Introduction to Inequalities

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Basic definitions

Suppose a and b are real numbers.
Then $a > b$ means a is strictly greater than b;
also $a < b$ means a is strictly less than b.
We write $a \geq b$ to mean “$a > b$ or $a = b$"
and we write $a \leq b$ to mean “$a < b$ or $a = b$".

(i) Logical implication and equivalence:
We consider statements to which we can assign the values true or false.
Many mathematical statements are of this type.
For example, statement A: $2+3=5$ is true, whereas statement B: $2\times 3=5$ is false.
Statements such as “Justin Bieber is cool" do not fall into this category since they are subjective and cannot be proven mathematically.

An important idea in mathematics is logical implication.
We write $A \Rightarrow B$ (read “A implies B")
to mean “if A is true then B is true"
(often read “if A then B").

For example, $x = 2 \Rightarrow x^2 = 4$ is true,
but $x^2 = 4 \Rightarrow x = 2$ is false (since $(-2)^2 = 4$ also).

If the implication holds both ways,
that is $A \Rightarrow B$ and $B \Rightarrow A$,
we write $A \Leftrightarrow B$, or sometimes A iff B;
this is read as "A if and only if B",
or sometimes "A is equivalent to B".
For example, $2x + 3 = 5 \Leftrightarrow x = 1$ is true,
since the statements $2x + 3 = 5 \Rightarrow x = 1$ and
$x = 1 \Rightarrow 2x + 3 = 5$ are both true.
Note that $x = 2 \Leftrightarrow x^2 = 4$ is false,
since one of the implications is false.
Another example of this:
the statement "$x > 0$ iff $x^2 > 0$ for all real numbers x"
is false since $9 > 0$ does not imply $-3 > 0$.

Find the flaw in the following:
First note that $(x + x)(x - x) = x^2 - x^2 = x(x - x)$
is true for all x.
Now cancelling a common factor $(x + x)\cancel{(x - x)} = x\cancel{(x - x)}$

$\Leftrightarrow x + x = x$

$\Leftrightarrow 2x = x$

$\Leftrightarrow 2 = 1.$

The flaw:
$(x + x)(x - x) = x(x - x) \not\Rightarrow x + x = x$,
since it requires division by zero, that is $(x - x)$.

(ii) Logical 'and' and 'or':

We say "A and B" is true if both A and B are true.

For example, $x^2 + y^2 = 0 \Rightarrow x = 0$ and $y = 0$.

We say "A or B" is true if at least one of A and B is true (they may both be true).

For example, $xy = 0 \Leftrightarrow x = 0$ or $y = 0$.

Another example is $x^2 = 9 \Leftrightarrow (x = 3)$ or $(x = -3)$.

We sometimes write this as $x = \pm 3$
(read x equals plus or minus 3).

We can combine and/or, as in the following example:

$(x^2 - 1)^2 + (y^2 - 4)^2 = 0 \Leftrightarrow x^2 = 1$ and $y^2 = 4$

$\Leftrightarrow (x = 1$ or $x = -1)$ and $(y = +2$ or $y = -2)$

$\Leftrightarrow ((x, y) = (1, 2))$ or $((x, y) = (1, -2))$ or $((x, y) = (-1, 2))$ or $((x, y) = (-1, -2))$.

An important example that we will use later follows:

Suppose $a > 0$. Then
$x^2 < a^2 \Leftrightarrow x^2 - a^2 < 0 \Leftrightarrow (x-a)(x+a) < 0$

$\Leftrightarrow ((x-a<0)$ and $(x+a>0))$ or $((x-a>0)$ and $(x+a<0))$

$\Leftrightarrow (-a<x<a)$ or $(x>a$ and $x<-a) \Leftrightarrow -a<x<a$.

Note that the second part of the 'or' is false (why?)

and so the statement is true iff the first part of the 'or' is true.

We summarise the example as $x^2 < a^2 \Leftrightarrow -a < x < a$.

Manipulating inequalities:

$a > b \Leftrightarrow a - b > 0 \Leftrightarrow b - a < 0$

$\Leftrightarrow b < a \Leftrightarrow -b > -a \Leftrightarrow -a < -b$.

Analogous statements can be written for $\geq, <$ and $\leq$.

