7. Lemoine Point.

Two lines AS and AT through the vertex A of an angle are said to be *isogonal* if they are equally inclined to the arms of $\widehat{A}$, or equivalently, to the bisector of $\widehat{A}$ (Figure 1).

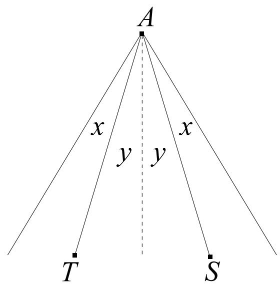


Figure 1:

The isogonals of the medians of a triangle are called *symmedians*. We will show in a little while that the symmedians are concurrent and their point of concurrency is called the *symmedian point*. It is also called the *Lemoine* point.

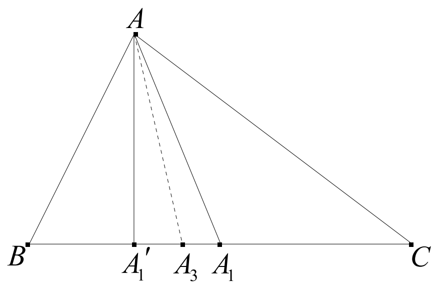


Figure 2:

As before, in a triangle ABC, the midpoint BC is denoted by A_1, the intersection of BC and the bisector of $\widehat{A}$ is A_3 and then the symmedian of AA_1 will be AA_1' (Figure 2). Thus

$$AA_1' = Sym_{AA_3}(AA_1).$$

Recall Steiner's theorem which states that in a triangle ABC, if AA_1 and AA_2 are isogonal (Figure 3), then

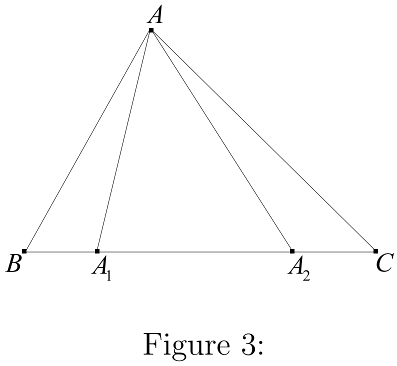


Figure 3:

$$\frac{|AB|^2}{|AC|^2} = \frac{|BA_1||BA_2|}{|CA_1||CA_2|}.$$

We now apply this to get the following.

Theorem 1 *A line AA_1' in a triangle ABC (Figure 4) is a symmedian if and only if*

$$\frac{|BA_1'|}{|CA_1'|} = \frac{|AB|^2}{|AC|^2} = \frac{c^2}{b^2}\ .$$

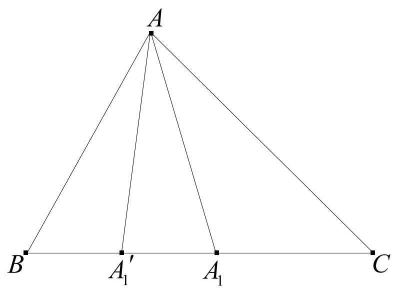


Figure 4:

Proof The line AA_1' is a symmedian if AA_1 is a median and

$$AA_1' = Sym_{AA_3}(AA_1).$$

Then $|BA_1| = |CA_1|$, so on applying Steiner's theorem, we get that AA_1' is a symmedian if and only if

$$\frac{|AB|^2}{|AC|^2} = \frac{|BA_1'||BA_1|}{|CA_1'||CA_1|} = \frac{|BA_1'|}{|CA_1'|}.$$

Remark It is well known that the bisector of an angle of a triangle divides the opposite side into the ratio of the sides about the angle. Then, be the above theorem, a symmedian does it in the ratio of the squares of the sides.

We can now apply the previous result to show that the symmedians are concurrent.

