Drawing a Straight Line Segment
Group Project 2 for Project Maths, Strand 2,
Leaving Certificate, Higher Level
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1 Introduction

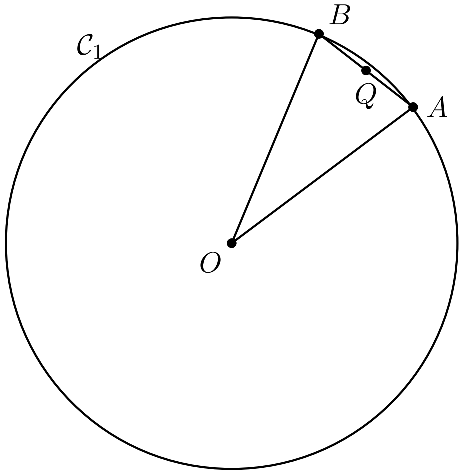
When we start geometry we are supplied with instruments that enable us to draw the figures involved more or less accurately. We should ponder how these can be made. Here we develop the theory of an instrument which enables straight line segments to be drawn. As regards drawing circles and arcs of circles, we have the use of a compass of course.

2 Building steps

We build up to our target in steps and start here with our first figure.

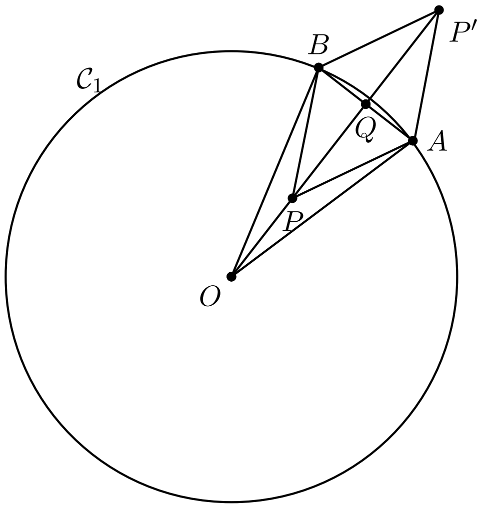
2.1 First figure

We start with a circle $\mathcal{C}_1$ with centre the point O and radius length r. On it we take two points A and B which are not diametrically opposite. We draw the segment $[A, B]$ and mark in the mid-point Q of A and B.



2.2 Second figure

We now take a point P on the segment $[O, Q]$ but not an endpoint, and then a point P' such that Q is the mid-point of P and P'. We join P to A and P to B. Similarly we join P' to A and P' to B.



Now $[AOB]$ is an isosceles triangle and Q is the mid-point of its base $[A, B]$ so the line OQ is perpendicular to the line AB. Then the triangles PAQ and PBQ are congruent by the SAS-principle of congruence. It follows that $|PA| = |PB|$. Moreover Q is on the line PP' so P' is on the line OPQ. By a similar argument $|P'A| = |P'B|$. Also the triangles $[A, P, Q]$ and $[A, P', Q]$ are congruent by the SAS-principle of congruence so $|P, A| = |P', A|$. On combining these we have that $|PA| = |AP'| = |P'B| = |BP|$. In fact, although we do not make use of this, $[P, A, P', B]$ is a parallelogram. For the triangles $[P, A, Q]$ and $[P', B, Q]$ are congruent by the SAS-principle and so $|\angle QPA| = |\angle QP'B|$. Then $AP \parallel P'B$ as these are alternate angles. Similarly $AP' \parallel BP$.

3 Using coordinate geometry

Now since we wish to use trigonometry and coordinate geometry we take a coordinate system with the origin O, draw the circle $\mathcal{C}_1$ and suppose that A and B have coordinates $(r\cos\theta_1, r\sin\theta_1)$ and $(r\cos\theta_2, r\sin\theta_2)$, respectively. Then the mid-point Q of A and B has coordinates

$$\left(\frac{r\cos\theta_1 + r\cos\theta_2}{2}, \frac{r\sin\theta_1 + r\sin\theta_2}{2}\right).$$

Now by formulae in our trigonometry courses we have that

$$\begin{aligned}\cos\theta_1+\cos\theta_2 =& 2\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_2-\theta_1}{2},\\ \sin\theta_1+\sin\theta_2 =& 2\sin\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_2-\theta_1}{2}.\end{aligned}$$

