THIRTIETH IRISH MATHEMATICAL OLYMPIAD

Saturday, 6 May 2017

Second Paper

Solutions

6. Proposed by Bernd Kreussler.

Does there exist an even positive integer n for which $n+1$ is divisible by 5 and the two numbers 2^n+n and 2^n-1 are co-prime?

(*Note:* Two integers are said to be *co-prime* if their greatest common divisor is equal to 1.)

Solution Because $(2^n+n)-(2^n-1)=n+1$, we have

$$\gcd(2^n+n, 2^n-1) = \gcd(n+1, 2^n-1)\,.$$

From $2^2 \equiv 4 \pmod 5$, $2^3 \equiv 3 \pmod 5$ and Fermat's Little Theorem we see that $2^n \equiv 1 \pmod 5$ iff n is divisible by 4. Hence, when $n \equiv -1 \pmod 5$ and $n \equiv 0 \pmod 4$, the two numbers 2^n+n and 2^n-1 are both divisible by 5. They can only be co-prime for $n \equiv 2 \pmod 4$.

Suppose $n = 4k+2$, then $n+1 = 4k+3$ and this number is divisible by 5 exactly when $k \equiv 3 \pmod 5$. Such k can be written as $k = 5m+3$ and so $n = 20m+14$. This means that the smallest candidates for n for which 2^n+n and 2^n-1 could be co-prime, are $n = 14, 34, 54, \ldots$.

Next we observe that $2^n \equiv (-1)^n \equiv 1 \pmod 3$ for all even numbers n. Hence, whenever $n+1$ is divisible by 3, the two numbers 2^n+n and 2^n-1 are both divisible by 3. This rules out $n = 14$.

Consider $n = 34$, then $n+1 = 35 = 5 \cdot 7$. As we have seen above, $2^4 \equiv 1 \pmod 5$ and so $2^{34} \equiv 2^2 \equiv 4 \pmod 5$ which means that 5 does not divide $\gcd(35, 2^{34}-1)$. Similarly, we have $2^3 \equiv 1 \pmod 7$ and so $2^{34} \equiv 2 \pmod 7$, which shows that 7 does not divide $\gcd(35, 2^{34}-1)$. Hence, $\gcd(35, 2^{34}-1) = 1$ and $n = 34$ is the smallest positive even integer for which $n+1$ is divisible by 5 and for which 2^n+n and 2^n-1 are co-prime.

7. Proposed by Gordon Lessells.

Five teams play in a soccer competition where each team plays one match against each of the other four teams. A winning team gains 5 points and a losing team 0 points. For a 0-0 draw both teams gain 1 point, and for other draws (1-1, 2-2, etc.) both teams gain 2 points. At the end of the competition, we write down the total points for each team, and we find that they form five consecutive integers. What is the minimum number of goals scored?

Solution 1

Ten matches are played each one contributing either 2, 4 or 5 points. Hence the total number of points is between 20 and 50.

If the team scores are five consecutive integers, then the total number of points must be a multiple of 5. If the total number of points is 20, all teams will score 4 and if the total number of points is 50 all team totals will be multiples of 5. Neither of these possibilities satisfy the conditions. Therefore, we need to consider the following five cases:

(a) scores are $3, 4, 5, 6, 7$,

(b) scores are $4, 5, 6, 7, 8$,

(c) scores are $5, 6, 7, 8, 9$,

(d) scores are $6, 7, 8, 9, 10$ and

(e) scores are $7, 8, 9, 10, 11$.

Case (a) Number of wins is odd. 3 wins yield 15 points and the other seven matches yield more than 10 points. The only possibility is one win, 8 0-0 draws and one score draw. But the team that wins must gain at least 3 points in the other matches. Hence this case is impossible.

Case (b) The number of wins is even and cannot be 4 as only three teams have a score of five or more and none have a score of 10. No wins means there are 5 score draws and 5 no score draws. The teams scoring 8 and 7 must be involved in 7 score draws to achieve these totals. Hence, the only possibility is 2 wins 2 score draws and 6 no score draws. The table

	A	B	C	D	E	total
A		5	1	1	1	8
B	0		1	1	5	7
C	1	1		2	2	6
D	1	1	2		1	5
E	1	0	2	1		4

realizes this possibility. The minimum number of goals in this case is $2+4 = 6$.

Case (c) The number of wins is odd. No team has more than one win. If the number of wins is five, each team must win one match and all other matches are no score draws. A total of 9 is now impossible. If the number of wins is 3, there must be 3 score draws and 4 no score draws. At least 9 goals are scored in this scenario. If the number of wins is 1, there are 6 score draws and three no score draws. This gives at least 13 goals.

Case (d) Again we can calculate the number of wins (W), score draws (S) and no score draws (N) yielding 40 points. The four possibilities are $(W, S, N) = (6, 1, 3)$ or $(4, 4, 2)$ or $(2, 7, 1)$ or $(0, 10, 0)$. The minimum number of goals is $W + 2S$ which in each case is bigger than 6.

