2. Equilateral Triangles

Recall the well-known theorem of van Schooten.

Theorem 1 *If ABC is an equilateral triangle and M is a point on the arc BC of $\mathcal{C}(ABC)$ then*

$$|MA| = |MB| + |MC|.$$

Proof Use Ptolemy on the cyclic quadrilateral $ABMC$. (Figure 1)

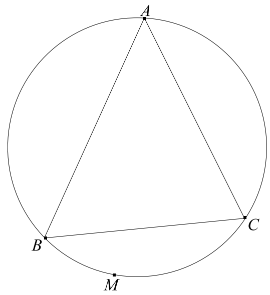


Figure 1:

In fact, using the Ptolemy inequality for quadrilaterals, we get the following van Schooten inequality.

Theorem 2 *Let ABC be an equilateral triangle. Then if M is any point in the plane of ABC we have*

$$|MA| \leq |MB| + |MC|.$$

1 Pompeiu Triangle

We get the following well-known theorem of D. Pompeiu as an immediate consequence of the previous inequality.

Theorem 3 *Let M be any point in the plane of an equilateral triangle ABC. Then the distances $|MA|, |MB|$ and $|MC|$ can be the sidelengths of a triangle.*

Proof It follows immediately since

$$|MA| \leq |MB| + |MC|$$

The triangle is degenerate if M lies on the arc BC of the circumcircle $\mathcal{C}(ABC)$.

A triangle with side lengths $|MA|, |MB|$ and $|MC|$ is called a *Pompeiu triangle*. When M is in the interior of ABC, then the pompeiu triangle can be explicitly constructed.

Locate N so that the triangle BNM is equilateral. Now consider the triangles

$$AMB \text{ and } BNC,$$

We have $AB = BC,$

$$BM = BN$$

and $M\widehat{B}A = 60^\circ - M\widehat{B}C = C\widehat{B}N.$

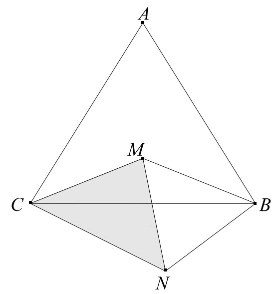


Figure 2:

Thus the triangles are similar, in fact CBN is got by rotating triangle ABM through 60° anti-clockwise in the diagram.

$$\text{Thus } AM = CN,$$

and so the triangle NMC has side lengths equal to $|MA|, |MB|$ and $|MC|$. Thus NMC is the Pompeiu triangle.

The measure of the angles of the Pompeiu triangle in terms of the angles subtended at M by the vertices of ABC and the area of the Pompeiu triangle are given by the following result of the distinguished Romanian born Sabin Tabirca

Theorem 4 *(Tabirca)* *If ABC is an equilateral triangle, and M is an interior point of ABC, then the angles of the Pompeiu triangle and its area are as follows:*

(a) the 3 angles are the angles $B\widehat{M}C - 60^\circ, C\widehat{M}A - 60^\circ$ and $A\widehat{M}B - 60^\circ$;

(b) *the area S (Pompeiu triangle) is*
$\frac{1}{3}$*(area of ABC)*$-\frac{\sqrt{3}}{4}|MO|^2$,
where O is circumcentre of the triangle ABC.

Proof

(a) In the triangle NMC,

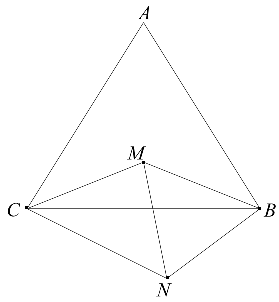


Figure 3:

$$\begin{aligned} C\widehat{M}N &= C\widehat{M}B - N\widehat{M}B \\ &= C\widehat{M}B - 60^\circ, \end{aligned}$$

$$\begin{aligned} C\widehat{N}M &= C\widehat{N}B - M\widehat{N}B \\ &= C\widehat{N}M - 60^\circ \qquad = A\widehat{M}B - 60^\circ, \end{aligned}$$

$$\begin{aligned} \text{and, finally, } M\widehat{C}N &= 180^\circ - \{C\widehat{M}N + C\widehat{N}M\} \\ &= 180^\circ - \{C\widehat{M}B - 60^\circ + A\widehat{M}B - 60^\circ\} \\ &= 300^\circ - \{360^\circ - A\widehat{M}C\} \\ &= \{A\widehat{M}C - 60^\circ\}. \end{aligned}$$