The mid-point Q then has coordinates

$$r\cos\frac{\theta_2-\theta_1}{2}\left(\cos\frac{\theta_1+\theta_2}{2},\sin\frac{\theta_1+\theta_2}{2}\right).$$

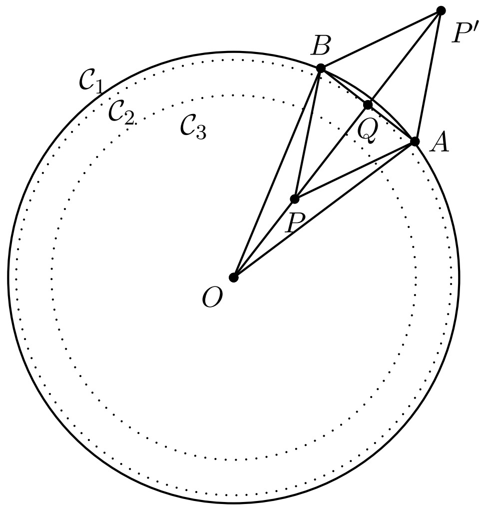
Now for some positive numbers s and s', P and P' will have coordinates

$$\begin{aligned}&sr\left(\cos\frac{\theta_1+\theta_2}{2},\sin\frac{\theta_1+\theta_2}{2}\right),\\ &s'r\left(\cos\frac{\theta_1+\theta_2}{2},\sin\frac{\theta_1+\theta_2}{2}\right).\end{aligned}$$

Then we will have that $|OP||OP'| = ss'r^2$ and we choose the positive number k so that $ss' = k^2$ from which it follows that $|OP||OP'| = k^2r^2$.

For comparison we denote by $\mathcal{C}_2$ the circle with centre O and radius length $\cos\frac{\theta_2-\theta_1}{2}r$ and include it in the third diagram. It clearly passes through the point Q and AB is the tangent to it at Q. We also draw the circle with centre O and length of radius kr and denote it by $\mathcal{C}_3$. As $|OP||OP'| = k^2r^2$, P and P' are called **inverse points with respect to the circle** $\mathcal{C}_3$.

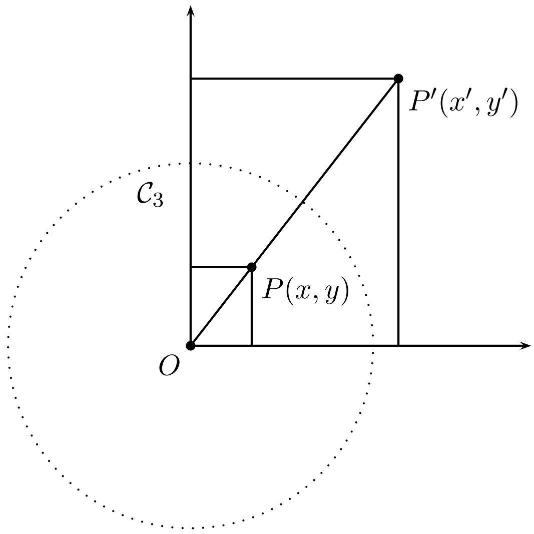
3.1 Third figure



4 Use of inverse points

4.1 Fourth diagram

We now draw the circle $\mathcal{C}_3$ and the point O and insert the original points P and P'. Let the points P and P' have the coordinates (x, y) and (x', y') respectively.



4.2 Inverse points in action

By familiar ratio results for similar triangles we have that

$$\frac{x}{x'} = \frac{y}{y'} = \frac{|OP|}{|OP'|} = \frac{|OP||OP'|}{|OP'|^2} = \frac{k^2r^2}{x'^2 + y'^2},$$