Notice how multiplying by -1 reverses the sense of the inequality, so for example $a \leq b \Leftrightarrow -a \geq -b$.

If $a > b$ and $k > 0$, then $ka > kb$.

If $a > b$ and $k < 0$, then $ka < kb$.

If $a > b > 0$, then $\dfrac{1}{a} < \dfrac{1}{b}$.

The square root function

If $x \geq 0$, then $\sqrt{x}$ is the unique non-negative number whose square is x.

That is, $y = \sqrt{x}$ means $y^2 = x$ and $y \geq 0$.

Sometimes $\sqrt{x}$ is written as $x^{\frac{1}{2}}$, which is consistent with the Law of Exponents since $x^{\frac{1}{2}} x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$.

Some properties of square roots:

(i) $\sqrt{x}$ is not defined for $x < 0$.

(ii) $\sqrt{x} \geq 0$.

(iii) $\sqrt{xy} = \sqrt{x}\sqrt{y}$, if $x, y \geq 0$.

(Warning: $\sqrt{(-3)(-5)}$ is defined even though $\sqrt{-3}$ and $\sqrt{-5}$ are not!)

(iv) $\sqrt{x} < x$, if $x > 1$; $\sqrt{x} > x$, if $0 < x < 1$.

(v) $\sqrt{x^2} = x$, if $x \geq 0$; $\sqrt{x^2} = -x$, if $x \leq 0$.

The absolute value function

We define the absolute value of x to be $|x| = \sqrt{x^2}$.

In the U.K. it is sometimes called the “modulus of x".

An equivalent definition is

$|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$.

Some properties of absolute values:

$|-x| = |x|$

$|x|^2 = x^2$

$|xy| = |x||y|$

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ (for $y \neq 0$)

$|x^n| = |x|^n$ for positive integers n.

Trivially $|x| = 0 \Leftrightarrow x = 0$.

Also $|x| = a \Leftrightarrow x = \pm a$.

We showed above that if $a > 0$,

then $x^2 < a^2$ iff $-a < x < a$.

Consequently $|x| < a$ iff $-a < x < a$.

We can think of the numbers x, with $|x| < a$, as the points that are within a distance a of 0.

Similarly if $a \geq 0$, then $|x| \leq a$ iff $-a \leq x \leq a$.

Some simple inequalities:

$x^2 \geq 0$, for all real numbers x, with equality iff $x = 0$.

$|x| \geq 0$, for all real numbers x, with equality iff $x = 0$.

$x \leq |x|$, for all real numbers x, with equality iff $x \geq 0$.

$-x \leq |x|$, for all real numbers x, with equality iff $x \leq 0$.

The first one can be generalised to a sum of squares to give: $a_1^2 + a_2^2 + \cdots + a_n^2 \geq 0$

and equals zero iff $a_1 = a_2 = \cdots = a_n = 0$.

Example

Suppose that x, y, z are positive real numbers. Prove that $x^2 + y^2 + z^2 \geq xy + yz + zx,$ with equality iff $x = y = z$.

It suffices to prove that

$2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx \geq 0$.

We can write this as

$(x^2 - 2xy + y^2) + (y^2 - 2yz + z^2) + (z^2 - 2zx + x^2) \geq 0$.

So it suffices to prove that

$(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0,$ which is true since it is a sum of squares, so the result follows. Equality occurs iff $x = y$ and $y = z$ and $z = x$, that is $x = y = z$.

The Triangle Inequality:

For all real numbers x and y, $|x + y| \leq |x| + |y|$.

Proof of the Triangle Inequality:

$|x + y| \leq |x| + |y|$

$\Leftrightarrow |x + y|^2 \leq (|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x||y|$

$\Leftrightarrow (x + y)^2 \leq x^2 + y^2 + 2|x||y|$

$\Leftrightarrow x^2 + y^2 + 2xy \leq x^2 + y^2 + 2|x||y|$

$\Leftrightarrow xy \leq |x||y| \Leftrightarrow xy \leq |xy|$.

But $z \leq |z|$ for all z, so the final statement is true and the result follows.

Some inequalities that reduce to a square being non-negative:

(a) $x + \frac{1}{x} \geq 2$ for all $x > 0$, with equality iff $x = 1$.