Notation: The area of a triangle ABC is denoted by $S(ABC)$.

(b) In the diagram, NMC is the Pompeiu triangle which we now denote by T_p. Then:

$$\begin{aligned} S(T_p) &= \frac{1}{2}(|CM|.|MN|)\sin(C\widehat{M}N) \\ &= \frac{1}{2}|CM|.|BM|\sin(C\widehat{M}B - 60^\circ) \\ &= \frac{1}{2}|CM|.|BM|\{\sin(C\widehat{M}B).\frac{1}{2} - \cos(C\widehat{M}B).\frac{\sqrt{3}}{2}\} \\ &= \frac{1}{4}|CM|.|BM|\sin(C\widehat{M}B) - \frac{\sqrt{3}}{4}|CM||MB|\cos(C\widehat{M}B) \\ &= \frac{1}{2}S(CMB) - \frac{\sqrt{3}}{8}\{|CM|^2 + |BM|^2 - a^2\}, \end{aligned}$$

where $a = |BC| = |CA| = |AB|$.

Thus $S(T_p) = S(CMB) - \frac{\sqrt{3}}{8}\{|CM|^2 + |BM|^2 - a^2\}$.

Similarly we can show that

$$S(T_p) = \frac{1}{2}S(CMA) - \frac{\sqrt{3}}{8}\{|CM|^2 + |MA|^2 - a^2\},$$

and $S(T_p) = \frac{1}{2}S(BMA) - \frac{\sqrt{3}}{8}\{|BM|^2 + |MA|^2 - a^2\}$.

Adding, we get

$$3S(T_p) = \frac{1}{2}S(ABC) - \frac{\sqrt{3}}{8}\{2(|MA|^2 + |MB|^2 + |MC|^2) - 3a^2\}.$$

Recall the Leibniz formula which states that for any triangle ABC with centroid G and point M

$$|MA|^2 + |MB|^2 + |MC|^2 = 3|MG|^2 + \frac{1}{3}\{|AB|^2 + |BC|^2 + |CA|^2\}$$

In the case of an equilateral triangle, $G = 0$, the centre of the circumcircle and $a^2 = |AB|^2 = |BC|^2 = |CA|^2$ so $|MA|^2+|MB|^2+|MC|^2 = 3|MO|^2+a^2$. Thus

$$3S(T_p) = \frac{1}{2}S(ABC) - \frac{\sqrt{3}}{8}\{6|MO|^2 + 2a^2 - 3a^2\}$$

$$= \frac{1}{2}S(ABC) - \frac{\sqrt{3}}{8}6|MO|^2 + \frac{\sqrt{3}}{8}a^2.$$

Since ABC is equilateral with side length a, $S(ABC) = \frac{\sqrt{3}}{4}a^2$, so

$$\begin{aligned} 3S(T_p) &= \frac{1}{2}S(ABC) - \frac{3\sqrt{3}}{4}|MO|^2 + \frac{1}{2}S(ABC) \\ &= S(ABC) - \frac{3\sqrt{3}}{4}|MO|^2. \end{aligned}$$

Thus $S(T_p) = \frac{1}{3}S(ABC) - \frac{3\sqrt{3}}{4}|MO|^2$, as required.

2 Fermat-Toricelli Point

Let ABC be any triangle and on each side construct externally three equilateral triangles ABC_1, BCA_1 and CAB_1.

Then we have the following theorem.