Theorem 2 *Let AA_1', BB_1' and CC_1' be the symmedians of a triangle. Then these lines are concurrent at a point L called the* Lemoine point *(Figure 5).*

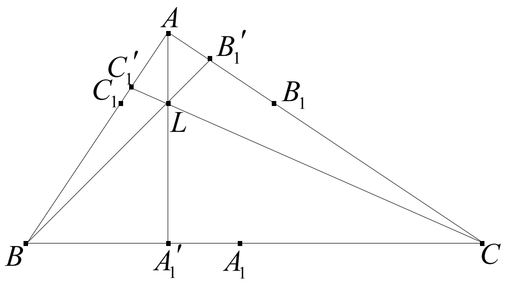


Figure 5:

Proof An easy application of Ceva's theorem and Theorem 1 above gives the result. We have

$$\frac{|A_1'B|}{|A_1'C|} = \frac{c^2}{b^2}, \quad \frac{|B_1'C|}{|B_1'A} = \frac{a^2}{c^2} \quad \text{and} \quad \frac{|C_1'A|}{|C_1'B} = \frac{b^2}{a^2}.$$

Then, by Ceva's theorem, the symmedians are concurrent since the product of the ratios is 1.

Using van Aubel's theorem we get the ratios in which L divides the symmedians AA_1'.

$$\frac{|LA|}{|LA_1'|} = \frac{|C_1'A|}{|C_1'B|} + \frac{|B_1'A|}{|B_1'C|} = \frac{b^2}{a^2} + \frac{c^2}{a^2} = \frac{b^2+c^2}{a^2}.$$

Theorem 3 *The tangents to the circumcircle $\mathcal{C}(ABC)$ of a triangle ABC at two of its vertices meet on the symmedian from the third vertex.*

Proof Let the tangents to the circumcircles at the points B and C meet at the point K. Join A to K and let A_1'' be the point of intersection of BC and AK (Figure 6). We need to show that AA_1'' is the symmedian from the vertex A, i.e.

$$\frac{|BA_1''|}{|CA_1''|} = \frac{c^2}{b^2}.$$

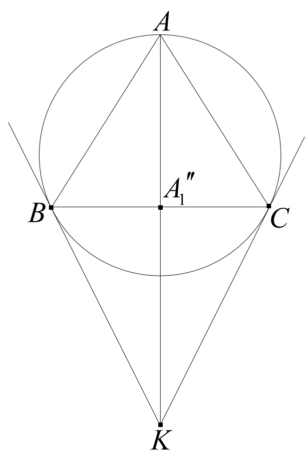


Figure 6:

Consider

$$\begin{aligned}
\frac{|BA_1''|}{|CA_1''|} &= \frac{area(ABA_1'')}{area(ACA_1'')} = \frac{area(BKA_1'')}{area(CKA_1'')} \\
&= \frac{area(ABA_1'') + area(BKA_1'')}{area(ACA_1'') + area(CKA_1'')} \\
&= \frac{area(ABK)}{area(ACK)} \\
&= \frac{|AB||BK|\sin(A\widehat{B}K)}{|AC||CK|\sin(A\widehat{C}K)} \qquad \ldots(i).
\end{aligned}$$

Now make some observations. We have $|KB| = |KC|$ since KB and KC are tangents from K to $\mathcal{C}(ABC)$. Furthermore, using the property that the angle between a tangent and a chord is equal to the angle in the segment on the opposite side of the chord, we have

$$K\widehat{B}C = K\widehat{C}B = \widehat{A}.$$

Thus

$$A\widehat{B}K = (\widehat{A} + \widehat{B}),$$

so

$$\sin(A\widehat{B}K) = \sin(\widehat{C}),$$

and

$$A\widehat{C}K = (\widehat{A} + \widehat{C}),$$

so

$$\sin(A\widehat{C}K) = \sin(\widehat{B}).$$

Thus, from (i) we have

$$\begin{aligned}
\frac{|BA_1''|}{CA_1''|} &= \frac{|AB|\sin(\widehat{C})}{|AC|\sin(\widehat{B})} \\
&= \frac{|AB|^2}{|AC|^2}, \text{ by sine rule.}
\end{aligned}$$

Then, by Theorem 1, AA_1'' is symmedian and so AK is the extension of a symmedian. $\square$

In a triangle, the median is the locus of the midpoints of the line segments joining points on two sides and parallel to the third side (Figure 7). In the case of symmedians, we take line segments antiparallel to the third side. This is the next result.

