Enrichment Lectures 2007

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1 Basic inequalities

$$R = (-\infty, \infty) = (-\infty, 0) \cup \{0\} \cup (0, \infty).$$

Here

$$(-\infty, 0) = \{x \in R : x < 0\}, \ (0, \infty) = \{x \in R : x > 0\}.$$

Law of Trichotomy: Every $x \in R$ is either positive, i.e., $x > 0$; (ii) zero, i.e., $x = 0$, or (iii) negative, i.e., $x < 0$.
N.B. If $a, b > 0$, then $a + b > 0$ and $ab > 0$. If $a, b < 0$, then $a + b < 0$ and $ab > 0$.
We write $a \geq b$ to mean that either $a - b > 0$ or $a = b$.

Lemma 1. *Suppose $A \geq C > 0$, $B \geq D > 0$. Then*

$$AB \geq CD,$$

with equality iff $A = C$ and $B = D$.

Proof. Since

$$AB - CD = (A - C)B + (B - D)C,$$

$A - C \geq 0$ and $B \geq 0$, the first term on the RHS is ≥ 0. Also, $B - D \geq 0$ and $C \geq 0$. Hence the second term is also ≥ 0. Since the sum of two nonnegative numbers is nonnegative, the inequality follows. If the equality holds, then

$$(A - C)B = (B - D)C = 0.$$

But B, C are positive numbers. Hence $A = C$ and $B = D$.

Theorem 1. *If $x \in R$, then $x^2 \geq 0$. In fact, $x^2 > 0$ unless $x = 0$.*

The graph of the square function $x \to x^2$ is a parabola that is symmetric about the vertical axis. The graph is strictly decreasing on $(-\infty, 0)$, takes its least value at $x = 0$ and is strictly increasing on $(0, \infty)$. The graph is convex, i.e., the arc joining any two points on it lies below the chord joining the points. The region $\{(x, y) : y > x^2\}$ has the property that the line segment connecting any two of its points is wholly contained in it.
Here's a proof of the convexity property of the square function. Let $A = (a, a^2)$ and $B = (b, b^2)$ be any two points on the graph of $y = x^2$. The equation of the line through A, B is given by

$$y = a^2 = \frac{b^2 - a^2}{b - a}(x - a) = (a + b)(x - a).$$

Thus, a point (x, y) lies on the chord joining A and B provided $a \le x \le b$ and

$$y = (a + b)(x - a) + a^2.$$

This chord lies above the arc of the parabola connecting A and B provided

$$x^2 \le (a + b)(x - a) + a^2,$$

whenever $a \le x \le b$. Now

$$\begin{aligned} x^2 &\le (a + b)(x - a) + a^2 \\ &\Leftrightarrow 0 \le (a + b)(x - a) + a^2 - x^2 \\ &\Leftrightarrow 0 \le (x - a)(a + b - x - a) \\ &\Leftrightarrow 0 \le (x - a)(b - x) \\ &\Leftrightarrow a \le x \le b. \end{aligned}$$

Note too that the parabola $\{(x, x^2) : x \in R\}$ is the set of points that are equidistant from the fixed point $(0, 1/4)$ and the fixed line $L = \{(x, y); y = -1/4\}$. Indeed, If $P = (x, y)$ is any point, then its distance from $(0, 1/4)$ is given by

$$\sqrt{(x^2 + (y - \frac{1}{4})^2} = \sqrt{x^2 + y^2 - \frac{y}{2} + \frac{1}{16}},$$

while its distance from L is given by $|y + 1/4|$. Hence P is equidistant from $(0, 1/4)$ and L iff

$$\sqrt{x^2 + y^2 - \frac{y}{2} + \frac{1}{16}} = |y + \frac{1}{4}|, \Leftrightarrow x^2 + y^2 - \frac{y}{2} + \frac{1}{16} = y^2 + \frac{y}{2} + \frac{1}{16}, \Leftrightarrow y = x^2.$$

The parabola is one of the conics, whose properties were uncovered by the ancient Greeks.

Theorem 2. *Suppose $x \ge 0$. Then there is a unique $y \ge 0$ so that*

$$y^2 = x.$$

The unique solution is denoted by $\sqrt{x}$.

The square-root function $x \to \sqrt{x}$ has the following properties.

1. Its domain of definition is $[0, \infty)$. Its range is the same set.
2. It is strictly increasing, i.e., $0 \le a < b$ implies that $\sqrt{a} < \sqrt{b}$.
3. It is multiplicative, i.e., if $a, b \in [0, \infty)$, then
$$\sqrt{ab} = \sqrt{a}\,\sqrt{b}.$$
4. It is sub-additive, ie., if $a, b \in [0, \infty)$, then
$$\sqrt{a+b} \le \sqrt{a} + \sqrt{b}.$$
5. It is concave on its domain, i.e., if $a, b \in [0, \infty)$ and $a \le x \le b$, then
$$\sqrt{a} + \frac{\sqrt{b} - \sqrt{a}}{b - a}(x - a) \le \sqrt{x}.$$

2 AM-GM inequality

If $a, b > 0$, then
$$AM(a, b) = \frac{a+b}{2},\ GM(a, b) = \sqrt{ab},$$
are the Arithmetic Mean and Geometric Mean of the two numbers a, b, respectively.

Theorem 3. *If $a, b > 0$, then*
$$GM(a, b) \le AM(a, b),$$
with equality iff $a = b$.

Proof. The statement we want to prove is equivalent to the assertion that
$$\begin{aligned} & \sqrt{ab} \le \frac{a+b}{2} \\ \Leftrightarrow\ & 0 \le a - 2\sqrt{ab} + b \\ \Leftrightarrow\ & 0 \le (\sqrt{a} - \sqrt{b})62, \end{aligned}$$
which holds because $\sqrt{a} - \sqrt{b} \in R$.

Example. *If $a, b, c > 0$, then*
$$8abc \le (a+b)(b+c)(c+a),$$
with equality iff $a = b = c$.

Solution. We have

$$2\sqrt{ab} \le a+b,\ 2\sqrt{bc} \le b+c,\ 2\sqrt{ca} \le c+a,$$

and so

$$8abc = (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca}) \le (a+b)(b+c)(c+a).$$

Clearly, equality holds if $a=b=c$. Suppose equality holds, so that

$$(2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca}) \le (a+b)(b+c)(c+a).$$

Apply the lemma, taking

$$A = a+b,\ C = \sqrt{ab},\ B = (b+c)(c+a),\ D = (2\sqrt{bc})(2\sqrt{ca}).$$

Since the conditions of the lemma are satisfied, it follows that $A = C$ and $B = D$. Thus, by the previous theorem, $a = b$. Together with $B = D$, this forces $(c+a)^2 = 4ca$, whence $c = a$.

Example.

$$x + \frac{1}{x} \ge 2,\ \forall x > 0,$$

with equality iff $x = 1$.