and thus we have the transformation

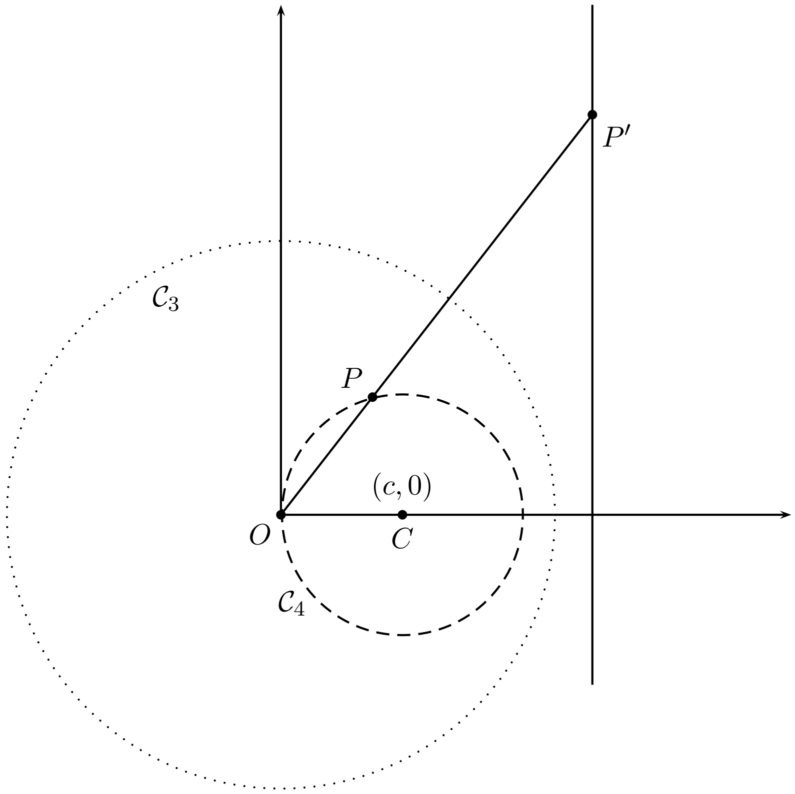
$$x = \frac{k^2r^2x'}{x'^2 + y'^2},\ y = \frac{k^2r^2y'}{x'^2 + y'^2}.$$

5 Transformation in action

We now take a fixed number c with $0 < c < \frac{1}{2}kr$, let C be the point with coordinates $(c, 0)$ and let $\mathcal{C}_4$ be the fixed circle with centre C and radius length c so that $\mathcal{C}_4$ passes through the origin O. We now let P vary on the fixed circle $\mathcal{C}_4$, which has the equation $(x - c)^2 + y^2 = c^2$, that is

$$x^2 - 2cx + y^2 = 0.$$

5.1 Fifth figure



It then follows that P' will vary on the locus with equation

$$\begin{aligned}\left(\frac{k^2r^2x'}{x'^2+y'^2}\right)^2 - 2c\frac{k^2r^2x'}{x'^2+y'^2} + \left(\frac{k^2r^2y'}{x'^2+y'^2}\right)^2 &= 0,\\ \frac{k^4r^4(x'^2+y'^2)}{(x'^2+y'^2)^2} - 2c\frac{k^2r^2x'}{x'^2+y'^2} &= 0,\\ k^2r^2 - 2cx' &= 0,\\ x' &= \frac{k^2r^2}{2c}.\end{aligned}$$

This is an equation of a line parallel to the y-axis.

5.2 Further information

To find the value of c we return to the special point P which we had and wish to find where the perpendicular bisector of $[O, P]$ cuts the x-axis. We recall that P had coordinates $sr(\cos\frac{\theta_1+\theta_2}{2}, \sin\frac{\theta_1+\theta_2}{2})$ so that the mid-point of O and P has coordinates

$$(\frac{1}{2}sr\cos\frac{\theta_1+\theta_2}{2}, \frac{1}{2}sr\sin\frac{\theta_1+\theta_2}{2}).$$

Now the slope of OP is

$$\frac{\sin\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1+\theta_2)},$$

so the slope of the perpendicular to OP is

$$-\frac{\cos\frac{1}{2}(\theta_1+\theta_2)}{\sin\frac{1}{2}(\theta_1+\theta_2)}.$$

Then the perpendicular bisector of $[O, P]$ has equation

$$y - \frac{1}{2}sr\sin\frac{\theta_1+\theta_2}{2} = -\frac{\cos\frac{1}{2}(\theta_1+\theta_2)}{\sin\frac{1}{2}(\theta_1+\theta_2)}\left(x - \frac{1}{2}sr\cos\frac{1}{2}(\theta_1+\theta_2)\right).$$