Proof: Suppose $x > 0$.

Then $x + \frac{1}{x} \geq 2 \Leftrightarrow x^2 + 1 - 2x \geq 0 \Leftrightarrow (x - 1)^2 \geq 0$, which is true.

For equality: $x + \frac{1}{x} = 2 \Leftrightarrow x^2 + 1 - 2x = 0 \Leftrightarrow x = 1$.

Analogously one can show that $x + \frac{1}{x} \leq -2$ for all $x < 0$, with equality iff $x = -1$.

(b) $\sqrt{xy} \leq \frac{x+y}{2}$ for all $x, y > 0$, with equality iff $x = y$.

(This is the $n = 2$ case of the *Arithmetic-Geometric Mean inequality*).

Proof: Suppose $x, y > 0$. Then

$$\sqrt{xy} \leq \frac{x+y}{2} \Leftrightarrow xy \leq \left(\frac{x+y}{2}\right)^2 \Leftrightarrow 4xy \leq x^2 + 2xy + y^2$$

$\Leftrightarrow 0 \leq x^2 - 2xy + y^2 = (x-y)^2$, which is true.

For equality, in a similar way

$$\sqrt{xy} = \frac{x+y}{2} \Leftrightarrow xy = \left(\frac{x+y}{2}\right)^2$$

$\Leftrightarrow 0 = x^2 - 2xy + y^2 \Leftrightarrow x = y$.

(c) $x^2 - xy + y^2 \geq xy$ for all x, y,

with equality iff $x = y$.

This reduces to $x^2 - 2xy + y^2 = (x-y)^2 \geq 0$.

(d) $\left(\frac{x+y}{2}\right)^2 \leq \frac{x^2+y^2}{2}$ for all $x, y,$ with equality iff $x = y$.

Simple algebra shows that this reduces to $(x-y)^2 \geq 0$.

This inequality is a simple case of the *Power-Mean inequality* and is sometimes written as $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$.

(e) $\frac{(x+y)^2}{a+b} \leq \frac{x^2}{a} + \frac{y^2}{b}$ for all x, y and with $a, b > 0$

(sometimes called the SQ inequality).

Proof: Multiply both sides by $ab(a+b)$ and bring everything to the right. The right side reduces to $(bx-ay)^2 \geq 0$.

(f) $ax + by \leq \sqrt{a^2+b^2}\sqrt{x^2+y^2}$ for all real a, b, x, y

(called the Cauchy-Schwarz inequality).

Proof: Square and combine both sides. It again reduces to $(bx-ay)^2 \geq 0$.

Example 0

Prove that $\sqrt{\frac{3}{4}} + \sqrt{\frac{4}{3}} > 2$.

Square it to obtain the equivalent statement $\frac{3}{4} + 2 + \frac{4}{3} > 4$, which is easily shown.

Alternatively use $x + \frac{1}{x} > 2$, for $x > 1$, with $x = \sqrt{\frac{4}{3}}$.

Alternatively use $x + y > 2\sqrt{xy}$, for $x, y > 0$ and $x \neq y$.

Here $x = \sqrt{\frac{3}{4}}$ and $y = \sqrt{\frac{4}{3}}$, so $\sqrt{xy} = \sqrt{\sqrt{\frac{3}{4}}\sqrt{\frac{4}{3}}} = \sqrt[4]{1} = 1$.

Example 1

Prove that the sum of the sides of a right angled-triangle never exceeds $\sqrt{2}$ times the hypotenuse of the triangle.

Proof: Suppose the sides are a and b and the hypotenuse is c.

Then from Pythagoras's Theorem,
$c^2 = a^2 + b^2 \geq \frac{(a+b)^2}{2}$, so $a + b \leq c\sqrt{2}$.

Example 2

(a) Prove that, of all the rectangles with the same perimeter P, the square encloses the greatest area.

(b) Prove that, of all the rectangles with the same area A, the one with the smallest perimeter is a square.

Suppose the sides of the rectangle are $a, b > 0$.

The perimeter is $P = 2a + 2b$ and the area is $A = ab$.

From the inequality (b) above $2\sqrt{ab} \leq a + b$, with equality iff $a = b$.
So $4\sqrt{A} \leq P$, with equality iff the rectangle is a square.

