation.

Here, we extend the existing literature on avoiding B-tipping to (i) analyse a sub-critical Hopf bifurcation, (ii) obtain additional results on a saddle-node bifurcation, and (iii) include R-tipping effects due to basin instability away from a bifurcation point. What is more, we compare three different results: the ecosystem model results, analysis of canonical forms for the two generic dangerous bifurcations of equilibria namely saddle-node and subcritical Hopf bifurcations, and the recent theoretical predictions for a saddle-node bifurcation from Ref. [21]. The canonical forms are modified (‘tilted’) normal forms to capture B-tipping near the bifurcation point as well as R-tipping away from the bifurcation point.

Specifically, we consider paths in one parameter \( \mu \). A path starts at \( \mu = \mu_- \) and may traverse the bifurcation at \( \mu = \mu_b \). Along a parameter path, we consider modified
monotone shifts that reach a maximum in finite time [green in Fig. 12(b)]:

\[
\mu(t) = \begin{cases} 
\mu_+ + \Delta \mu \text{sech}(\varepsilon t), & t \leq 0, \\
\mu_+ + \Delta \mu, & t > 0,
\end{cases}
\]

and are parametrised by the magnitude \(\Delta \mu\) and rate \(\varepsilon > 0\). The parameter-shift reversal of (6.1) can, in general, have two additional parameters \(c, \tau > 0\):

\[
\mu(t) = \begin{cases} 
\mu_+ + \Delta \mu \text{sech}(\varepsilon t), & t \leq 0, \\
\mu_+ + \Delta \mu, & 0 < t < \tau \\
\mu_+ + \Delta \mu \text{sech}(c \varepsilon (t - \tau)), & t \geq \tau,
\end{cases}
\]

where \(c \neq 1\) allows for different rates of shifting back and forth, and \(\tau > 0\) allows for some 'waiting time' before turning around [22]. Here, we consider a special case, obtained by setting \(c = 1\) and \(\tau = 0\) in (6.2), which corresponds to the parameter shift (4.3) used in the previous section [red in Fig. 12(b)]. For each path, we obtain (\(\Delta \mu, \varepsilon\)) combinations where tipping can or cannot be prevented by the parameter-shift reversal. In this way, we uncover four possible regions in the \((\Delta \mu, \varepsilon)\) tipping diagram:

- **Points of tracking** are defined as \((\Delta \mu, \varepsilon)\) settings where the system avoids tipping for monotone and non-monotone shifts. This is the safe region of tracking, sometimes referred to as the “safe operating space” [57].

- **Points of return** are defined as \((\Delta \mu, \varepsilon)\) settings where the system tips for monotone shifts, but does not tip for non-monotone shifts. Here, an otherwise imminent tipping is prevented by the parameter-shift reversal.

- **Points of no return** are defined as \((\Delta \mu, \varepsilon)\) settings where the system tips for monotone and non-monotone shifts. Here, tipping is not prevented by the parameter-shift reversal.

- **Points of return tipping** are defined as \((\Delta \mu, \varepsilon)\) settings where the system does not tip for monotone shifts, but tips for non-monotone shifts. Here, the parameter-shift reversal inadvertently induces tipping in an otherwise safe situation.

Note that the existence, shape and location of the four regions in the \((\Delta \mu, \varepsilon)\) tipping diagram will, in general, depend on the geometric form of the shift \(\mu(t)\), on the difference between the rates for shifting back and forth \((c \neq 0)\), and on the waiting time \((\tau > 0)\). These dependencies are not addressed here and are left for future study.

To facilitate comparisons with other works that express their results in terms of the exceedance time \(t_e\), which is the time the system spends past a dangerous bifurcation, we give the formula for \(t_e\) in terms of the magnitude \(\Delta \mu\) and rate \(\varepsilon\) of the shift (6.2):

\[
t_e = \frac{c + 1}{c \varepsilon} \text{sech}^{-1} \left( \frac{\mu_0 - \mu_-}{\Delta \mu} \right) + \tau,
\]

where \(\text{sech}^{-1} x \geq 0\) for \(0 < x \leq 1\).