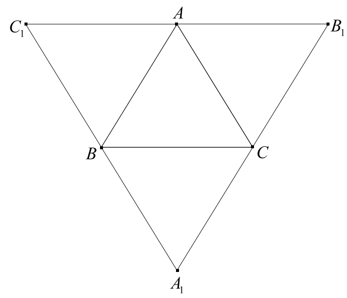


Figure 4:

Theorem 5

(a) The three circumcircles of the equilateral triangles intersect in a point T, i.e.
$\mathcal{C}(ABC_1) \cap \mathcal{C}(BCA_1) \cap \mathcal{C}(CAB_1) = \{T\}$.

(b) The lines AA_1, BB_1 and CC_1 are concurrent at T, i.e.
$AA_1 \cap BB_1 \cap CC_1 = \{T\}$.

(c) $|AA_1| = |BB_1| = |CC_1| = |TA| + |TB| + |TC|$.

(d) For all points M in the plane of ABC,
$|MA| + |MB| + |MC| \geq |AA_1| = |TA| + |TB| + |TC|$.

i.e. the point T minimises the expression $|MA| + |MB| + |MC|$. The point T is called the Toricelli-Fermat *point.*

Proof

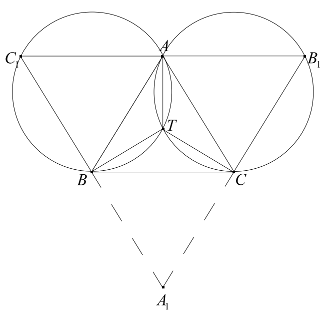


Figure 5:

(a) Let $\{A, T\}$ be the intersection points of the circles $\mathcal{C}(ABC_1)$ and $\mathcal{C}(ACB_1)$.

Then since C_1BTA is cyclic,
$A\widehat{T}B = 180^\circ - A\widehat{C_1}B = 120^\circ$.

Because B_1ATC is cyclic,
$A\widehat{T}C = 180^\circ - AB_1C = 120^\circ$.

Thus $BTC = 120^\circ$ also.
Thus $B\widehat{T}C + B\widehat{A}_1C = 180^\circ$.

and so BA_1CT is cyclic, i.e. $T \in \mathcal{C}(A_1BC)$

(b) We claim that C_1, T, C are collinear points.

$$A\widehat{T}C = 120^\circ, \qquad A\widehat{T}C_1 = A\widehat{B}C_1 = 60^\circ$$

giving $ATC + ATC_1 = 180^\circ$, i.e. C, T and C_1 are collinear. Similarly A, T, A_1 and B, T, B_1 are collinear.

(c) We claim that $|CC_1| = |TA| + |TB| + |TC|$.

Since $T \in \mathcal{C}(AC_1B)$ and AC_1B is equilateral, then by van Schooten's theorem

$$|TC_1| = |TA| + |TB|.$$

Thus $|CC_1| = |CT| + |TC_1| = |TC| + |TA| + |TB|$, as required. Similarly for $|AA_1|$ and $|BB_1|$.

(d) Now let M be any point in the plane of ABC. Then, since ABC_1 is equilateral:

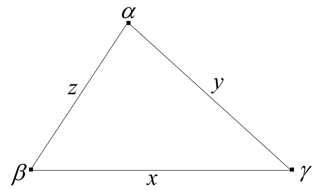
$$|MC_1| \leq |MA| + |MB|$$

Thus
$$\begin{aligned} |MA| + |MB| + |MC| &\geq |MC| + |MC_1| \\ &\geq |CC_1| \\ &= |TA| + |TB| + |TC| \end{aligned}$$

So the point of a triangle which minimises the sum of the distances to the three vertices is the Toricelli-Fermat point. One could ask the question of weighted distances to the vertices and ask which point(s) minimise weighted sums. This is the question we now investigate.