Solution. Apply the theorem with $a = x, b = 1/x$.
Note that the graph

$$\{(x,y) : y = x + \frac{1}{x}, x > 0\}$$

is convex. Indeed, if $a, b > 0$, the equation of the chord joining $A = (a, a+\frac{1}{a})$ and $B = (b, b+\frac{1}{b})$ is given by

$$y = a + \frac{1}{a} + \frac{b+\frac{1}{b}-a-\frac{1}{a}}{b-a}(x-a) = a + \frac{1}{a} + (1-\frac{1}{ab})(x-a),\ a \le x \le b.$$

Hence the chord lies above the arc of the graph connecting A and B iff whenever $a \le x \le b$, then

$$x + \frac{1}{x} \le a + \frac{1}{a} + (1-\frac{1}{ab})(x-a).$$

Equivalently, iff

$$\begin{array}{rcl} 0 & \le & (a-x) + (\frac{1}{a}-\frac{1}{x}) + (1-\frac{1}{ab})(x-a) \\ & \Leftrightarrow & \\ 0 & \le & (x-a)[-1+\frac{1}{ax}+1-\frac{1}{ab}] \\ & \Leftrightarrow & \\ 0 & \le & (x-a)\left(\dfrac{b-x}{axb}\right) \\ & \Leftrightarrow & \\ 0 & \le & \dfrac{(x-a)(b-x)}{axb}, \end{array}$$

i.e., iff $a \le x \le b$.

Example. *If $x, y \in R$, then*
$$2xy \leq x^2 + y^2$$
with equality iff $x = y$.

Example. *If $a, b, c \in R$, then*

$$ab + bc + ca \leq a^2 + b^2 + c^2,$$

with equality iff $a = b = c$.

Proof. This follows from the identity

$$2ab + 2bc + 2ca - 2a^2 - 2b^2 - 2c^2 = (a-b)^2 + (b-c)^2 + (c-a)^2.$$

Theorem 4. *If $a, b, c, d > 0$, then*

$$GM(a, b, c, d) = \sqrt[4]{abcd} \leq \frac{a+b+c+d}{4} = AM(a, b, c, d),$$

with equality iff $a = b = c = d$.

Proof. Let $x = \sqrt{ab}$, $y = \sqrt{cd}$. Then $x, y > 0$ and so

$$\begin{aligned} \sqrt[4]{abcd} &= \sqrt{xy} \\ &\leq \frac{x+y}{2} \\ &= \frac{\sqrt{ab} + \sqrt{cd}}{2} \\ &\leq \frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \\ &= \frac{a+b+c+d}{4}. \end{aligned}$$

Hence the inequality holds. On the other hand, if equality holds throughout in this chain of inequalities, then $x = y$, $a = b$ and $c = d$. Hence $a = b = c = d$. Trivially, there is equality when $a = b = c = d$. Hence the theorem is proved.

Theorem 5. *If $a, b, c > 0$, then*

$$GM(a, b, c) = \sqrt[3]{abc} \leq \frac{a+b+c}{3} = AM(a, b, c),$$

with equality iff $a = b = c$.

Proof. Introduce the dummy variable $d = AM(a, b, c)$. Then (?) $AM(a, b, c, d) = d$ and by the previous theorem,

$$\sqrt[4]{abcd} = GM(a, b, c, d) \leq AM(a, b, c, d) = d, \ (abcd)^{1/4} \leq d, \ abcd \leq d^4,$$

i.e., $abc \le d^3$, or $\sqrt[3]{abc} \le d$. In other words,

$$GM(a, b, c) \le AM(a, b, c).$$

Moreover, there is equality iff

$$a = b = c = d = \frac{a + b + c}{3},$$

i.e., $a = b = c$.
Study the strategy used in the last two theorems. Use it to extend the result of Theorem 4 to prove that

$$GM(a_1, a_2, \ldots, a_8) \le AM(a_1, a_2, \ldots, a_8),$$

with equality iff $a_1 = a_2 = \cdots = a_8$. Now repeatedly employ the tactic used in Theorem 5 to prove that, if $5 \le i \le 7$, then

$$GM(a_1, a_2, \ldots, a_i) \le AM(a_1, a_2, \ldots, a_i),$$

with equality iff $a_1 = a_2 = \cdots = a_i$.

Exercise. *Suppose* $0 < x, y < \pi$. *Prove that*

$$\sqrt{\sin x \sin y} \le \frac{\sin x + \sin y}{2} \le \sin \frac{x + y}{2},$$

with equality iff $x = y$.

Exercise. *Suppose* $0 < x, y, z < \pi$. *Prove that*

$$\sqrt[3]{\sin x \sin y \sin z} \le \frac{\sin x + \sin y + \sin z}{3} \le \sin \frac{x + y + z}{3},$$

with equality iff $x = y = z$.

Exercise. *Suppose ABC is a triangle. Prove that*

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2},$$

with equality iff ABC is equilateral.

Exercise. *Suppose ABC is a triangle. Prove that*

$$\sin A \sin B \sin C \le \frac{3\sqrt{3}}{8},$$

with equality iff ABC is equilateral.

Exercise. *Suppose $ABCD$ is a quadrilateral. Prove that*

$$\sin A + \sin B + \sin C + \sin D \le 4,$$

with equality iff $ABCD$ is a rectangle.

3 Geometric inequalities

Given two rectangles, A, B with dimensions a, b, and c, d, respectively, form the rectangle C with dimensions $a+c, b+d$. How is the area of C related to those of A, B?
Since

$$(a+c)(b+d) = ab + cd + (ad+bc) > ab + cd$$

it is clear that the area of C is bigger than the sum of the areas of A and B. Can we improve on this?
Denote by $\Delta(S)$ the area of a region S.

Theorem 6.

$$\sqrt{\Delta(C)} \geq \sqrt{\Delta(A)} + \sqrt{\Delta(B)},$$

with equality iff A, B are proportional.

Proof. The claim is that, if $a, b, c, d > 0$, then

$$\sqrt{ab} + \sqrt{cd} \leq \sqrt{(a+c)(b+d)},$$

with equality iff

$$\frac{a}{b} = \frac{c}{d}.$$

Squaring both sides the desired result follows if

$$ab + 2\sqrt{abcd} + cd \leq ab + cd + ad + bc, \; 2\sqrt{(ad)(bc)} \leq ad + bc.$$

The latter holds by Theorem 3, with equality iff $ad = bc$, i.e.,$a/b = c/d$. Thus our claim is true.
Alternatively, we can proceed as follows. We can rewrite the stated result in the form

$$\sqrt{\frac{a}{a+c}\frac{b}{b+d}} + \sqrt{\frac{c}{a+c}\frac{d}{b+d}} \leq 1.$$

And,

$$\begin{aligned}
\sqrt{\frac{a}{a+c}\frac{b}{b+d}} + \sqrt{\frac{c}{a+c}\frac{d}{b+d}} &\leq \frac{\frac{a}{a+c} + \frac{b}{b+d}}{2} + \frac{\frac{c}{a+c} + \frac{d}{b+d}}{2} \\
&= \frac{\frac{a}{a+c} + \frac{c}{a+c} + \frac{b}{b+d} + \frac{b}{b+d}}{2} \\
&= \frac{1+1}{2} \\
&= 1,
\end{aligned}$$

with equality iff

$$\frac{a}{a+c} = \frac{b}{b+d}, \; \frac{c}{a+c} = \frac{d}{b+d},$$

i.e.,

$$\frac{a}{b} = \frac{a+c}{b+d} = \frac{c}{d}.$$

Build on this to establish the following exercises.