### 6.1. The Ecosystem Model

For the ecosystem model (2.5)–(2.6), we consider two different parameter paths giving rise to two different diagrams in Fig. 22. The \((\Delta \mu, \varepsilon)\) tipping diagram in Fig. 22(a) is obtained for a parameter path with a fixed \(m = 0.075\), \(r_- = 1\), and \(r(t) > 1\) such that the path crosses the boundary \(BI\) of the basin instability \(BI(e_3, p_1)\), but does not cross any bifurcations. Thus, Fig. 22(a) describes points of return and no return for R-tipping alone. Points of no return are
bounded by the tipping-tracking transition curve $c^\uparrow$ for the non-monotone shift (4.3). Points of return are located between $c^\downarrow$ and the tipping-tracking transition curve $c^\uparrow$ for the monotone shift (6.1) with $\mu = r$. At higher $\varepsilon$, (green) points of return extend over the entire $\Delta_r$ interval past $BI$. This is indicative of R-tipping occurring after the input $r(t)$ reaches its maximum. Here, the natural timescales of $H(t)$ and $P(t)$ are slower than $e_3(t)$, and the system is slow to respond to changes in $r(t)$. However, as $\varepsilon$ is decreased, $c^\uparrow$ and $c^\downarrow$ approach each other so that the (green) points of return shrink and appear to vanish at $\varepsilon \approx 0.2$. Overlapping of $c^\uparrow$ and $c^\downarrow$ gives rise to apparently direct transitions from (white) tracking to (pink) points of no return. This is indicative of R-tipping occurring before the input $r(t)$ reaches its maximum. Here, the natural timescales of $H(t)$ and $P(t)$ become comparable to $e_3(t)$, the system R-tips to $e_2(t)$ during the upshift in $r(t)$, and the parameter-shift reversal has no effect on the overall response of the system. Note that $e_2(t)$ is basin stable on any parameter path within regions 5 and 7.

The $(\Delta_r, \varepsilon)$ tipping diagram in Fig. 22(b) is obtained for the parameter path $\Delta_r$ from Fig. 19(a) with a fixed $m = 0.12$, $r_\text{crit} = 0.5$, and $r(t) > 0.5$ such that the path crosses the boundary $BI$ of the basin instability $BI(e_3, p_1)$ as well as the dangerous subcritical Hopf bifurcation $H_e$. Thus, Fig. 22(a) describes points of return and no return for the interplay between B-tipping and R-tipping. At higher $\varepsilon$, R-tipping is the dominant tipping mechanism. Indeed, the part of the tipping diagram between $BI$ and $H_e$ at higher $\varepsilon$ is the same as in Fig. 22(a), including the vanishing (green) region with points of return. At intermediate $\varepsilon$, the competition between B-tipping and R-tipping gives rise to a deep wedge in $c^\downarrow$, which opens up another (green) region with points of return. At lower $\varepsilon$, B-tipping is the dominant tipping mechanism. Here, the natural timescales of $H(t)$ and $P(t)$ become faster than $e_3(t)$, and the problem resembles a slow passage through a Hopf bifurcation. The associated bifurcation delay is responsible for the lower boundary of the deep wedge in $c^\downarrow$, and for the 'slow'
approach of $c^\uparrow$ towards $c^\downarrow$ (or towards $H_e$) as $\varepsilon \to 0$.

Overall, the intricate ($\Delta r, \varepsilon$) tipping diagram for the ecosystem model is partitioned into regions of tracking, points of return and points of no return. In particular, there appear to be two different regions of points of return separated by direct transitions from tracking to points of no return. This leads us to the final question: How typical is the intricate tipping diagram from Fig. 22(b)? To answer this question we analyse tipping diagrams for a (slow) passage through the two generic dangerous bifurcations of equilibria, namely saddle-node and subcritical Hopf bifurcations.

6.2. The Two Generic Dangerous Bifurcations of Equilibria. From among different dangerous bifurcations of equilibria, only saddle-node and subcritical Hopf bifurcations are generic in the sense that they persist under arbitrarily small perturbations of the vector field. Here, we consider modified (‘tilted’) versions of the saddle-node and subcritical Hopf normal forms to study typical effects of non-monotone shifts across a dangerous bifurcation. The modification involves an additional parameter $s$ that quantifies the ‘tilt’ of the branches of solutions in the one-parameter bifurcation diagram; see Fig 23. Both bifurcations occur at $\mu_b = 0$, and the regular normal forms are recovered when $s = 0$. As there is no basin instability in the regular normal forms, there can be no R-tipping from the stable equilibrium when $s = 0$. For the unmodified subcritical Hopf normal form, that is Eq. (6.5) with $s = 0$, both B-tipping and R-tipping from the stable equilibrium can be excluded because the branch of equilibria $z = 0$ is a flow-invariant line in the $(z, t)$ phase space of the non-autonomous system. For the unmodified saddle-node normal form, that is Eq. (6.10) with $s = 0$, R-tipping can be excluded by [46, Th.3.2(1)] because the stable equilibrium is basin stable and the system is one-dimensional. However, the dynamics change when $s \neq 0$. In particular, R-tipping can be observed when the ‘tilt’ is sufficient enough to give basin instability along the chosen parameter path. In the following, we use $\mu_*$ to denote the basin instability boundary.