Generalised Fermat-Toricelli Theorem

Let x, y and z be the side length of a triangle $\alpha\beta\gamma$ with x the length of the side opposite vertex α, y the length of the side opposite β and z the length of the side opposite γ.



On an arbitrary triangle ABC construct externally 3 triangles similar to $\alpha\beta\gamma$ with vertices positioned as indicated in Figure 6.

(a) Then their circumcircles intersect at a point T_1, i.e.
$\mathcal{C}(ABC_1) \cap \mathcal{C}(BCA_1) \cap \mathcal{C}(CAB_1) = \{T_1\}$

(b) The lines AA_1, BB_1 and CC_1 are concurrent, i.e.
$AA_1 \cap BB_1 \cap CC_1 = \{T_1\}$

(c) $x|AA_1| = y|BB_1| = z|CC_1|$
$= x|AT_1| = y|BT_1| = z|CT_1|$

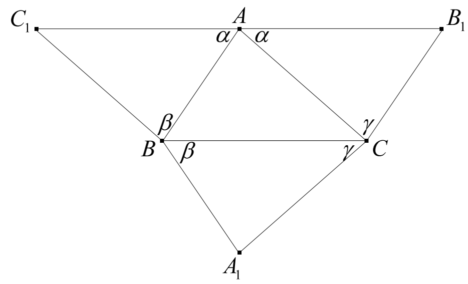


Figure 6:

(d) For any point M in the plane of ABC,

$$x|MA| + y|MB| + z|MC| \geq$$
$$x|AA_1| = x|AT_1| + y|BT_1| + z|CT_1|$$

Thus the point T_1 minimises the weighted distances of a point to the vertices.

Proof The construction of the proof is similar to the proofs in the special case when $x = y = z$.

(a) Let $\mathcal{C}(ABC_1) \cap \mathcal{C}(ACB_1) = \{A, T_1\}$.

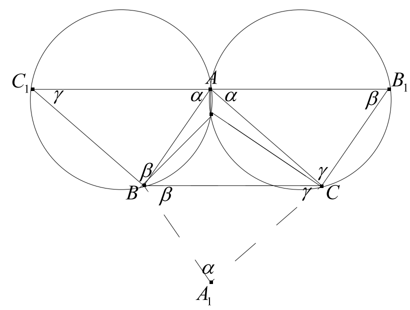


Figure 7:

Since AT_1BC_1 is cyclic,
$A\widehat{T_1}B = 180° - \widehat{\gamma}$,

and AT_1CB_1 is cyclic, so
$A\widehat{T_1}C = 180° - \widehat{\beta}$.

$$\begin{aligned} \text{Thus} \quad B\widehat{T_1}C &= 360° - \{A\widehat{T_1}B + A\widehat{T_1}C\} \\ &= 360° - \{180° - \widehat{\gamma} + 180° - \widehat{\beta} \\ &= 180° - \{\widehat{\gamma} + \widehat{\beta}\} = \widehat{\alpha}. \end{aligned}$$

Thus T_1BA_1C is cyclic, i.e. $T_1 \in \mathcal{C}(BA_1C)$, as required.

(b) We claim that $A\widehat{T_1}C_1 + A\widehat{T_1}C = 180°$ and from this it follows that T_1 lies on CC_1. In a similar way, we get that T_1 also belongs to the line segments BB_1 and AA_1.

To show that $A\widehat{T_1}C_1 + A\widehat{T_1}C = 180°$, we have, since, AT_1CB_1 is cyclic,

$$AT_1C = 180° - \widehat{\beta}.$$

Also, $A\widehat{T_1}C_1 + A\widehat{B}C_1 = 180°$, as required.