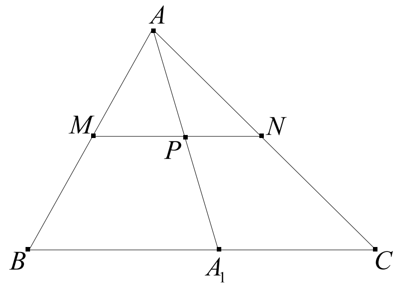


Figure 7:

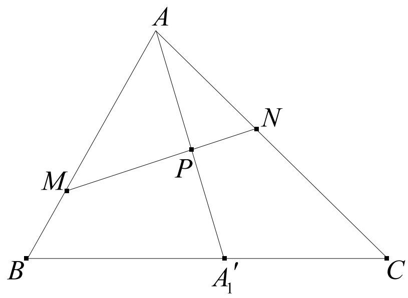


Figure 8:

Theorem 4 *In a triangle ABC, if M and N are points on the sides AB and AC respectively, such that MN is antiparallel to BC, then the midpoint P of MN lies on the symmedian AA_1' (Figure 8).*

Proof Let AA_3 be the bisector of the angle $\widehat{A}$. Points M' and N' on sides AC and AB, respectively are images of M and N under reflection in the line AA_3 (Figure 9),

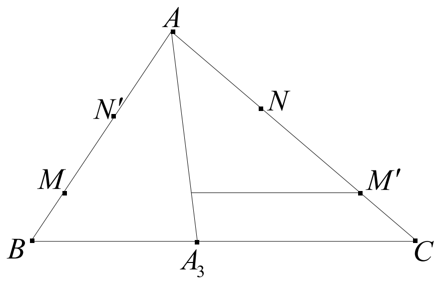


Figure 9:

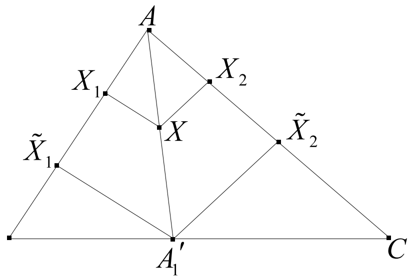
$$M' = Sym_{AA_3}(M), \qquad N' = Sym_{AA_3}(N).$$

Then $|AN| = |AN'|$, $|AM| = |AM'|$ and it follows that the triangles $AN'M'$ and ANM are congruent. Thus $A\widehat{N'}M' = A\widehat{N}M = A\widehat{B}C$, since MN is antiparallel to BC. Thus $N'M'$ is parallel to BC. Then the midpoint of $M'N'$ lies on the median AA_1 and so the midpoint of MN will lie on the symmedian AA_1' since mapping $Sym_{AA_3}(\quad)$ maps midpoints of segments to midpoints of images.

1 Properties of Lemoine Point.

Theorem 5 *If X is a point on the symmedian from the vertex A of a triangle ABC, then the distances from X to the sides AB and AC are in the ratios of the lengths of these sides.*

Proof Let AA_1' be the symmedian from A and let X be a point on AA_1'. Drop perpendiculars XX_1 and XX_2 to the sides AB and AC respectively. Also, drop perpendiculars $A_1'\widetilde{X_1}$ and $A_1'\widetilde{X_2}$ to the sides AB and AC, respectively, from A_1' (Figure 10).