6.2.1. Modified Subcritical Hopf Normal Form. To make direct comparisons with the ecosystem model, first consider a system in $\mathbb{R}^2$ akin to the normal form of the subcritical Hopf bifurcation [32, Sec.3.4] written in terms of a complex variable $z = x + iy$:

$$
\dot{z} = \left(\mu + i \left[\omega + \alpha |z - \mu s|^2\right]\right) (z - \mu s) + |z - \mu s|^2 (z - \mu s).
$$

where $\mu$ is the bifurcation parameter, $\omega$ is the angular frequency of small-amplitude oscillations, $\alpha$ quantifies the amount of shear or amplitude-phase coupling and $s$ is the ‘tilt’ parameter. The subcritical Hopf normal form is recovered when we set $s = 0$ and apply a change of coordinates to transform away the term proportional to $\alpha$ [32, Sec.3.4]. There is one branch of equilibria

$$
e(\mu, s) = \mu s + 0i,
$$

that is stable for $\mu < 0$ and unstable for $\mu > 0$, and one branch of unstable limit cycles

$$l(\mu, s, t) = \mu s + \sqrt{-\mu} e^{i(\omega - \alpha \mu)t},
$$

that exists for $\mu < 0$. The real part of the limit cycle solution oscillates between

$$l_x^- (\mu, s) = -\sqrt{-\mu} + \mu s \quad \text{and} \quad l_x^+ (\mu, s) = \sqrt{-\mu} + \mu s,$$
Figure 23. One-parameter bifurcation diagrams for the modified (tilted) subcritical Hopf normal form Eq. (6.4) with (a) $s > 0$ and (b) $s < 0$, and for the modified (tilted) saddle-node normal form Eq. (6.7) with (c) $s > 0$ and (d) $s < 0$. Shown are branches of (solid) stable and (dashed) unstable equilibria $e$, branches of the maxima $l^+$ and minima $l^-$ of the $x$-component of the unstable limit cycle, parameter paths $\Delta \mu$ from $p_1 = \mu_-$, and the corresponding basin instability boundary $\mu^\pm$ of (a-b) BI($e, \mu_-$) and (c-d) BI($e^+, \mu_-$).

As shown in Fig. 23(a)–(b). For every $s \neq 0$, there are two basin instability boundaries. They are obtained by fixing $\mu_-$ and solving

$$\text{Re}[e(\mu_-, s)] = l_+^- (\mu_+, s) \quad \text{and} \quad \text{Re}[e(\mu_-, s)] = l_+^+ (\mu_+, s),$$

for $\mu_+$, which gives

$$\mu_- = \mu_- - \frac{1 + \sqrt{1 - 4s^2 \mu_-}}{2s^2} < \mu_- \quad \text{and} \quad \mu_+ = \mu_- - \frac{1 - \sqrt{1 - 4s^2 \mu_-}}{2s^2} > \mu_-.$$

Since we restrict to small enough and positive shift magnitudes $\Delta \mu > 0$, the relevant basin instability boundary is $\mu_+^+ > \mu_-$; see Fig. 23(a).

Now, consider the corresponding non-autonomous system

(6.5) \[ \dot{z} = \left( \mu(t) + i \left[ \omega + \alpha |z - s\mu(t)|^2 \right] \right) \left( z - s\mu(t) \right) + |z - s\mu(t)|^2 \left( z - s\mu(t) \right), \]

initialised at \[ z(t_0) = e(\mu(t_0), s), \quad t_0 = \frac{1}{\varepsilon} \text{sech}^{-1} (10^{-3}). \]
Firstly, we analyse R-tipping for non-monotone $\mu(t)$ given by Eq. (6.2) with $\mu_\ast = -1$, $\Delta_\mu > 0, c = 1$, $\tau = 0$ and different values of $s$ [Fig. 24(a)]. Note that the line $e = \mu s + 0i$ is flow-invariant when $s = 0$, but not when $s \neq 0$. Therefore, tipping from the stable equilibrium $e$ requires nonzero $s$. For $s = 10^{-4}$, we obtain $\mu_\ast^{\uparrow} \approx -10^{-8}$, meaning that the region of basin instability between $\mu_\ast^{\uparrow}$ and $H_e$ is negligible. The only tipping that occurs in the non-autonomous system is B-tipping for $\Delta_\mu > 1$, as evidenced by the tipping-tracking transition curve $c^{\uparrow}$ in the $(\Delta_\mu, \varepsilon)$ tipping diagram.