Exercise. *Suppose $a, b, c, d, e, f > 0$. Prove that*

$$\sqrt{ab} + \sqrt{cd} + \sqrt{ef} \le \sqrt{(a+c+e)(b+d+f)},$$

with equality iff

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

Exercise. *Suppose a, b, c, x, y, z are real numbers. Prove that*

$$|ax + by + cz| \le \sqrt{a^2 + b^2 + c^2}\,\sqrt{x^2 + y^2 + z^2}.$$

Can you determine the cases of equality?

Exercise. *Suppose $a, b, c, d, e, f > 0$. Prove that*

$$\sqrt[3]{abc} + \sqrt[3]{def} \le \sqrt[3]{(a+d)(b+e)(c+f)},$$

with equality iff

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}.$$

Exercise. *Suppose $a, b, c, d, e, f, g, h > 0$. Prove that*

$$\sqrt[4]{abcd} + \sqrt[4]{efgh} \le \sqrt[4]{(a+e)(b+f)(c+g)(d+h)},$$

with equality iff

$$\frac{a}{e} = \frac{b}{f} = \frac{c}{g} = \frac{d}{h}.$$

4 Heron's formula

Is there a result corresponding to Theorem 6 for triangles? Recall that three positive numbers a, b, c are the side lengths of a triangle iff any one is less than the sum of the other two:

$$a < b + c,\ \ b < c + a,\ c < a + b.$$

There are a number of equivalent ways of stating this. One is the requirement that

$$|a - b| < c < a + b.$$

Introducing the symbol s by

$$s = \frac{a+b+c}{2},$$

a, b, c are the side lengths of a triangle ABC iff

$$\max(a, b, c) < s.$$

If this is satisfied, s is the semi-perimeter of ABC.
Two other ways are given in the next two theorems,

Theorem 7. *Three positive numbers a, b, c are the side lengths of a triangle iff*

$$(a+b-c)(b+c-a)(c+a-b) > 0.$$

Proof. Since the product of three positive numbers is positive, the displayed inequality holds if a, b, c are the side lengths of a triangle.
Conversely, if the displayed inequality holds, then either all the factors are positive, in which case we're done, or at most two of the factors are negative. Suppose the latter happens and suppose, for definiteness, that $a+b-c<0$ and $b+c-a<0$. Then, by addition, $a+2b+c<a+c$, i.e., $b<0$, which is false. So, each factor is positive, after all.

Theorem 8. *Three positive numbers a, b, c are the side lengths of a triangle iff*

$$a^4+b^4+c^4 < 2(a^2b^2+b^2c^2+c^2a^2).$$

Proof. By the previous theorem, a, b, c are the side lengths of a triangle iff

$$\begin{array}{rcl}
0 & < & (a+b+c)(a+b-c)(b+c-a)(c+a-b) \\
\Leftrightarrow & & \\
0 & < & ((a+b)^2-c^2)(c^2-(a-b)^2) \\
\Leftrightarrow & & \\
0 & < & (a+b)^2c^2-(a+b)^2(a-b)^2+c^2(a-b)^2-c^4 \\
\Leftrightarrow & & \\
0 & < & -(a^2-b^2)^2+c^2((a+b)^2+(a-b)^2)-c^4 \\
\Leftrightarrow & & \\
0 & < & -a^4+2a^2b^2-b^4+c^2(2a^2+2b^2)-c^4 \\
\Leftrightarrow & & \\
0 & < & 2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4,
\end{array}$$

whence the result.

Exercise. *Suppose a, b, c are the side lengths of a triangle. Prove that*

$$a^2|b^2+c^2-a^2|,\ b^2|c^2+a^2-b^2|,\ c^2|a^2+b^2-c^2|$$

are the side lengths of another triangle.

Exercise. *Suppose a, b, c are the side lengths of a triangle. Prove that*

$$\sqrt{2(b^2+c^2)-a^2},\ \sqrt{2(c^2+a^2)-b^2},\ \sqrt{2(a^2+b^2)-c^2}$$

are the side lengths of another triangle.

Exercise. *Suppose a, b, c, d are positive numbers such that*

$$a+b+c > d,\ b+c+d > a,\ c+d+a > b,\ d+a+b > c.$$

Prove that

$$b,\ c,\ \sqrt{\frac{(ac+bd)(ab+cd)}{(bc+ad)}},$$

are the side lengths of a triangle.

Theorem 9 (Heron, 1st c). *Suppose a, b, c are the side lengths of a triangle. Then its area is given by*

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof. Denote by ABC the triangle with side lengths $|BC| = a$, $|CA| = b$, $|AB| = c$. Then

$$\Delta = \frac{1}{2} ab \sin C.$$

Hence, by the Cosine Rule,

$$\begin{aligned}
16\Delta^2 &= a4^2b^2(1-\cos^2 C) \\
&= 4a^2b^2(1-\cos C)(1+\cos C) \\
&= (2ab-(a^2+b^2-c^2)(2ab+(a^2+b^2-c^2) \\
&= (c^2-(a^2-2ab+b^2))((a^2+2ab+b^2)-c^2) \\
&= (c^2-(a-b)^2)((a+b)^2-c^2) \\
&= (c+(a-b))(c-(a-b))((a+b)+c)((a+b)-c) \\
&= (c+a-b)(c+b-a)(a+b+c)(a+b-c) \\
&= (2s-2b)(2s-2a)(2s)(2s-2c) \\
&= 16s(s-a)(s-b)(s-c),
\end{aligned}$$

whence the result.

Exercise. *Suppose that Δ is the area of a triangle with side lengths a, b, c. Prove that*

$$\Delta = \frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4}.$$

Exercise. *Let a, b, c be the side lengths of a triangle with area Δ, and let a', b', c' be the side lengths of a triangle with area Δ'. Show that $a+a', b+b', c+c'$ are the side lengths of a triangle. Denoting the area of this by Δ'' prove that*

$$\sqrt{\Delta} + \sqrt{\Delta'} \leq \sqrt{\Delta''},$$

with equality iff the triangles are congruent.

Exercise (Bramahgupta, 6th c). *Let p, q, r be positive numbers such that $pg > r^2$. Prove that the numbers*

$$p(q^2 + r^2),\ q(p^2 + r^2),\ (p + q)(pq - r^2)$$

are the side lengths of a triangle whose area is

$$pqr(p + q)(pq - r^2).$$

Note that if p, q, r are positive integers then the side lengths and the area of the triangle are also positive integers.

5 Some trigonometrical formulae

Following on from the last section, and keeping the same notation,

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B,$$

whence

$$\frac{2\Delta}{abc} = \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This is the Sine Rule.
Moreover,

$$8\Delta^3 = a^2b^2c^2 \sin A \sin B \sin C.$$

Exercise. *Deduce that*

1. $$\Delta \le \frac{\sqrt{3}\sqrt[3]{a^2b^2c^2}}{4},$$

   with equality iff ABC is equilateral.

2. $$\Delta \le \frac{a^2 + b^2 + c^2}{4\sqrt{3}},$$

   with equality iff ABC is equilateral.

3. $$\Delta \le \frac{s^2}{3\sqrt{3}},$$

   with equality iff ABC is equilateral.