We claim that

$$\frac{d(X,AB)}{|AB|} = \frac{d(X,AC)}{|AC|}, \text{ i.e. } \frac{|XX_1|}{|AB|} = \frac{|XX_2|}{|AC|}.$$

Figure 10:

$$\text{Consider } \frac{d(X,AB)}{d(X,AC)} = \frac{d(A_1',AB)}{d(A_1',AC)}$$

$$= \frac{|BA_1'|\sin\widehat{B}}{|CA_1'|\sin\widehat{C}} = \frac{|AB|^2\sin\widehat{B}}{|AC|^2\sin\widehat{C}}$$

$$= \frac{|AB|}{|AC|},$$

as required.

Theorem 6 *(Grebe's first.)* *If L is the Lemoine point of a triangle ABC, then*

$$\frac{d(L,BC)}{|BC|} = \frac{d(L,AC)}{|AC|} = \frac{d(L,AB)}{|AB|}.$$

This follows immediately from Theorem 5 since L lies on all 3 symmedians.

Theorem 7 *(Grebe's second.)* *The point X in the plane of a triangle ABC which minimisses the quantity*

$$d^2(X,BC) + d^2(X,AC) + d^2(X,AB)$$

is the Lemoine point.

Proof In proving this we shall apply the Cauchy-Schwarz inequality. Recall that if $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are sequences of real numbers then

$$(\sum_{i=1}^n a_i b_i)^2 \le (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$$

with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Let X be a point of the plane and drop perpendiculars from X to the sides AB, BC and CA. Let X_a, X_b and X_c be the feet of the perpendiculars (Figure 11).

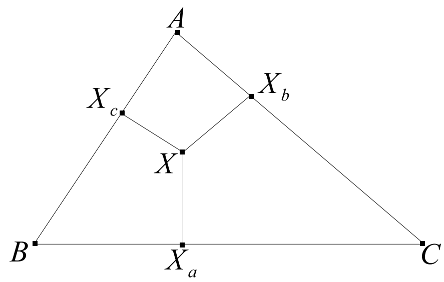


Figure 11:

Consider $a|XX_a| + b|XX_b| + c|XX_c|$
$= 2\,area(ABC)$,
if X is inside ABC.

Then by the Cauchy-Schwarz inequality,

$$4[area(ABC)]^2 \quad \le (a^2+b^2+c^2)\{|XX_a|^2 + |XX_b|^2 + |XX_c|^2\}$$

Thus

$$|XX_a|^2 + |XX_b|^2 + |XX_c|^2 \ge \frac{4(area(ABC))^2}{a^2+b^2+c^2},$$

with equality if and only if

$$\frac{|XX_a|}{a} = \frac{|XX_b|}{b} = \frac{|XX_c}{c} = \text{ constant},$$

and this is true if and only if $X = L$, the Lemoine point.

Theorem 8 *(Rigby?)* *The Lemoine point of a triangle is the centroid of its pedal triangle.*

Proof Let the tangents to the circumcircle, $\mathcal{C}(ABC)$, of the triangle ABC at the points B and C meet at K. From K drop perpendiculars K_a, K_b and K_c to the sides BC, AC(extended) and AB(extended) (Figure 11). We claim that $KK_bK_aK_c$ is a parallelogram.

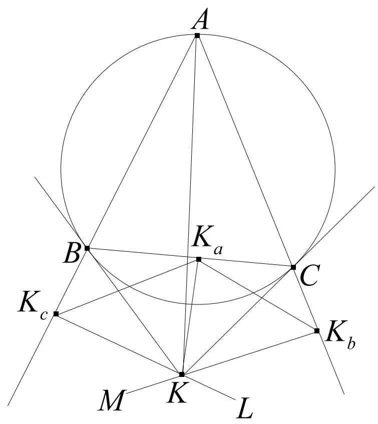


Figure 12:

First extend the line segment K_cK beyond K to a point L and extend K_bK beyond K to a point M.