When $s = 0.5$, the basin instability boundary moves to $\mu_\ast^{\uparrow} = 2\sqrt{2} - 3 \approx -0.17$ or $\Delta_\mu \approx 0.83$, and the region of basin instability becomes non-negligible. As a result, the curve $c^{\uparrow}$ deviates from the case $s = 0$ in different ways. While R-tipping still does not occur, basin instability gives rise to a fold on $c^{\uparrow}$ and a range of shift magnitudes $\Delta_\mu$ with three critical rates. When the ‘tilt’ is increased to $s = 2$, the basin instability boundary moves to $\mu_\ast^{\uparrow} \approx -0.61$ or $\Delta_\mu \approx 0.39$. Now, in addition to B-tipping and a range of $\Delta_\mu$ with three critical rates, there is R-tipping for $\Delta_\mu < 1$. The tracking-tipping transition curve $c^{\uparrow}$ closely resembles the tipping diagram for the ecosystem model from Fig. 20. The R-tipping tongue at higher rates is the result of basin instability.

The ‘slow’ approach (and possibly lack of convergence) of the $c^{\uparrow}$ curves towards $H_e$ as $\varepsilon \to 0$ is the result of a surprising property of the slow passage through a Hopf bifurcation. Namely, the distance the solution tracks the unstable equilibrium past the bifurcation point is independent of the rate of parameter change and does not tend to zero as $\varepsilon \to 0$ [52–54]. In other words, the system tracks the unstable equilibrium past the bifurcation point for a noticeable amount of time, making it possible to turn around and avoid tipping even for vanishing rates of parameter change. The most noticeable difference from the ecosystem model is the absence of the “deep tracking wedge” at the intermediate rates. Instead, there is a characteristic kink on the $c^{\uparrow}$ curves near $\varepsilon = 10^{-2}$ in Fig. 24(a), possibly with multiple wiggles such as those shown in the inset of Fig. 21(b). The origin of the kink and the wiggles, as well as the scaling law for $c^{\uparrow}$ in the limit $\varepsilon \to 0$, are left for future study.

The agreement with the ecosystem model extends to “points of return” and...
“points of no return” as shown in Figs. 22(b) and 25(b1), where the tracking-tipping transition curve \( c^\uparrow \) is obtained for the monotone parameter shift (6.1). Interestingly, for sufficiently high ‘tilt’ parameter \( s \), a new region of “points of return tipping” appears in the diagram [Fig. 25(c1)] that is not present in the ecosystem model. This means that, in general, all four regions identified in the beginning of Sec. 6 can be present for a non-monotone passage through a subcritical Hopf bifurcation. What is more, the \( c^\uparrow \) and \( c^\downarrow \) curves need not approach each other like they do in the ecosystem model in Fig. 22(b). Finally, the rotational symmetry in the phase space of the (modified) Hopf normal form implies a symmetry in the basin instability boundaries

\[
\mu^+_{\ast}(s) = \mu^+_{\ast}(-s),
\]

meaning that the system has the same basin instability properties for \( s \) and \(-s\).

According to the R-tipping criterion from Sec. 4.2, given a suitable \( \mu(t) \) that increases over time, R-tipping for \( s \) and \(-s\) requires the same shift magnitude. Similarly, given a suitable \( \mu(t) \) that decreases over time, R-tipping for \( s \) and \(-s\) requires the same shift magnitude. Thus, we obtain the same tipping diagrams for \( s \) and \(-s\) in the left column of Fig. 25. For a fixed \( s \neq 0 \), R-tipping for an increasing \( \mu(t) \) requires a smaller shift magnitude than R-tipping for the decreasing \( \mu(-t) \). This is why the region of “points of return tipping” in Fig. 25(c1) is small.