The third part of this exercise is an example of an iso-perimetric result, which we state as

Theorem 10. *Among all triangles having the same perimeter, the equilateral triangle has the greatest area. Equivalently, among all triangles with the same area, the equilateral triangle has the smallest perimeter.*

Proof. Consider the class of all triangles ABC with the same perimeter L. If a, b, c are the side lengths of one of them, then its area is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

where $2s = a + b + c = L$. Hence

$$\begin{aligned}
\frac{\Delta^2}{s} &= (s-a)(s-b)(s-c) \\
&= (GM(s-a, s-b, s-c))^3 \\
&\leq (AM(s-a, s-b, s-c))^3 \\
&= \left(\frac{(s-a)+(s-b)+(s-c)}{3}\right)^3 \\
&= (\frac{s}{3})^3,
\end{aligned}$$

with equality iff

$$s - a = s - b = s - c, \ a = b = c = \frac{L}{3}.$$

Thus,

$$\frac{\Delta^2}{s} \leq \frac{s^3}{27}, \ \Delta \leq \frac{L^2}{12\sqrt{3}},$$

with equality holding throughout only when ABC is equilateral of side length $L/3$.

Exercise. *Suppose a, b, c are the side lengths of a triangle. Show that $(a+b)/2, (a+b)/2, c$ are the side lengths of a triangle having the same perimeter, but a bigger area.*

Also, we have the Cosine Rule, which we've already applied to derive Heron's formula. This tells us that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

with similar expressions for $\cos B, \cos C$ obtained from this by permuting the symbols a, b, c.
Note that

$$\begin{aligned}
2\sin^2\frac{A}{2} &= 1 - \cos A \\
&= \frac{2bc - (b^2 + c^2 - a^2)}{2bc} \\
&= \frac{a^2 - (b-c)^2}{2bc} \\
&= \frac{a - (b-c))(a + (b-c)}{2bc}
\end{aligned}$$

$$= \frac{(a+c-b)(a+b-c)}{2bc}$$
$$= \frac{2s-2c)(2s-2b)}{2bc}$$
$$= \frac{2(s-b)(s-c)}{bc}.$$

Hence

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Similarly,

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

and so

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

6 Quadrilaterals

Four positive numbers a, b, c, d form a quadrilateral with these as side lengths iff

$$2\sigma \equiv a+b+c+d > 2\max(a,b,c,d).$$

How come? Let $ABCD$ be a quadrilateral with side lengths given by

$$a = |BC|,\ b = |CD|,\ c = |DA|,\ d = |AB|.$$

Let x, y denote the lengths of the 'diagonals' AC and BD: $x = |AC|$, $y = |BD|$. There are infinitely many such quadrilaterals. (Imagine that $ABCD$ is a flexible frame hinged at A, B, C, D.) Which of them has the largest area? A little thought shows that to answer this question we can confine our attention to those that are convex.

Theorem 11. *Consider the class of all convex quadrilaterals $ABCD$ whose side lengths are given by the fixed numbers $a = |BC|$, $b = |CD|$, $c = |DA|$, $d = |AB|$. The one with the largest area is cyclic, and its area is*

$$\sqrt{(\sigma-a)(\sigma-b)(\sigma-c)(\sigma-d)}.$$

Proof. Let $x = |AC|$. By Heron's formula,

$$\Delta(ABC) = \frac{1}{4}\sqrt{((a+d)^2-x^2)(x^2-(a-d)^2)},$$

and

$$\Delta(CDA) = \frac{1}{4}\sqrt{((b+c)^2-x^2)(x^2-(b-c)^2)}.$$

Also,

$$ABCD = ABC \cup CDA$$

and

$$ABC \cap CDA = AC,$$

which has zero area. Hence, we see that

$$\begin{aligned}
\Delta(ABCD) &= \Delta(ABC) + \Delta(CDA) \\
&= \frac{1}{4}\sqrt{(a+d)^2 - x^2)(x^2 - (a-d)^2} + \frac{1}{4}\sqrt{((b+c)^2 - x^2)(x^2 - (b-c)^2)} \\
&= \frac{1}{4}[\sqrt{(a+d)^2 - x^2)(x^2 - (a-d)^2} + \sqrt{((b+c)^2 - x^2)(x^2 - (b-c)^2)}],
\end{aligned}$$

where x, being a side length of the triangles ABC and CDA, satisfies two sets of inequalities

$$|a-d| \leq x \leq a+d, \ |b-c| \leq x \leq |b+c,$$

i.e.,

$$\alpha = \max(|a-d|, |b-c|) \leq x \leq \min(a+d, b+c) = \beta.$$

What value of x maximizes the area of $ABCD$? Anticipating some algebraic manipulations, introduce the variables

$$p = (a+d)^2, q = (a-d)^2, r = (b+c)^2, s = (b-c)^2$$

and let $t = x^2$. The quantity we want to maximize is now

$$\sqrt{(p-t)(t-q)} + \sqrt{(r-t)(t-s)},$$

where $\alpha^2 \leq t \leq \beta^2$. To handle this we appeal to the fact that

$$\sqrt{ab} + \sqrt{cd} \leq \sqrt{(a+c)(b+d)},$$

with equality iff

$$\frac{a}{b} = \frac{c}{d},$$

a result we established to prove Theorem 6. We employ this with

$$a = p-t, \ b = t-q, \ c = t-s, \ d = r-t,$$

giving us that

$$\sqrt{(p-t)(t-q)} + \sqrt{(r-t)(t-s)} \leq \sqrt{(p-t+t-s)(t-q+r-t)} = \sqrt{(p-s)(r-q)},$$

with equality iff

$$\frac{p-t}{t-q} = \frac{t-s}{r-t}, \ \Leftrightarrow (p-t)(r-t) = (t-q)(t-s),$$

i.e.,

$$pr - t(r+p) + t^2 = t^2 - t(s+q) + sq, \;\Leftrightarrow t = \frac{pr - sq}{r+p-s-q}.$$

In terms of the a, b, c, d the condition for equality is that

$$\begin{aligned}
x^2 &= \frac{pr - sq}{r+p-s-q} \\
&= \frac{(a+d)^2(b+c)^2 - (a-d)^2(b-c)^2}{(a+d)^2 + (b+c)^2 - (a-d)^2 - (b-c)^2} \\
&= \frac{(a+d)((b+c)+(a-d)(b-c))((a+d)(b+c)-(a-d)(b-c)}{(a+d)^2 - (a-d)^2 + (b+c)^2 - (b-c)^2} \\
&= \frac{4(ac+bd)(ab+cd)}{4(bc+ad)} \\
&= \frac{(ac+bd)(ab+cd)}{(bc+ad)}.
\end{aligned}$$

To learn what this tells us about the convex quadrilateral $ABCD$ the length of whose diagonal AC is x, we recast this identity as follows. It's equivalent to the following:

$$x^2bc + x^2ad = a^2bc + b^2ad + c^2ad + d^2bc = (a^2+d^2)bc + (b^2+c^2)ad,$$

or,

$$-bc(a^2+d^2-x^2) = ad(b^2+c^2-x^2) \;\Leftrightarrow -\frac{a^2+d^2-x^2}{2ad} = \frac{b^2+c^2-x^2}{2bc},$$

i.e., $-\cos B = \cos D$, which is the same as saying that $B + D = \pi$. Thus $ABCD$ is cyclic.
Concerning the area of $ABCD$, now assuming the latter to be cyclic, in the same notation, this is equal to

$$\Delta(ABC) + \Delta(CDA) = \frac{1}{4}[\sqrt{(p-t)(t-q)} + \sqrt{r-t)(t-s)}] = \frac{1}{4}\sqrt{(p-s)(r-q)}.$$

Now

$$p - s = (a+d)^2 - (b-c)^2 = (a+d-(b-c))(a+d+(b-c)) = (a+d+c-b)(a+d+b-c) = 4(\sigma-c)(\sigma-b),$$

while

$$r - q = (b=c)^2 - (a-d)^2 = (b+c-(a-d))(b+c+(a-d)) = (b+c+d-a)(b+c+a-d) = 4(\sigma-a)(\sigma-d).$$

Thus

$$\Delta(ABCD) = \frac{1}{4}\sqrt{16(\sigma-a)(\sigma-b)(\sigma-c)(\sigma-d)},$$

as stated.