(c) Since the triangles A_1BC and $\alpha\beta\gamma$ are similar, then

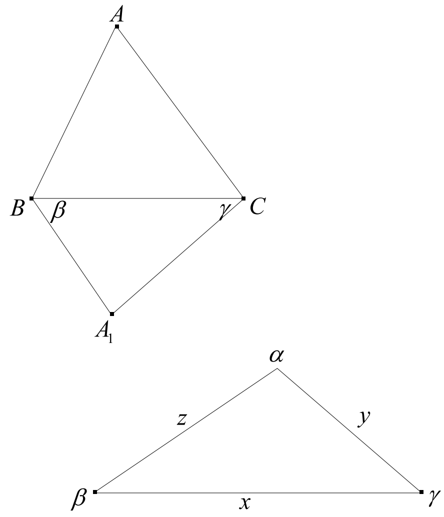


Figure 8:

$$\frac{|A_1B|}{z} = \frac{|A_1C|}{y} = \frac{|BC|}{x}.$$

Thus $|A_1B| = |BC|\dfrac{z}{x}$,

and $|A_1C| = |BC|\dfrac{y}{x}$.

Since $T_1 \in \mathcal{C}(BCA_1)$, then, by Ptolemy,

$$\begin{aligned}|T_1A_1|.|BC| &= |BT_1||CA_1| + |CT_1||BA_1| \\ &= |BT_1|.|BC|\,\frac{y}{x} + |CT_1||BC|\,\frac{z}{x}\end{aligned}$$

Dividing across by $|BC|$ and multiplying by x, we get $x|T_1A_1| = y|BT_1| + z|CT_1|$.

$$\text{Thus} \quad x|T_1A_1| + y|BT_1| + z|CT_1| \\ = x|T_1A| + x|T_1A_1| = x|AA_1|.$$

Similarly, we can show that $x|T_1A| + y|T_1B| + z|T_1C| = y|BB_1| = z|CC_1|$

(d) Now take a point $M \notin \mathcal{C}(BCA_1)$. Then, by the Ptolemy inequality,

$$|BM||CA_1| + |CM|.|BA_1| > |MA_1||BC|$$

Proceeding as in (c) above, we get

$$x|MA| + y|MB| + z|MC| > x|AA_1| = x|T_1A| + y|T_1B| + z|T_1C|$$

as required.

Remarks

1. Now suppose that the positive weights x, y and z are not the sides of a triangle, i.e. suppose

$$x \geq y + z$$

What then is the point which minimises the quantity

$$x|MA| + y|MB| + z|MC|?$$

where M is any point in the plane of ABC.

To decide this, consider

$$\begin{aligned} x|MA|+y|MB|+z|MC| &\geq y(|MA|+|MB|)+z(|MA|+|MC|), \text{ since } x \geq y+z \\ &\geq y|AB|+z|AC| \\ &= x|AA|+y|AB|+z|AC| \end{aligned}$$

Thus the point A(vertex) minimises the quantity $x|MA|+y|MB|+z|MC|$

2. Suppose we take

$$\begin{aligned} x &= \sin(B\widehat{A}C) = \sin\widehat{A}, \\ y &= \sin(A\widehat{B}C) = \sin\widehat{B}, \\ z &= \sin(B\widehat{C}A) = \sin\widehat{C}, \end{aligned}$$

then the weighted expression

$$\sin(\widehat{A})|MA|+\sin(\widehat{B})|MB|+\sin(\widehat{C})|MC|$$

is minimised when $M=O$ the centre of the circumcircle of ABC

3. If we take $x=\sin(\widehat{A}), y=\sin(\widehat{B})$ and $z=\sin(\widehat{C})$, then

$$\sin(\widehat{A})|MA|+\sin(\widehat{B})|MB|+\sin(\widehat{C})|MC|$$

is minimised when $M=H$, the orhthocentre.

4. If $x=\sin(\frac{\widehat{A}}{2}), y=\sin(\frac{\widehat{B}}{2})$ and $z=\sin(\frac{\widehat{C}}{2})$ then

$$\sin(\frac{\widehat{A}}{2})|MA|+(\frac{\sin\widehat{B}}{2})|MB|+\sin(\frac{\widehat{C}}{2})|MC|$$

is minimised when $M=I$, the incentre.