6.2.2. Modified Saddle-Node Normal Form. To make comparisons with the other generic dangerous bifurcation of equilibria, consider a system in \( \mathbb{R} \) akin to the normal form of the saddle-node bifurcation [32, Sec.3.2]:

\[
\dot{x} = -(x - \mu) - \mu,
\]

where \( \mu \) is the bifurcation parameter and \( s \) is the ‘tilt’ parameter. The branches of stable \( e^+ \) and unstable \( e^- \) equilibria exist for \( \mu \leq 0 \) and are given by

\[
e^+(\mu, s) = \mu s + \sqrt{-\mu}, \quad \text{and} \quad e^-(\mu, s) = \mu s - \sqrt{-\mu},
\]
as shown in Fig. 23(c)–(d). The basin instability boundary is obtained by fixing \( \mu_- \) and solving

\[
e^+(\mu_-, s) = e^-(\mu_+, s),
\]

for \( \mu_\ast \). Region of basin instability exists for

\[
s < 0 \quad \text{or} \quad s > 1/\sqrt{-\mu_-},
\]

and its boundary is given by

\[
(\mu_\ast)^2 = -\left(\sqrt{-\mu_-} - \frac{1}{s}\right)^2.
\]

Now, consider the corresponding non-autonomous system

\[
\dot{x} = -(x - \mu(t)s)^2 - \mu(t),
\]

initialised at

\[
x(t_0) = e^+(\mu(t_0), s), \quad t_0 = \frac{1}{\varepsilon} \operatorname{sech}^{-1} (10^{-3}).
\]

Firstly, we analyse R-tipping for non-monotone \( \mu(t) \) given by Eq. (6.2) with \( \mu_- = -1, \Delta \mu > 0, c = 1, \tau = 0 \) and different values of \( s \) [Fig. 24(b)]. When \( s = 0 \), there is no
basin instability and R-tipping cannot occur. The only tipping that occurs for $s = 0$ is B-tipping for $\Delta_\mu > 1$. For $s = 0$, the tracking-tipping transition curve $c^\uparrow$ in the $(\Delta_\mu, \varepsilon)$ tipping diagram is in very good agreement with the critical “exceedance time” inverse square law

$$t_c \approx \frac{2}{\sqrt{\Delta_\mu + \mu_-}}.$$
derived in Ref. [21] for $s \varepsilon$ small enough. To demonstrate the agreement, we use
Eq. (6.3) with $c = 1$, $\tau = 0$ and $\mu_b = 0$ to rewrite the $t_c$ formula above in terms of $\varepsilon$
and $\Delta_\mu$:

\[
\varepsilon = \sqrt{\Delta_\mu + \mu_\varepsilon} \cdot \text{sech}^{-1}\left(\frac{-\mu_\varepsilon}{\Delta_\mu}\right),
\]

and plot condition (6.11) as a dashed red curve in Fig. 24(b); see Appendix B for
more details. However, for non-zero $s$ the the tracking-tipping transition curve $c^\dagger$ can
deviate from the inverse square law noticeably and qualitatively, especially at higher
rates $\varepsilon$. When $s = 2$, there is a basin instability boundary at $\mu_\varepsilon = -1/4$, meaning
that the stable equilibrium $e^\dagger$ is basin unstable for $\Delta_\mu > \mu_\varepsilon - \mu_- = 3/4$. Although
the tracking-tipping transition curve $c^\dagger$ deviates from the case $s = 0$ noticeably (up
to an order of magnitude in $\varepsilon$), the changes are quantitative and R-tipping still does
not occur. When the ‘tilt’ is increased to $s = 3$, the basin instability boundary moves
to $\mu_\varepsilon = -4/9$, meaning that $e^\dagger$ is basin unstable for $\Delta_\mu > \mu_\varepsilon - \mu_- = 5/9$. This
results in two significant changes to the tracking-tipping transition curve $c^\dagger$. Firstly,$
\Delta_\mu$ with three different critical rates. Secondly, in addition to B-tipping, there is an
R-tipping tongue for $\Delta_\mu < 1$. In contrast to the subcritical Hopf bifurcation, different
$c^\dagger$ curves appear to converge to $S_\varepsilon$ as $\varepsilon \rightarrow 0$. This is because the distance the solution
overshoots the saddle-node bifurcation point vanishes as the rate of parameter change
tends to zero [58, 59]. In other words, in the limit $\varepsilon \rightarrow 0$, the solution jumps off the
branch of stable equilibria at the bifurcation point with no time to turn around and
avoid tipping.