Theorem 12. *Among all* cyclic *quadrilaterals with the same perimeter, the square contains the largest area.*

Proof. In the notation above, the area of a cyclic quadrilateral with side lengths a, b, c, d is

$$\sqrt{(\sigma-a)(\sigma-b)(\sigma-c)(\sigma-d)},$$

where σ is its semi perimeter, and so fixed. By Theorem 4,

$$\sqrt{(\sigma-a)(\sigma-b)(\sigma-c)(\sigma-d)} \le \left(\frac{(\sigma-a)+(\sigma-b)+(\sigma-c)+(\sigma-d)}{4}\right)^2 = \frac{\sigma^2}{4},$$

with equality iff $a = b = c = d = \sigma/2$.

7 The in-radius of a triangle

What point, if any, within a triangle is equidistant from the sides? To answer this, let P be any point interior or on the sides of a triangle ABC. First, what do me mean by the distance between P and $BC \cup BC \cup AB$? Denoting by x, y, z the perpendicular distances from P to the sides BC, CA, AB, respectively, it makes sense to define the distance, $d(P)$, from P to the sides to be the minimum of (x, y, z):

$$d(P) = \min(x, y, z).$$

As P varies within ABC, the sum of the areas of BPC, CPA, APB remains constant, since this sum is equal to the area of ABC. In other words,

$$\Delta(ABC) = \Delta(BPC) + \Delta(CPA) + \Delta(APB) = \frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz,$$

or

$$2\Delta(ABC) = ax + by + cz.$$

Hence

$$2\Delta(ABC) \ge (a+b+c)\min(x, y, z) = (a+b+c)d(P),$$

so that

$$d(P) \le \frac{2\Delta(ABC)}{a+b+c} = \frac{\Delta(ABC)}{s}.$$

Thus, the distance from any point to the sides doesn't exceed the ratio

$$\frac{\Delta(ABC)}{s}.$$

There is equality in the inequality iff

$$ax + by + cz = d(P)a + d(P)b + d(P)c, \ \ a(x - d(P)) + b(y - d(P)) + c(z - d(P)) = 0.$$

But a sum of three non-negative numbers is zero iff each of them is equal to zero, which means that the equality occurs iff

$$x = y = z = d(P) = \frac{\Delta(ABC)}{s}.$$

Since the internal bisectors of ABC are concurrent, and their point of intersection is equidistant from the sides, there is a point within the triangle which is equidistant from the sides. This is the in-centre, the centre of the inscribed circle, whose radius is usually denoted by r, and so

$$r = \frac{\Delta(ABC)}{s}.$$

Theorem 13. *Among all triangles with the same perimeter, the equilateral triangle has the largest in-radius. In fact, in any triangle,*

$$r \leq \frac{s}{3\sqrt{3}},$$

with equality iff the triangle is equilateral.

Proof. This follows from the proof of Theorem 10 which shows that

$$\frac{\Delta^2}{s} \leq \frac{s^3}{27},$$

whence

$$r^2 = \frac{\Delta^2}{s^2} \leq \frac{s^2}{27}, \ r \leq \frac{s}{3\sqrt{3}}.$$

Exercise. *What point within a triangle has the property that the sum of its distances from the sides is a minimum?*

Theorem 14. *There is a point within a triangle, known as the Lemoine point of the triangle, which has the property that the sum of the squares of its distances from the sides is a minimum.*

Proof. To explore this, in the usual notation, let P be a point within ABC, and let x, y, z be its distances to the sides BC, CA, AB, respectively. Then

$$2\Delta(ABC) = ax + by + cz.$$

Subject to this constraint, the quantity we wish to minimize as P varies is $x^2+y^2+z^2$. But, by the first exercise following Theorem 6,

$$ax + by + cz = \sqrt{a^2x^2} + \sqrt{b^2y^2} + \sqrt{c^2z^2} \leq \sqrt{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)},$$

with equality iff

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \frac{a^2 + b^2 + c^2}{2\Delta}.$$

Thus,

$$x^2 + y^2 + z^2 \geq \frac{4\Delta^2}{a^2 + b^2 + c^2},$$

with equality iff

$$x = \frac{2\Delta a}{a^2 + b^2 + c^2}, \; y = \frac{2\Delta b}{a^2 + b^2 + c^2}, \; z = \frac{2\Delta c}{a^2 + b^2 + c^2}.$$

This analytical argument establishes a lower bound for the sums of the squares $x^2+y^2+z^2$ subject to the restriction that $2\delta = ax+by+cz$. It remains to verify that there is actually a point within the triangle whose distances x, y, z are proportional to the side lengths a, b, c in the manner shown in the displayed equations. Can you fill in the rest?

Corollary 1.

$$r \geq \frac{2\Delta}{\sqrt{3(a^2 + b^2 + c^2)}},$$

with equality iff $a = b = c$.

Proof. The in-centre of the triangle is equidistant from the sides. Hence the sum of the squares of these distances exceeds

$$\frac{4\Delta^2}{a^2 + b^2 + c^2},$$

whence

$$3r^2 \geq \frac{4\Delta^2}{a^2 + b^2 + c^2},$$

as claimed.

Exercise. *Prove the last result directly without appealing to Theorem 14.*

8 The circum-radius of a triangle

Is there a point in the plane of a triangle which is equidistant from the vertices? The answer is "yes": the point of intersection of the bisectors of the sides is equidistant from the vertices. This point is the centre—the circum-centre—of a circle—the circum-circle—that passes through the three vertices. The radius of this circle is denoted by R.

To evaluate R in terms of the sides and angles, let O denote the circum-centre, and consider the isosceles triangle BOC. Since BC is a chord of the circle, $\angle BOC = 2\angle A$. Hence, by the Cosine Rule,

$$\cos 2A = \cos \angle BOC = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2}.$$

In other words,

$$1 - 2\sin^2 A = 1 - \frac{a^2}{2R^2},\ 2R = \frac{a}{\sin A}.$$

Hence

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This is one expression for the circum-radius of a triangle. Others can be derived from it. For instance, it easily follows that

$$R = \frac{abc}{4\Delta} = \frac{abc}{4rs}.$$

Since the in-circle is a subset of the triangle, and this in turn is a subset of the circum-circle, we see that

$$\pi r^2 \leq \Delta \leq \pi R^2,$$

whence $r \leq R$. This raises the question: How much bigger is R than r? Computing these quantities for an equilateral triangle, it's tempting to conjecture that $2r \leq R$. This turns out to be true.