In part (a), P is fixed, so the greatest area is $\frac{P^2}{16}$ and this occurs iff the rectangle is a square.

In part (b), A is fixed so the least perimeter is $4\sqrt{A}$ and this occurs iff the rectangle is a square.

Example 3

Suppose x, y and z are positive real numbers. Prove that $\frac{4}{x+y} \leq \frac{1}{x} + \frac{1}{y}$ and find when equality occurs.
Hence prove that $\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \leq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$ and find when equality applies.

$$\frac{4}{x+y} \leq \frac{1}{x} + \frac{1}{y}$$
$$\Leftrightarrow \quad \frac{4}{x+y} \leq \frac{x+y}{xy}$$

$\Leftrightarrow \; 4xy \leq (x+y)^2$

$\Leftrightarrow \; 4xy \leq x^2 + 2xy + y^2$

$\Leftrightarrow \; 0 \leq x^2 - 2xy + y^2$

$\Leftrightarrow \; 0 \leq (x-y)^2,$ which is true.

Equality occurs if and only if $x = y$.

To prove the second inequality, we use this inequality three times.

$$\frac{4}{x+y} + \frac{4}{y+z} + \frac{4}{z+x} \leq \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{y} + \frac{1}{z} + \frac{1}{z} + \frac{1}{x}\right).$$

That is $\frac{4}{x+y} + \frac{4}{y+z} + \frac{4}{z+x} \leq \frac{2}{x} + \frac{2}{y} + \frac{2}{z}$.

Dividing by 4, the result follows.

Equality occurs if and only if $x = y$ and $y = z$ and $z = x$. That is $x = y = z$.

Statement of the Arithmetic-Geometric Mean Inequality

Suppose we have n strictly positive real numbers $a_1, a_2, \ldots, a_n > 0$.

We define the **arithmetic mean** of these numbers to be
$AM = \frac{a_1 + a_2 + \cdots + a_n}{n}$
and we define the **geometric mean** to be
$GM = \sqrt[n]{a_1 a_2 \cdots a_n}$.

Then the AGM states that

$AM \geq GM$ with equality iff $a_1 = a_2 = \cdots = a_n$.

The proof of the general case is not given (though we proved the $n = 2$ case before when we showed $\sqrt{a_1 a_2} \leq \frac{1}{2}(a_1 + a_2)$, with equality iff $a_1 = a_2$).

For $n = 3$, it states:
If $a, b, c > 0$, then $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$,
with equality iff $a = b = c$.

Proof 1: Write $a = x^3$, $b = y^3$ and $c = z^3$, where $x, y, z > 0$.

It suffices to prove that $x^3 + y^3 + z^3 \geq 3xyz$, with equality iff $x = y = z$. Observe that
$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
$= \frac{1}{2}(x + y + z)((x - y)^2 + (y - z)^2 + (z - x)^2) \geq 0$
and equality occurs iff $x = y = z$.

Proof 2: Again we prove that $x^3+y^3+z^3 \geq 3xyz$, with equality iff $x=y=z$.

Observe that

$x^3+y^3 = (x+y)(x^2-xy+y^2) \geq (x+y)xy = x^2y+xy^2,$

with equality iff $x=y$.

Similarly $y^3+z^3 \geq y^2z+yz^2$ and $z^3+x^3 \geq z^2x+zx^2$.

Adding these three inequalities gives

$$2(x^3+y^3+z^3) = x^2y+xy^2+y^2z+yz^2+z^2x+zx^2$$
$$= x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)$$
$$\geq x(2yz)+y(2zx)+z(2xy) = 6xyz.$$

The result follows and clearly equality occurs iff $x=y=z$.

Example 1

Prove that if $a,b,c>0$, then $(a+b)(b+c)(c+a) \geq 8abc$, with equality iff $a=b=c$.

Proof 1: Apply the AGM for two numbers (three times) to get

$a+b \geq 2\sqrt{ab}$ and $b+c \geq 2\sqrt{bc}$ and $c+a \geq 2\sqrt{ca}$,

with equality iff $a=b$ and $b=c$ and $c=a$.