Apart from some differences at small $\varepsilon$ owing to the different character of the
bifurcation delay, the analysis of “points of return” and “points of no return” near
a saddle-node bifurcation reveals much similarity to the subcritical Hopf bifurcation
when $s > 0$ [Fig. 25(b2)]. However, the dynamics for $s < 0$ are rather different.
The striking difference for $s = -3$ is the large region of “points of return tipping”,
where there is R-tipping for non-monotone $\mu(t)$, but not for monotone increasing
$\mu(t)$ [Fig. 25(c2)]. This difference is a consequence of asymmetry in the (modified)
saddle-node normal form. To be more specific,

\[
\mu_\ast(s) \neq \mu_\ast(-s),
\]

meaning that the system has different basin instability properties for $s$ and $-s$. Ac-

According to the R-tipping criterion from Sec. 4.2, given a suitable $\mu(t)$ that increases
over time, the system is guaranteed to R-tip for $s > 0$, but not for $s < 0$. Conversely,$
given a suitable $\mu(t)$ that decreases over time, the system is guaranteed to R-tip for
$s < 0$, but not for $s > 0$. Thus, “points of return tipping” cannot occur for $s > 0$, and
are expected to occur for $s < 0$, which explains the diagrams for $s = 3$ and $s = -3$ in
Fig. 25(b2) and (c2).

6.2.3. Universal Properties of Non-monotone Passage Through a Danger-
ous Bifurcation. A comparison between the tracking-tipping transition curves
$c^\dagger$ for the modified subcritical Hopf [Fig. 24(a)] and saddle-node [Fig. 24(b)] nor-
mal forms reveals some universal qualitative properties of a non-monotone passage
through a dangerous bifurcation that are independent of the bifurcation type. In
both systems, the tracking-tipping transition curve $c^\dagger$ becomes S-shaped, gives rise
to three critical rates and develops an R-tipping tongue as the ‘tilt’ parameter $s$ is
increased. On the other hand, there are differences between the two systems that are also worth pointing out. Multiple critical rates and R-tipping are achieved for a smaller ‘tilt’ parameter $s$ in the modified Hopf normal form, whereas the approach of $c^\downarrow$ towards the bifurcation as $\varepsilon \to 0$ is much faster and follows a different scaling law in the modified saddle-node normal form. What is more, owing to its its different basin instability properties for $s$ and $-s$ (6.12), a saddle-node bifurcation may give rise to a larger region of “points of return tipping”.

7. Conclusion. In this paper we analyse nonlinear tipping phenomena using examples of an ecological model [7] and modified saddle-node and subcritical Hopf normal forms with smooth parameter shifts. The mathematical work is motivated and inspired by two scientific concerns. One is Article 2 of the United Nations Framework Convention for Climate Change (UNFCCC) [23] highlighting two critical factors for real-world tipping points: critical levels and critical rates (time frames) of changing environmental conditions. This was later extended to become the Kyoto Protocol [60] and the current Paris Agreement [61]. The other is the question of whether tipping can be prevented by a parameter trend reversal. We combine classical bifurcation analysis with the concept of basin instability to give new insight into critical rates, uncover non-trivial effects arising from the interplay between critical levels (B-tipping) and critical rates (R-tipping), and extend the existing literature on preventing tipping by a parameter trend reversal.

We begin with classical bifurcation analysis of the corresponding autonomous ecosystem model with fixed in time parameters and identify a codimension-three degenerate Bogdanov-Takens bifurcation as the organising centre for B-tipping and the source of a dangerous subcritical Hopf bifurcation. We give testable criteria for B-tipping in the non-autonomous system in terms of parameter paths that cross a subcritical Hopf in the corresponding autonomous system. Next, we perform basin instability analysis to reveal and give testable criteria for R-tipping in the non-autonomous system in terms of parameter paths that do not cross any bifurcation in the corresponding autonomous system. Finally, we produce a single diagram encompassing criteria for both B-tipping and R-tipping by superimposing regions of basin instability on a classical two-parameter bifurcation diagram of the plant growth rate vs. the herbivore mortality rate. This approach gives new insight into system stability, beyond traditional bifurcation analysis and adiabatic effects of a parameter change, as it gives basic information on R-tipping and captures non-trivial non-adiabatic effects of a parameter change. Most importantly, it guides tipping analysis in the non-autonomous system including genuine non-autonomous instabilities.

In the non-autonomous system with time-varying parameters we obtain tipping diagrams in the plane of the rate and magnitude of parameter shift and show that:

- R-tipping transitions in the tipping diagram correspond to canard-like solutions in the phase space that, rather surprisingly, track a moving unstable state.
- R-tipping transition curves in the tipping diagram for non-monotone parameter shifts that cross a basin instability boundary alone and then turn around can form R-tipping tongues with two critical rates. This means that the system switches from tracking to tipping and back to tracking again as the rate of the parameter shift increases. R-tipping tongues are reminiscent of resonance tongues in the sense of enhanced response to optimally timed external inputs.
- The interplay between critical levels and critical rates (or between B-tipping
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and R-tipping) for non-monotone parameter shifts that cross a basin instability boundary and a dangerous bifurcation and then turn around gives rise to an S-shaped tipping-tracking transition curve in the tipping diagram with one critical level and multiple critical rates. This means the system exhibits inverted behaviour to an R-tipping tongue and switches from tipping to tracking and back to tipping again as the rate of the parameter shift increases.