Exercise. *Suppose ABC is isosceles. Prove that $2r \leq R$.*

Theorem 15 (Euler). *In any triangle, $2r \leq R$, with equality only for an equilateral triangle.*

Proof. We have

$$r = \frac{\Delta}{s},\ R = \frac{abc}{4\Delta}.$$

Hence

$$\frac{r}{R} = \frac{4\Delta^2}{abcs} = \frac{4(s-a)(s-b)(s-c)}{abc}.$$

So, we must prove that

$$8(s-a)(s-b)(s-c) \leq abc.$$

But this is a consequence of the worked example after Theorem 3! And there is equality iff

$$s - a = s - b = s - c,\ a = b = c.$$

Thus Euler's theorem is true.

Is there a point in the plane of a triangle such that the sum of its distances from the vertices is a minimum? This was first asked by Fermat. Torricelli provided a solution. It's next to impossible to treat this without adverting to Napoleon's theorem. This states the following

Theorem 16 (Napoleon). *If three equilateral triangles are erected externally (or internally) on the sides of a triangle, then their centroids form an equilateral triangle.*

Exercise. *Try proving this!*

Exercise. *Let $BA'C$, $CB'A$, $AC'B$ be three equilateral triangles erected externally on the sides of a triangle ABC. Prove that*

$$|AA'| = |BB'| = |CC'| = \sqrt{\frac{a^2+b^2+c^2}{2} + 2\sqrt{3}\Delta}.$$

Here's a clue to doing Fermat's problem.

Theorem 17. *Let P be any point within a triangle ABC. Then*

$$|PA| + |PB| + |PC| \geq \sqrt{\frac{a^2+b^2+c^2}{2} + 2\sqrt{3}\Delta}.$$

Proof. Rotate the triangle BPC counterclockwise about the vertex C through an angle of 60 degrees. Let the image of P be denoted by P'. Then, in the notation of the last exercise, $A'P'C$ is the image of BPC. Since $PP'C$ is equilateral, so that $|CP| = |PP'|$, and $|BP| = |A'P'|$, by construction, we see that

$$|AP| + |BP| + |CP| = |AP| + |A'P'| + |P'P| = |AP| + |PP'| + |P'A'|.$$

This is the length of the broken line made up of the segments $A'P', P'P, PA$ that joins A and A'. Since the shortest distance between two points is a straight line, we deduce that

$$|AP| + |BP| + |CP| \geq |A'A|,$$

which proves the result.

Exercise. *Figure out from this the solution to Fermat's problem.*

Exercise. *Deduce that*

$$3R \geq \sqrt{\frac{a^2+b^2+c^2}{2} + 2\sqrt{3}\Delta}.$$

Theorem 18. *The centroid (centre of gravity) of a triangle has the property that the sum of the squares of its distances from the vertices is a minimum. This minimum is given by*

$$\frac{a^2+b^2+c^2}{3}.$$

Proof. It's convenient to use Coordinate Geometry to show this. To this end, let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ denote the vertices of a triangle. Let $P = (x, y)$ be any point in the plane. Then

$$|PA| = \sqrt{(x-a_1)^2 + (y-a_2)^2},$$

with similar expressions for $|PB|, |PC|$. The quantity we're interested in is

$$(x-a_1)^2 + (y-a_2)^2 + (x-b_1)^2 + (y-b_2)^2 + (x-c_1)^2 + (y-c_2)^2 \equiv f(x) = g(y),$$

where

$$f(x) = (x-a_1)^2 + (x-b_1)^2 + (x-c_1)^2, \ g(y) = (y-a_2)^2 + (y-b_2)^2 + (y-c_2)^2.$$

The functions f, g are quadratics in x, y, respectively. Expanding f and collecting terms we see that

$$f(x) = 3x^2 - 2x(a_1+b_1+c_1) + a_1^2 + b_1^2 + c_1^2,$$

and completing the square on x

$$\begin{aligned}
f(x) &= 3(x - \frac{a_1+b_1+c_1}{3})^2 + a_1^2 + b_1^2 + c_1^2 - 3(\frac{a_1+b_1+c_1}{3})^2 \\
&= 3(x - \frac{a_1+b_1+c_1}{3})^2 + \frac{3(a_1^2+b_1^2+c_1^2) - (a_1^2+b_1^2+c_1^2+2a_1b_1+2ab_1c_1+2c_1a_1)}{3} \\
&= 3(x - \frac{a_1+b_1+c_1}{3})^2 + \frac{2(a_1^2+b_1^2+c_1^2) - (2a_1b_1+2ab_1c_1+2c_1a_1)}{3} \\
&= 3(x - \frac{a_1+b_1+c_1}{3})^2 + \frac{(a_1-b_1)^2 + (b_1-c_1)^2 + (c_1-a_1)^2}{3} \\
&\geq \frac{(a_1-b_1)^2 + (b_1-c_1)^2 + (c_1-a_1)^2}{3},
\end{aligned}$$

with equality iff

$$x = \frac{a_1+b_1+c_1}{3}.$$

Similarly, g attains its minimum when

$$y = \frac{a_2+b_2+c_2}{3},$$

and its minimum value is

$$\frac{(a_2-b_2)^2 + (b_2-c_2)^2 + (c_2-a_2)^2}{3}.$$

In other words, the expression

$$|AP|^2 + |BP|^2 + |CP|^2$$

achieves its minimum value when the coordinates of P are

$$x = \frac{a_1+b_1+c_1}{3}, \ y = \frac{a_2+b_2+c_2}{3},$$

the coordinates of the centroid of ABC. Moreover, the minimum value is given by

$$\begin{aligned}
&\frac{(a_1-b_1)^2 + (b_1-c_1)^2 + (c_1-a_1)^2}{3} + \frac{(a_2-b_2)^2 + (b_2-c_2)^2 + (c_2-a_2)^2}{3} \\
= &\frac{[(a_1-b_1)^2 + (a_2-b_2)^2] + [(b_1-c_1)^2 + (b_2-c_2)^2] + [(c_1-a_1)^2 + (c_2-a_2)^2]}{3} \\
= &\frac{|AB|^2 + |BC|^2 + |CA|^2}{3} \\
= &\frac{a^2+b^2+c^2}{3}.
\end{aligned}$$

Thus, for all points P in the plane of ABC,

$$|AP|^2 + |BP|^2 + |CP|^2 \geq \frac{a^2 + b^2 + c^2}{3},$$

with equality holding iff P is the centroid of ABC.

Corollary 2.

$$R \geq \frac{\sqrt{a^2 + b^2 + c^2}}{3}.$$

9 Reflections

Given a line L and two points A, B on the same side of it, what point $P \in L$ has the property that $|AP| + |PB|$ is as short as possible?

Theorem 19 (Heron). *Let A, B be two points on the same side of a line L. Let P be the intersection of L and the line joining A to the reflection, B', of B in L. Then, if X is any other point in L,*

$$|AX| + |XB| \geq |AP| + |PB|.$$

Proof. By definition of B', $|PB| = |PB'|$ and $|XB| = |XB'|$. Hence, applying the triangle inequality in the (possibly degenerate) triangle AXB',

$$|AX| + |XB| = |AX| + |XB'| \geq |AB'| = |AP| + |PB'| = |AP| + |PB|,$$

as claimed.
This is the Reflection Principle.