Since KK_cAK_b is a cyclic quadrilateral then since exterior angles are equal to interior opposites, we have

$$L\widehat{K}K_b = \widehat{A} \text{ and } M\widehat{K}K_c = \widehat{A}.$$

The quadrilateral KK_cBK_a is cyclic so

$$K_c\widehat{B}K = K_c\widehat{K_a}K \text{ (chord } K_cK)$$

and

$$L\widehat{K}K_a = K_c\widehat{B}K \text{ (exterior equal to opposite interior).}$$

The last equation can be written as

$$L\widehat{K}K_b + K_b\widehat{K}K_a = K_c\widehat{K_a}K + K\widehat{B}K_a$$
$$= K_c\widehat{K_a}K + \widehat{A} \text{ (angle between tangent and chord).}$$

But $L\widehat{K} = \widehat{A}$ so we get

$$K_b\widehat{K}K_a = K_c\widehat{K_a}K.$$

Thus lines K_cK_a is parallel to KK_b.
Similarly, by considering the cyclic quadrilateral KK_bCK_a it can be shown that $K_c\widehat{K}K_a = K\widehat{K_a}K_b$, and so lines K_cK and K_aK_b are parallel.

Thus $KK_bK_aK_c$ is a parallelogram, as claimed.

It follows that KK_a bisects K_cK_b and so KK_a passes through the midpoint of K_cK_b.

Now drop perpendicular lines from the Lemoine point L to the sides BC, CA and AB. Let L_a, L_b and L_c be the feet of these perpendiculars (Figure 13).

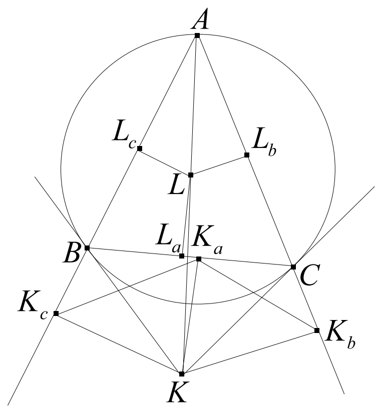


Figure 13:

Clearly LL_c is parallel to KK_c and LL_b is parallel to KK_b.

The triangles AL_cL and AK_cK are similar so

$$\frac{AL_c}{AK_c} = \frac{AL}{AK}.$$

The triangle ALL_b and AKK_b are similar so

$$\frac{AL}{AK} = \frac{AL_b}{AK_b}.$$

Thus, combining both equalities,

$$\frac{AL_c}{AK_c} = \frac{AL_b}{AK_b}.$$

Thus L_cL_b is parallel to K_cK_b.

So we have that the triangles L_cL_bL and K_cK_bK are similar.

Since KK_a is parallel to LL_a and KK_a is a median of the triangle KK_cK_b, then L_aL, when extended, is a median of the triangle LL_cL_b. Thus L lies on the median of the triangle $L_aL_bL_c$ from the vertex L_a. Similarly it can be shown that L also lies on the other medians of the triangle $L_aL_bL_c$.
Result follows since $L_aL_bL_c$ is pedal triangle of the point L. □

Recall that if ABC is a triangle X, Y, Z are points of the sides BC, CA and AB, then the perimeter of the triangle XYZ is a minimum if XYZ is the orthic triangle. Now suppose we wish to minimise the quantity

$$|XY|^2 + |YZ|^2 + |ZX|^2.$$

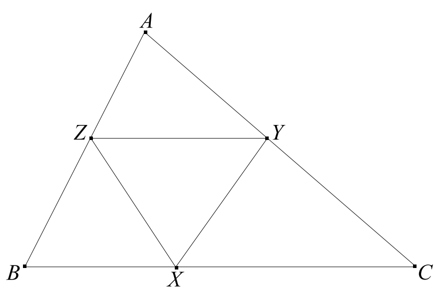


Figure 14:

The next theorem tells us when that is done.

Theorem 9 *If X, Y and Z are 3 points on the sides BC, CA and AB, respectively, then the quantity*

$$|XY|^2 + |YZ|^2 + |ZX|^2.$$

is a minimum when XYZ is the pedal triangle of the Lemoine point L.