Taking the product of the three inequalities gives

$(a+b)(b+c)(c+a) \geq 2^3\sqrt{abbcca} = 8abc,$

with equality iff $a=b=c$.

Proof 2: Expand out the left side and subtract $2abc$ to show that the inequality is equivalent to

$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \geq 6abc$.

Apply the AGM to the six numbers

$a^2b,\ ab^2,\ b^2c,\ bc^2,\ c^2a,\ ca^2$ to get

$$\frac{a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2}{6} \geq \sqrt[6]{a^6b^6c^6} = abc$$

which gives the result.

Equality occurs iff the six numbers are equal iff $a = b = c$.

Example 2

Prove that if $a > b > 0$, then $a^5 - 5a^3b^2 + 5a^2b^3 - b^5 > 0$.

Proof: $a^5 - 5a^3b^2 + 5a^2b^3 - b^5 > 0 \Leftrightarrow a^5 - b^5 > 5a^3b^2 - 5a^2b^3$

$\Leftrightarrow (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) > 5(a-b)a^2b^2$

$\Leftrightarrow \dfrac{a^4 + a^3b + a^2b^2 + ab^3 + b^4}{5} > a^2b^2$ (since $(a-b) > 0$).

The left side is the arithmetic mean AM of the five numbers $a^4, a^3b, a^2b^2, ab^3, b^4$.

Their geometric mean is $GM = \sqrt[5]{a^{10}b^{10}} = a^2b^2$, which is the right side.

Since equality does not occur, $AM > GM$ and so the result follows.

Note that this example can be generalised to show that $a^n - b^n > n(a-b)(ab)^{\frac{n-1}{2}}$.

Example 3

Prove that if $a, b, c, d > 0$ *and* $abcd = 1$*, then* $a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10$.

Proof: Apply the AGM to the ten terms of the sum on the left side.

Their product is $a^5 b^5 c^5 d^5 = 1$.

So $AM \geq GM$ gives $\dfrac{LHS}{10} \geq \sqrt[10]{1} = 1$ and the result follows.

Example 4

Prove that $n! < \left(\frac{n+1}{2}\right)^n$*.*

Recall $n! = n(n-1)(n-2)\cdots(3)(2)(1)$.

Apply the AGM to the numbers $1, 2, 3, \ldots, n-1, n$.

$$AM = \frac{1+2+3+\cdots+(n-1)+n}{n} = \frac{\frac{1}{2}n(n+1)}{n} = \frac{n+1}{2},$$

by summing the arithmetic series.

Next $GM = (1.2.3\ldots(n-1).n)^{\frac{1}{n}} = (n!)^{\frac{1}{n}}$.

So $AM > GM \Rightarrow \frac{n+1}{2} > (n!)^{\frac{1}{n}}$.

Taking the nth power of both sides proves the result.

Example 5

Prove that $2(a+\frac{1}{a})^2 + 2(b+\frac{1}{b})^2 \geq 25$

for all $a, b > 0$*, with* $a+b=1$*.*

Proof: Since $2x^2+2y^2 \geq (x+y)^2$, we get

$$2(a+\frac{1}{a})^2 + 2(b+\frac{1}{b})^2 \geq (a+\frac{1}{a}+b+\frac{1}{b})^2$$

$$= (a+b+\frac{a+b}{ab})^2 = (1+\frac{1}{ab})^2.$$

Now by the AGM, $\sqrt{ab} \leq \frac{a+b}{2} = \frac{1}{2}$,

so since $ab > 0$, this gives $\frac{1}{ab} \geq 4$,

so $(1+\frac{1}{ab})^2 \geq (1+4)^2 = 25$ and the result follows.

Example 6

Consider an obtuse triangle $\triangle ABC$, where $\theta = \angle ABC > 90^\circ$.

Prove that $|AC|^6 \geq -54|AB|^3|BC|^3 \cos\theta$

and find for which triangles equality occurs.

Proof: The Cosine Rule gives
$|AC|^2 = |AB|^2 + |BC|^2 - 2|AB||BC|\cos\theta$.