Theorem 20. *Among all triangles with the same area the equilateral triangle has the smallest perimeter.*

Consider all those triangles with the same area Δ. Suppose ABC is a non-equilateral triangle with area Δ. Suppose $a = \max(a, b, c)$. Consider the isosceles triangle standing on the same base BC with the same height as ABC. This too has area Δ. By the Reflection Principle its perimeter does not exceed that of ABC. Now repeat the same argument using one of the equal sides of the isosceles triangle as base to construct an equilateral one with the given area.
Returning to Theorem 19, it seems clear that as X moves along L towards P, the expression $|AX| + |XB|$ decreases. To confirm this, we prove a preliminary result.

Lemma 2. *Suppose P is an interior point of a triangle ABC. Then*

$$|BP| + |PC| \leq |BA| + |AC|,$$

with equality iff P coincides with A.

Proof. Let the line through B and P meet AC at Q. By the triangle inequality,

$$|BA| + |AQ| \geq |BQ|.$$

Also, for the same reason,

$$|PQ| + |QC| \geq |PC|.$$

Hence

$$\begin{aligned} |BP| + |PC| &\leq |BP| + |PQ| + |QC| \\ &= |BQ| + |QC| \\ &\leq |BA| + |AQ| + |QC| \\ &= |BA| + |AC|. \end{aligned}$$

Theorem 21. *Let A, B be two points on the same side of a line L. Let P be the intersection of L and the line joining A to the reflection, B', of B in L. Let X be any other point in L, and Y a point in L that is between X and P. Then*

$$|AX| + |XB| \geq |AY| + |YB|.$$

Proof. By hypothesis, Y is an interior point of the triangle AXB', and so, by the last lemma,

$$\begin{aligned} |AX| + |XB| &= |AX| + |XB'| \\ &\geq |AY| + |YB'| \\ &= |AY| + |YB|, \end{aligned}$$

by the Reflection Principle.

Theorem 22. *Suppose ABC is acute-angled and $X \in BC$. Denote by X', X'' the reflections of X in the sides CA, AB, respectively. Then*

$$\angle X''AX' = 2\angle A,$$

and $|X''X'|$ is least when X is the foot of the perpendicular from A on BC.

Proof. Let XX' intersect AC at P. Let XX'' intersect AB at Q. Since the right-angled triangles XAP and $PX'A$ are congruent, $\angle XAP = \angle PAX'$. In the same way we see that $\angle XAQ = \angle QAX''$. Hence

$$\angle X''AX' = \angle X''AQ + \angle QAX + \angle XAP + \angle PAX' = 2(\angle QAX + \angle XAP) = 2\angle A.$$

Since also, by reflection,

$$|X''A| = |AX| = |AX'|,$$

we deduce that

$$\begin{aligned} |X''X'|^2 &= |X''A|^2 + |AX'|^2 - 2|X''A|\,|AX'| \cos \angle X''AX' \\ &= 2|AX|^2 - 2|AX|^2 \cos 2A \\ &= 4|AX|^2 \sin^2 A \\ &\geq 4|AD|^2 \sin^2 A, \end{aligned}$$

where D is the foot of the perpendicular from A on BC. Thus the least value of $|X''X'|$ is $2|AD|\sin A$.

Theorem 23 (Fejér). *Let ABC be acute-angled. Fix $X \in BC$. If $Y \in CA$, $Z \in AB$, then*

$$|XY| + |YZ| + |ZX| \geq |X''X'|,$$

with equality iff $Y, Z \in X''X'$.

Proof. By Reflection,

$$|XZ| + |ZY| + |YX| = |X''Z| + |ZY| + |YX'| \geq |X''X'|,$$

since $X''Z \cup ZY \cup YX'$ is a broken line joining the points X'' and X'. There is equality iff $Z = Q$, $Y = P$, with P, Q as in the previous theorem.
The *pedal triangle* of an acute-angled triangle ABC is the inscribed triangle whose vertices are the feet of the altitudes of ABC.
You now have enough evidence to finish off Fejér's proof of Fagnano's theorem which tells that

Theorem 24. *Suppose ABC is an acute-angled triangle. Of all inscribed triangles in ABC, the pedal triangle has the smallest perimeter.*

Here's a different Proof. Let XYZ be an arbitrary inscribed triangle with $X \in BC, Y \in CA, Z \in AB$. Let $x = |BX|, y = |CY|, z = |AZ|$. Then $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

$$\begin{aligned} |ZX|^2 &= (c-z)^2 + x^2 - 2x(c-z)\cos B \\ &= (c-z)^2 + x^2 + 2x(c-z)\cos(A+C) \\ &= (x\cos A + (c-z)\cos C)^2 + (x\sin A - (c-z)\sin C)^2. \end{aligned}$$

Hence,

$$|ZX| \geq x\cos A + (c-z)\cos C,$$

which is positive by hypothesis, with equality iff $x\sin A = (c-z)\sin C$, i.e., iff

$$ax + cz = c^2, \tag{1}$$

by the Sine Rule. Similarly,

$$|XY| \geq y\cos B + (a-x)\cos A,$$

with equality iff

$$ax + by = a^2. \tag{2}$$

And

$$|YZ| \geq z\cos C + (b-y)\cos B,$$

with equality iff

$$by + cz = b^2. \tag{3}$$

Thus

$$|XY| + YZ| + |ZX| \geq a\cos A + b\cos B + c\cos C,$$

with equality iff equations (1), (2) and (3) hold. Now the solution of these equations is given by

$$x = c\cos B,\ y = a\cos C,\ z = b\cos A,$$

which is equivalent to saying that X, Y, Z are the feet of the altitudes from A, B, C, respectively, onto the opposite sides, i., XYZ is the pedal triangle. Conversely, if XYZ is the pedal triangle, then, for instance,

$$\begin{aligned}
|ZX|^2 &= (c-z)^2 + x^2 - 2x(c-z)\cos B \\
&= (c - b\cos A)^2 + (c\cos B)^2 - 2c(c - b\cos A)\cos^2 B \\
&= -c(c - 2b\cos A)\cos^2 B + (c - b\cos A)^2 \\
&= (c^2 - 2bc\cos A)\sin^2 B + b^2\cos^2 A \\
&= (a^2 - b^2)\sin^2 B + b^2 - b^2\sin^2 A \\
&= b^2(1 - \sin^2 B) \\
&= b^2\cos^2 B,
\end{aligned}$$

by the Sine and Cosine Rules. Hence

$$|ZX| = b|\cos B| = b\cos B.$$

Similarly,

$$|XY| = c\cos C,\ |YZ| = a\cos A,$$

whence

$$\begin{aligned}
a\cos A + b\cos B + c\cos C &= \frac{a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)}{2abc} \\
&= \frac{2(a^2b^2 + b^2c^{+}c^2a^2) - a^4 - b^4 - c^4}{2abc} \\
&= \frac{8\Delta^2}{abc},
\end{aligned}$$

is the perimeter of the pedal triangle, and no other inscribed triangle has a smaller one.

A related result is the following, whose proof we sketch, omitting a proof of a crucial property of the Lemoine point of a triangle, namely, that it is the centroid of its pedal triangle, the triangle whose vertices are the feet of the perpendiculars from the Lemoine point onto the sides.