Proof First we show that there is a unique set of points X_0, Y_0, Z_0, on the sides such that

$$|X_0Y_0|^2 + |Y_0Z_0|^2 + |Z_0X_0|^2$$

is a minimum. Let $x = |BX|, y = |CY|$ and $z = |AZ|$ (Figure 15).

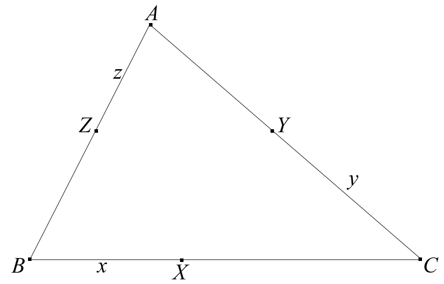


Figure 15:

Now consider the function $P(x, y, z)$ whose value is the quantity

$$|ZY|^2 + |YX|^2 + |XZ|^2.$$

$$\begin{aligned} \text{Then } P(x,y,z) &= z^2 + (b-y)^2 - 2z(b-y)\cos(A) \\ &= x^2 + (c-z)^2 - 2x(c-z)\cos(B) \\ &= y^2 + (a-x)^2 - 2y(a-x)\cos(C) \end{aligned}$$

$$\begin{aligned} &= 2(x^2+y^2+z^2) + (a^2+b^2+c^2) - 2by - 2cz - 2ax \\ &-2bz\cos(A) - 2cx\cos(B) - 2ay\cos(C) \\ &+2yz\cos(A) + 2yx\cos(B) + 2xz\cos(C) \end{aligned}$$

$$\begin{aligned} &= 2(x^2+y^2+z^2) + (a^2+b^2c^2) \\ &+2\{2xy\cos(C) + 2yz\cos(A) + 2zx\cos(B)\} \\ &-2b(y+z\cos(A)) - 2c(z+x\cos(B)) + 2a(x+y\cos(C)). \end{aligned}$$

Since $P(x, y, z)$ represents a sphere or an ellipsoid then there exists a unique solution (x_0, y_0, z_0) which minimises $P(x, y, z)$. This gives corresponding points X_0, Y_0, Z_0 on the sides of the triangle XYZ (Figure 16).

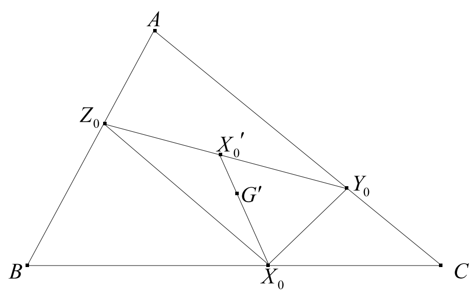


Figure 16:

Now let $X_0Y_0Z_0$ be the triangle which minimises $|X_0Y_0|^2 + |Y_0Z_0|^2 + |Z_0X_0|^2$.

Let G' be the centroid of $X_0Y_0Z_0$ and X_0X_0' be the median from the point X_0.

By the median property of triangles

$$\begin{aligned} |X_0Z_0|^2 + |X_0Y_0| &= 2|X_0X_0'|^2 + 2|X_0'Z_0|^2 \text{ (use cosine rule.)} \\ &= 2|X_0X_0'|^2 + |Z_0Y_0|^2/2. \end{aligned}$$

Thus $|Z_0Y_0|^2 + |Z_0X_0|^2 + |X_0Y_0|^2 = 2|X_0X_0'|^2 + \frac{3}{2}(|Z_0Y_0|^2)$.

This is minimised if X_0X_0' is perpendicular to the side BC. Similarly we need the other two medians of $X_0Y_0Z_0$ to be perpendicular to the other two sides of the triangle. Thus the centroid G' of $X_0Y_0Z_0$ has $X_0Y_0Z_0$ as its pedal triangle. It follows that G' is the Lemoine point L of the triangle ABC. Result follows.