Note that each of the three terms on the right is strictly positive, so the AGM can be applied to the them, giving

$$\frac{|AB|^2 + |BC|^2 - 2|AB||BC|\cos\theta}{3}$$
$$\geq \sqrt[3]{|AB|^2|BC|^2(-2|AB||BC|\cos\theta)}$$

with equality iff $|AB|^2 = |BC|^2 = -2|AB||BC|\cos\theta$;

that is $\frac{1}{3}|AC|^2 \geq \sqrt[3]{-2|AB|^3|BC|^3\cos\theta}$
with equality iff $|AB| = |BC|$ and $\cos\theta = -\frac{1}{2}$;

that is $|AC|^6 \geq -54|AB|^3|BC|^3\cos\theta$ with equality iff $\triangle ABC$ is isosceles, with $|AB| = |BC|$, and $\angle ABC = 120^\circ$.

Example 7

Suppose that $a, b, c > 0$. *Prove that*

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) \geq 8$$

and find when equality occurs.

Applying the AGM to each pair we get

$$a + \frac{1}{b} \geq 2\sqrt{\frac{a}{b}}, \qquad b + \frac{1}{c} \geq 2\sqrt{\frac{b}{c}}, \qquad c + \frac{1}{a} \geq 2\sqrt{\frac{c}{a}}.$$

Taking the product of the left sides we get

$$LHS \geq 8\sqrt{\frac{a}{b}}\sqrt{\frac{b}{c}}\sqrt{\frac{c}{a}} = 8\sqrt{\frac{abc}{bca}} = 8.$$

Equality occurs when $a = \frac{1}{b}, \qquad b = \frac{1}{c}, \qquad c = \frac{1}{a}.$

That is $a = b = c = 1$.

Example 8

Suppose that $a, b, c > 0$ *and that* $a + b + c = 1$.

Prove that $ab + bc + ca \geq 9abc$

and find when equality occurs.

Applying the AGM to the left side gives

$$ab + bc + ca \geq 3\sqrt[3]{abbcca} = 3\sqrt[3]{a^2b^2c^2}.$$

Also by the AGM, $a + b + c \geq 3\sqrt[3]{abc}$, so $1 \geq 3\sqrt[3]{abc}$.

So

$$ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2}(1) \geq 3\sqrt[3]{a^2b^2c^2}(3\sqrt[3]{abc}) = 9abc.$$

Equality occurs when $ab = bc = ca$, that is $a = b = c = \frac{1}{3}$.

Example 9

Suppose that $a, b, c > 0$. *Prove that*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c$$

and find when equality occurs.

Solution 1:

By the AGM, $\dfrac{a^2}{b} + b \geq 2\sqrt{\dfrac{a^2}{b}b} = 2a$.

Likewise $\dfrac{b^2}{c} + c \geq 2b$ and $\dfrac{c^2}{a} + a \geq 2c$.

Adding these three inequalities we get

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + a + b + c \geq 2a + 2b + 2c,$$

which gives the desired inequality.

Equality occurs iff $\dfrac{a^2}{b} = b, \quad \dfrac{b^2}{c} = c, \quad \dfrac{c^2}{a} = a,$

that is $a = b = c$.

Solution 2: Observe that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{a^2 - ab + b^2}{b} + \frac{b^2 - bc + c^2}{c} + \frac{c^2 - ca + a^2}{a}.$$

Since for all x, y we have $x^2 - xy + y^2 \geq xy$, it follows that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{ab}{b} + \frac{bc}{c} + \frac{ca}{a} = a + b + c.$$

Equality occurs iff $a = b$ and $b = c$ and $c = a$, that is $a = b = c$.

Arithmetic-Geometric-Harmonic Mean Inequalities

Suppose $a_1, a_2, \ldots, a_n > 0$. We define the *harmonic mean* of these numbers to be

$$HM = \left(\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\right)^{-1}.$$

Then $GM \geq HM$, with equality iff $a_1 = a_2 = \cdots = a_n$.

To prove this, apply the AGM inequality to the (positive) numbers $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$ and cross multiply (check this works).

We can combine this result with the AGM inequality to get the *Arithmetic-Geometric-Harmonic Mean Inequalities:*

$AM \geq GM \geq HM$ with equality iff $a_1 = a_2 = \cdots = a_n$.

Since $GM \geq HM$ is so easily derived from $AM \geq GM$ it is rarely used in its own right.