When this meets the x-axis we have

$$\begin{aligned}
-\frac{1}{2}sr\sin\frac{1}{2}(\theta_1+\theta_2) &= -\frac{\cos\frac{1}{2}(\theta_1+\theta_2)}{\sin\frac{1}{2}(\theta_1+\theta_2)}\left(x - \frac{1}{2}sr\cos\frac{1}{2}(\theta_1+\theta_2)\right)\\
x - \frac{1}{2}sr\cos\frac{1}{2}(\theta_1+\theta_2) &= \frac{1}{2}sr\frac{\sin^2\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1+\theta_2)}\\
x &= \frac{1}{2}sr\cos\frac{1}{2}(\theta_1+\theta_2) + \frac{1}{2}sr\frac{\sin^2\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1+\theta_2)}\\
&= \frac{\frac{1}{2}sr}{\cos\frac{1}{2}(\theta_1+\theta_2)}\left(\cos^2\frac{1}{2}(\theta_1+\theta_2) + \sin^2\frac{1}{2}(\theta_1+\theta_2)\right)\\
&= \frac{\frac{1}{2}sr}{\cos\frac{1}{2}(\theta_1+\theta_2)}.
\end{aligned}$$

Hence

$$c = \frac{\frac{1}{2}sr}{\cos\frac{1}{2}(\theta_1+\theta_2)},\ 2c = \frac{sr}{\cos\frac{1}{2}(\theta_1+\theta_2)}$$

With P' having coordinates,

$$(\frac{1}{2}s'r\cos\frac{\theta_1+\theta_2}{2}, \frac{1}{2}s'r\sin\frac{\theta_1+\theta_2}{2}),$$

the locus of P' has equation

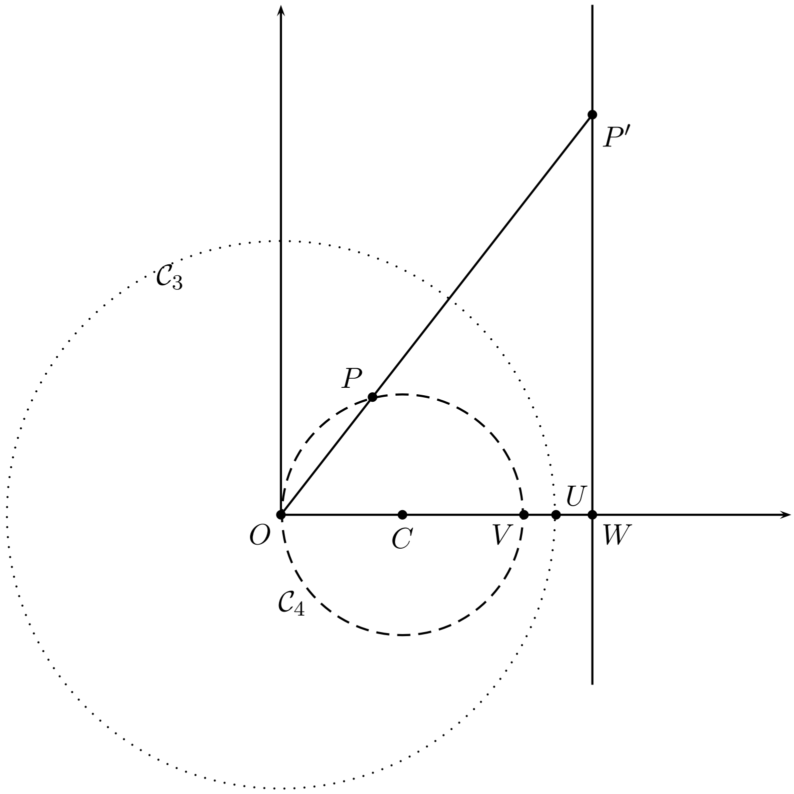
$$x' = \tfrac{1}{2}s'r\cos\frac{\theta_1+\theta_2}{2}.$$

From this it follows that if U is the point where $\mathcal{C}_3$ meets the x-axis, V the point where $\mathcal{C}_4$ meets the x-axis and so has coordinates $(2c, 0)$, and W the foot of the perpendicular from P' onto the x-axis, then we have that

$$\begin{aligned}|OV||OW| =& 2c|OW| = \frac{sr}{\cos\frac{1}{2}(\theta_1+\theta_2)}s'r\cos\frac{\theta_1+\theta_2}{2}\\ =& ss'r^2 = k^2r^2 = |OU|^2,\end{aligned}$$

as expected.

5.3 Sixth figure



6 Background

The physical implementation of this as a linkage has been generally known in Europe as **Peaucellier's Cell** after an officer in the French army. Details of this should be looked up in *Wikipedia*, where there dynamic models in which P is moved on a circular arc and in response P' moves on a segment. There are also several practical implementations and uses included in articles in Google.