- Given a monotone parameter shift and its non-monotone reversal, tipping diagrams can be partitioned into points of tracking, points of return where tipping can be prevented by the reversal, points of no return where tipping cannot be prevented by the reversal, and points of return tipping where tipping is inadvertently induced by the reversal. This partitioning provides an alternative way to categorise tipping phenomena.

Our results on the ecosystem model give new insight into the sensitivity of ecosystems to the magnitudes and rates of environmental change. More generally, the method of superimposing regions of basin instability on traditional bifurcation diagrams can be extended to regions of threshold instability for tipping thresholds that do not separate the phase space into different basins of attraction [26, 48]. Such approach would capture, in addition to B-tipping due to dangerous bifurcations, both irreversible and transient reversible R-tipping, and facilitate systematic in-depth analysis of tipping phenomena in any nonlinear system. This is evidenced further by a comparison of the ecosystem model with the modified saddle-node and subcritical Hopf normal forms that reveals some universal features of non-monotone parameter shifts that cross a basin instability boundary and a dangerous bifurcation and then turn around.

Acknowledgments. P.E.O’K thanks M. Mortell for useful discussions on singular perturbations. S.W. has received funding from the the CRITICS Innovative Training Network via the European Union’s Horizon 2020 research and innovation programme under Grant Agreement No. 643073.

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Appendix A. Asymptotic Approximations for $P_{\text{opt}}$, $P_3$ and $P_4$. The optimal plant biomass $P_{\text{opt}}$ defines the maximum herbivore growth (2.4) and satisfies

$$
\epsilon P_{\text{opt}}^3 + \epsilon^2 P_{\text{opt}} - a^2 = 0,
$$

where $\epsilon = (b + b_c)/2$ is the small parameter. We rewrite this singular perturbation problem for $P_{\text{opt}}$ as a regular perturbation problem for the stretched variable $\tilde{P}_{\text{opt}} = \epsilon \frac{1}{3} P_{\text{opt}}$:

$$
\tilde{P}_{\text{opt}}^3 + \epsilon^2 a^2 \tilde{P}_{\text{opt}} - a^2 = 0,
$$

and seek solutions in the form of an asymptotic expansion

$$
\tilde{P}_{\text{opt}} = \tilde{P}_{\text{opt},0} + \epsilon \frac{1}{3} \tilde{P}_{\text{opt},1} + \epsilon^2 \tilde{P}_{\text{opt},2} + O(\epsilon^3).
$$

Substituting the expansion into Eq. (A.2), and successively equating terms of the same order in $\epsilon$ on both sides gives

$$
\tilde{P}_{\text{opt}} = a^2 - \frac{1}{3} \epsilon^2 a^4 + \frac{1}{81} \epsilon^2 a^8 + O(\epsilon^5).
$$

Going back to the original variable gives

$$
P_{\text{opt}} = \epsilon^{-\frac{1}{3}} \left(a^2 - \frac{1}{3} \epsilon^2 a^4 + \frac{1}{81} \epsilon^2 a^8 + O(\epsilon^5)\right),
$$

which can be rewritten in terms of the original parameters

$$
P_{\text{opt}} = \left(\frac{2a^2}{b + b_c}\right)^{\frac{1}{3}} + \frac{1}{3} \left(\frac{4(b + b_c)}{2}\right)^{\frac{1}{3}} + \frac{1}{81} \left(\frac{8(b + b_c)}{2}\right)^{\frac{1}{3}} + O\left(\frac{b + b_c}{2}\right).
$$

The $P$-components $P_3$ of the herbivore-dominating equilibrium $e_3$ and $P_4$ of the plant-dominating equilibrium $e_4$ are the positive roots of Eq. (3.4), which can be written as

$$
P^2 e^{-\epsilon P} - P^2 \tilde{c} - \tilde{d} = 0,
$$

in terms of $\tilde{c} = m/(E c_{\text{max}})$, $\tilde{d} = a^2 \tilde{c}$ and the small parameter $\epsilon = b + b_c$.