However we give two examples. Both use the *harmonic series* $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$.

Example 1

Prove that $H_n > n(n+1)^{\frac{1}{n}} - n$, *for* $n > 1$.

Proof: $n + H_n = 1 + 1 + 1 + \frac{1}{2} + 1 + \frac{1}{3} + \cdots + 1 + \frac{1}{n}$ (splitting up n into ones).

$n + H_n = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n}$.

Now apply $AM \geq GM$ to these numbers to find

$$\frac{n + H_n}{n} > \sqrt[n]{\left(\frac{2}{1}\right) \cdot \left(\frac{3}{2}\right) \cdot \left(\frac{4}{3}\right) \cdots \left(\frac{n+1}{n}\right)}$$

$$= \left(\frac{n+1}{1}\right)^{\frac{1}{n}}$$ (this is called a *telescoping product*).

So $n + H_n > n(n+1)^{\frac{1}{n}}$, which proves the result.

Example 2: *Prove that* $n - H_n > (n-1)(n)^{-\frac{1}{n-1}}$, *for* $n > 2$.

The proof is left as an exercise but repeats many of the ideas from the previous example.

First, split the n into ones and form a sum of $n - 1$ terms, to which the AGM is applied, giving another telescoping product.

The Power Mean

Given a set of numbers $a_1, a_2, \ldots, a_n > 0$, we define $P(r)$ as follows:

$P(0) = GM$,

$$P(r) = \left(\frac{1}{n}\left(a_1^r + a_2^r + \cdots + a_n^r\right)\right)^{\frac{1}{r}} \text{ for } r \neq 0.$$

So $P(1) = AM$ and $P(-1) = HM$.

We extend the definition by defining
$P(\infty) = \max\{a_1, a_2, \ldots, a_n\}$ and
$P(-\infty) = \min\{a_1, a_2, \ldots, a_n\}$.

The Power Mean Inequality

The statement of the The Power-Mean Inequality is:

If $-\infty \leq r < s \leq \infty$, then $P(r) \leq P(s)$ with equality iff $a_1 = a_2 = \cdots = a_n$.

It is in effect a continuum of inequalities.

A useful case is $P(1) \leq P(2)$, which when squared gives

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2 \leq \left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}\right),$$

with equality iff $a_1 = a_2 = \cdots = a_n$.

We had that inequality before for $n = 2$. Although we only get it here for positive numbers, in fact we will see that it applies for **all real numbers**.

That is, for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$, we have

$(a_1 + a_2 + \cdots + a_n)^2 \leq n(a_1^2 + a_2^2 + \cdots + a_n^2)$,

with equality iff $a_1 = a_2 = \cdots = a_n$.

This is sometimes written as

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

in which case it's written $QM \geq AM$, that is

Quadratic Mean $\geq$ Arithmetic Mean.

Example 1

Suppose that $a, b, c > 0$. *Prove that*

$(a^2 + b^2 + c^2)^2 \geq 3(a + b + c)abc$

and find when equality occurs.

Applying $P(2) \geq P(1)$ gives

$$(a^2 + b^2 + c^2)^2 \geq \frac{1}{9}(a + b + c)^4 = \frac{1}{9}(a + b + c)(a + b + c)^3$$

$$\geq \frac{1}{9}(a + b + c)(27abc) = 3(a + b + c)abc$$

(the last step following from the AGM).

Equality in both steps follows iff $a = b = c$.

Example 2

Let a, b, c, d, e be real numbers satisfying the simultaneous equations $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$.

Find the maximum possible value of e.

Clearly $e \leq 4$, but $e = 4$ is not achievable (why?).

We apply the previous inequality to a, b, c, d to get
$$\left(\frac{a+b+c+d}{4}\right)^2 \leq \left(\frac{a^2+b^2+c^2+d^2}{4}\right),$$
with equality iff $a = b = c = d$.

So $\frac{1}{16}(8 - e)^2 \leq \frac{1}{4}(16 - e^2)$.

A little algebra shows that $5e^2 - 16e \leq 0,$ which is true iff $0 \leq e \leq \frac{16}{5}$.

The maximum is then $e = \frac{16}{5}$ and this occurs when $a = b = c = d = \frac{6}{5}$.