Theorem 25. *Among all triangles inscribed in a given one ABC, the pedal triangle of the Lemoine point has the property that the sum of the squares of its sides is a minimum. In fact, in the usual notation, if XYZ is inscribed in ABC, then*

$$|XY|^2 + |YZ|^2 + |ZX|^2 \geq \frac{4\Delta^2}{a^2+b^2+c^2},$$

and this is best possible.

Proof. We use two important properties of the Lemoine point, which we denote by L. One, according to Theorem 14, the sum of the squares of its distances from the sides of ABC is a minimum. Two, which we're assuming, it is the centroid of its pedal triangle, and is the only point with this property.
To proceed, let X, Y, Z be points on the sides BC, CA, AB, respectively. Let O be the centroid of XYZ. Denote by E, F, G the feet of the perpendiculars from O onto the sides BC, CA, AB, and by P, Q, R the feet of the perpendiculars from L onto the sides BC, CA, AB. The triangle PQR is the pedal triangle of L, and, denoting the length of the median from P by m_p, etc., $m_p = \frac{3}{2}|LP|$, by the second property, and so, since

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}[a^2+b^2+c^2],$$

we see that, by Theorem 14,

$$\begin{aligned}
\frac{4\Delta^2}{a^2+b^2+c^2} &= |PQ|^2 + |QR|^2 + |RP|^2 \\
&= \frac{4}{3}[m_p^2 + m_r^2 + m_q^2] \\
&= 3[|LP|^2 + |LR|^2 + |LQ|^2] \\
&\leq 3[|OE|^2 + |OF|^2 + |OG|^2] \\
&\leq 3[|OX|^2 + |OY|^2 + |OZ|^2] \\
&= \frac{4}{3}[m_x^2 + m_y^2 + m_z^2] \\
&= |XY|^2 + |YZ|^2 + |ZX|^2.
\end{aligned}$$

10 Euler's theorem revisited

We sketch a variational approach to Euler's result that $R \geq 2r$, due to Kazarinoff. The relation

$$r = \frac{\Delta}{s}$$

tells us that among all triangles with the same area, r is greatest when s is least.

Theorem 26. *Consider all triangles PBC with the same base BC and with variable vertex P that belongs to a line L parallel to the base, and denote by ABC the isosceles triangle, with $A \in L$. Then, as a function of P, $r(PBC)$ increases as P moves towards A.*

Proof. Assume $P, Q \in L$ with P between A and Q. The claim is that

$$r(QBC) \leq r(PBC) \leq r(ABC),$$

equivalently,

$$s(QBC) \geq s(PBC) \geq s(ABC),$$

Or

$$|BQ| + |QC| \geq |BP| + |PC| \geq |BA| + |AC|.$$

To see this, let C' be the reflection of C in L. Then P is an interior point of BQC', and so, by the previous lemma,

$$|BP| + |PC| = |BP + |PC'| \leq |BQ| + |QC'| = |BQ| + |QC|,$$

as required.

Theorem 27. *Suppose P belongs to the side AC of a triangle ABC. Then*

$$r(BPC| \leq r(BAC).$$

Proof. Let $x = |BP|$, $y = |PC|$. Since

$$r(ABC) = \frac{ab\sin C}{a+b+c},\ r(ABC) = \frac{ay\sin C}{a+x+y},$$

the claim is that

$$\frac{y}{a+x+y} \leq \frac{b}{a+b+c} \Leftrightarrow a(b-y) + bx - cy \geq 0.$$

But, $b \geq y$, and so this holds as long as

$$\frac{b}{c} \geq \frac{y}{x}.$$

But, by the Sine Rule,

$$\frac{y}{x} = \frac{\sin PBC}{\sin A} \leq \frac{\sin C}{\sin C} = \frac{b}{c}.$$

The result follows.

Theorem 28 (Euler). *Suppose ABC is a triangle. Then*

$$R(ABC) \geq 2r(ABC),$$

with equality iff ABC is equilateral.

Proof. A calculation shows that there is equality when the triangle is equilateral. Suppose ABC is not equilateral, and let $b < \min(a, c)$. Let Γ be the circum circle of ABC. With C as one vertex inscribe an equilateral triangle $B'A'C$ in Γ. Let the line through A that is parallel with BC meet $A'C$ at A''. Then

$$r(BA''C) > r(BAC)$$

by Theorem 26. Next,

$$r(BA'C) > r(BA''C),$$

by Theorem 27. In other words, $r(ABC) < r(BA'C)$.
To complete the argument, let the line through B that is parallel to $A'C$ intersect $B'A'$ at B''. Applying Theorem 26 once more, we see that

$$r(A'BC) < r(A'B''C).$$

And, then, by Theorem 27,

$$r(B'A'C) < r(B'A'C).$$

Thus

$$r(ABC) < r(B'A'C) = \frac{1}{2}R(ABC).$$

11 The Erdos-Mordell theorem

Theorem 29. *Suppose P is any point within a triangle ABC, and let X, Y, Z be the feet of the perpendiculars from P onto the sides BC, CA, AB, respectively. Then*

$$|PA| + |PB| + |PC| \geq 2(|PX| + |PY| + |PZ|),$$

with equality iff ABC is equilateral and P is its in-centre.

Proof. Let $x = |PX|$, $y = |PY|$, $z = |PZ|$. Since $AZPY$ is a cyclic quadrilateral, and AP is a diameter of the circumscribing circle that passes through the vertices of AZY,

$$|PA| = \frac{|ZY|}{\sin A}.$$

Similarly,

$$|PB| = \frac{|ZX|}{\sin B}, \ |PC| = \frac{|XY|}{\sin C}.$$

But also,

$$\cos ZPY = -\cos A,$$

hence

$$\begin{aligned} |ZY|^2 &= |PZ|^2 + |PY|^2 - 2|PZ|\,|PY| \cos ZPY \\ &= z^2 + y^2 + 2zy \cos A \\ &= z^2 + y^2 - 2yz \cos(B + C) \\ &= (z \sin B + y \sin C)^2 + (z \cos B - y \cos C)^2. \end{aligned}$$

Hence

$$|ZY| \geq z\sin B + y\sin C,$$

with equality iff $z\cos B = y\cos C$.
Similarly,

$$|ZX| \geq z\sin A + x\sin C,$$

with equality iff $z\cos A = x\cos C$, and

$$|XY| \geq y\sin A + x\sin B,$$

with equality iff $y\cos A = x\cos B$. It follows that

$$\begin{aligned}
|PA| + |PB| + |PC| &= \frac{|ZY|}{\sin A} + \frac{|XZ|}{\sin B} + \frac{YX|}{\sin B} \\
&\geq \frac{z\sin B + y\sin C}{\sin A} + \frac{z\sin A + x\sin C}{\sin B} + \frac{y\sin A + x\sin B}{\sin C} \\
&= z(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}) + y(\frac{\sin C}{\sin A} + \frac{\sin A}{\sin C}) + x(\frac{\sin C}{\sin B} + \frac{\sin B}{\sin C}) \\
&\geq 2(z + y + x) \\
&= 2(|PX| + |PY| + |PZ|).
\end{aligned}$$

Equality prevails in the last step iff

$$\sin A = \sin B = \sin C$$

i.e., iff ABC is equilateral. Hence there is equality throughout iff the triangle is equilateral and $x = y = z$. Thus the equality holds, iff P is the in-centre of an equilateral triangle.