The root $P_3$ can be approximated by regular perturbations. We first expand $e^{-\epsilon P_3}$ so that Eq.(A.3) becomes

$$
- \epsilon P_3^3 + (1 - \tilde{c}) P_3^2 - \tilde{d} + O(\epsilon) = 0,
$$

and seek solutions in the form of an asymptotic expansion

$$
P_3 = P_{3,0} + \epsilon P_{3,1} + O(\epsilon^2).
$$

Substituting the expansion into Eq.(A.4) and successively equating terms of the same order in $\epsilon$ on both sides gives

$$
P_3 = \sqrt{\frac{\tilde{d}}{1 - \tilde{c}}} + \frac{\tilde{d}}{2(1 - \tilde{c})^2} \epsilon + O(\epsilon^2),
$$
which can be rewritten in terms of the original parameters

\[ P_3 = \sqrt{\frac{a^2 m}{E c_{\text{max}} - m} + \frac{E c_{\text{max}} a^2 m}{2(E c_{\text{max}} - m)^2}(b + b_c) + O((b + b_c)^2)}. \]

The root \( P_4 \) requires singular perturbations. To approximate \( P_4 \), we first reframe Eq. (3.4) as a regular perturbation problem for the rescaled variable \( \tilde{P}_4 = \epsilon P_4 \):

\begin{equation}
(A.5) \quad (e^{-\tilde{P}_4} - \tilde{c})\tilde{P}_4^2 = \epsilon^2 \tilde{d},
\end{equation}

and seek solutions in the form of an asymptotic expansion for \( \tilde{P}_4 \):

\[ \tilde{P}_4 = \tilde{P}_{4,0} + \epsilon \tilde{P}_{4,1} + \epsilon^2 \tilde{P}_{4,2} + O(\epsilon^3). \]

Substituting the expansion into Eq. (A.5) and successively equating terms of the same order in \( \epsilon \) on both sides gives

\[ \tilde{P}_4 = \ln \frac{1}{\tilde{c}} - \epsilon^2 \frac{\tilde{d}}{\tilde{c}(\ln \frac{1}{\tilde{c}})^2} + O(\epsilon^3), \]

which can be rewritten in terms of the original parameters

\[ P_4 = \frac{1}{b + b_c} \left( \ln \frac{E c_{\text{max}}}{m} - \frac{a^2(b + b_c)^2}{(\ln \frac{E c_{\text{max}}}{m})^2} + O((b + b_c)^3) \right). \]

Appendix B. Inverse Square Law for Preventing Tipping Across Saddle-Node Bifurcation from Ref. [21].

Ritchie et al. [21] considered the problem where the system parameter \( \mu \) drifts past the saddle-node bifurcation at \( \mu = \mu_b \) and then turns around. For \( \varepsilon \) small enough, they obtained an asymptotic formula to prevent tipping in terms of the distance \( \Delta_\mu + \mu_- \) the parameter travels past the bifurcation and the time \( t_e \) the system spends past the bifurcation. More specifically, their criterion corresponding to our tracking-tipping transition curve \( c \downarrow \) has the form [21, Eq.(2.10)]:

\begin{equation}
(B.1) \quad d^b/(\Delta_\mu + \mu_-)t_e^2 = 16,
\end{equation}

where

\[ d^b = \lim_{\mu \nearrow \mu_b} \frac{[\lambda(\mu)]^2}{\mu_b - \mu}, \]

is the system-dependent factor and \( \lambda(\mu) \) is the leading eigenvalue of the stable equilibrium.

To compare our computations of \( c \downarrow \) with the result derived in Ref. [21] we obtain the exceedence time \( t_e \) for the non-monotone parameter shift Eq. (6.2) with \( \mu_- = -1, \Delta_\mu > 0, c = 1 \) and \( \tau = 0 \):

\[ t_e = \frac{2}{\varepsilon} \text{sech}^{-1}(-\mu_-/\Delta_\mu) > 0, \]

note that the modified saddle-node normal form (6.7) has \( \mu_b = 0 \) and \( \lambda(\mu) = -2\sqrt{-\mu} \) which give

\[ d^b = \lim_{\mu \nearrow \mu_b} \frac{[\lambda(\mu)]^2}{\mu_b - \mu} = 4, \]

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and rewrite Eq. (B.1) in terms of \( \varepsilon \) and \( \Delta \mu \):

\[
\varepsilon = \sqrt{\Delta \mu + \mu - \text{sech}^{-1} \left( \frac{-\mu - \Delta \mu}{\Delta \mu} \right)}.
\]

Condition (B.2) is plotted as a red dashed curve in Fig. 24(b).

REFERENCES