Case (e) Calculating possible values of (W, S, N) we obtain $(7, 2, 1)$, $(5, 5, 0)$ giving more than 6 goals.

Thus the minimum number of goals scored in the tournament is 6.

Solution 2

If the five consecutive scores of the teams are $a-2, a-1, a, a+1, a+1$, the total number of points is $5a$. Each match contributes $5, 4$ or 2 points to this total.

Let w be the number of matches that did not end in a draw, d_0 the number of 0-0 draws and d_1 the number of draws with goals. The minimal possible number of goals scored is $g = w + 2d_1$. The number of matches is $\binom{5}{2} = 10$, so we obtain

$$\begin{aligned} w + d_0 + d_1 &= 10 \\ 5w + 2d_0 + 4d_1 &= 5a\,. \end{aligned}$$

Eliminating d_0 from these equations we obtain $3w + 2d_1 = 5(a-4)$. This equation implies $3w \equiv -2d_1 \pmod 5$, hence $w \equiv d_1 \pmod 5$.

None of w, d_1 and $w + d_1$ can exceed 10. Hence, $d_1 = w + 5t$ with $-2 \leq t \leq 2$. We obtain $g = w + 2d_1 = 3w + 10t$.

- If $t = -2$, we have $w = d_1 + 10 \leq 10$, hence $w = 10, d_1 = 0$ and $g = 10$.
- If $t = -1$, we have $d_1 = w - 5 \geq 0$ and $w + d_1 = 2w - 5 \leq 10$, hence $5 \leq w \leq 7$. If $w \geq 6$, we have $g = 3w - 10 \geq 8$. For $w = 5$ we get $g = 5$.
- If $t = 0$, and $w \geq 3$ we have $g = 3w + 10t \geq 9$.
- If $t \geq 1$, we have $g \geq 10$.

Hence, there are only four cases in which $g \leq 6$. They are all shown in the table below. The values of d_0, a and g are obtained from the equations above.

case	w	d_1	d_0	a	g
1	0	0	10	4	0
2	1	1	8	5	3
3	5	0	5	7	5
4	2	2	6	6	6

Case 1. If $d_0 = 10$ all matches were no score draws, hence all teams scored 5 points contradicting the conditions.

Case 2. If $a = 5$, the winning team achieved 7 points. But, if $w = 1$ the team who has won one match cannot have lost any of their matches, hence would have scored at least 8 points.

Case 3. If $a = 7$, the maximum number of points a team scored is 9, hence no team won two matches. As $d_1 = 0$, the top team can have scored at most 3 points from the matches it didn't win, hence cannot have got 9 points in total.

Case 4. This case is actually possible, as the following table shows.

	A	B	C	D	E	total
A		5	1	1	1	8
B	0		1	1	5	7
C	1	1		2	2	6
D	1	1	2		1	5
E	1	0	2	1		4

Therefore, the minimum number of goals scored is 6.

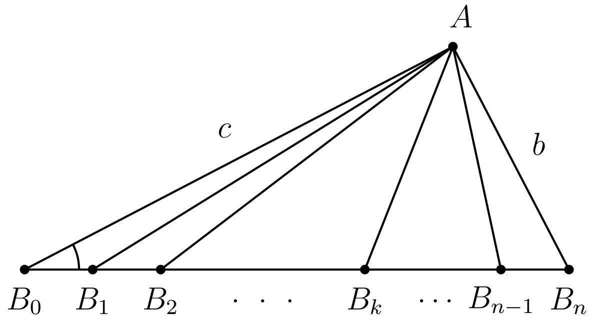
8. Proposed by Jim Leahy.

A line segment B_0B_n is divided into n equal parts at points $B_1, B_2, \ldots, B_{n-1}$ and A is a point such that $\angle B_0AB_n$ is a right angle. Prove that

$$\sum_{k=0}^{n} |AB_k|^2 = \sum_{k=0}^{n} |B_0B_k|^2 .$$

Solution 1

Let $a = |B_0B_n|$, $b = |AB_n|$ and $c = |AB_0|$. Then $|B_0B_k| = ka/n$. Because $\angle B_0AB_n = 90°$, we have $\cos\angle AB_0B_n = c/a$.



Hence, for $k = 1, 2, \ldots, n-1$, the Cosine Rule for triangle AB_0B_k gives

$$|AB_k|^2 = c^2 + |B_0B_k|^2 - \frac{2c^2}{a}|B_0B_k| = |B_0B_k|^2 + c^2\left(1 - \frac{2k}{n}\right) .$$

Therefore, using that $|AB_0|^2 = c^2$, $|AB_n|^2 = b^2$ and $c^2 + b^2 = a^2 = |B_0B_n|^2$, we obtain

$$\sum_{k=0}^{n} |AB_k|^2 = c^2+b^2+\sum_{k=1}^{n-1}\left(|B_0B_k|^2 + c^2\left(1 - \frac{2k}{n}\right)\right) = \sum_{k=0}^{n}|B_0B_k|^2+c^2\sum_{k=1}^{n-1}\left(1-\frac{2k}{n}\right) .$$

Now using $\sum_{k=1}^{n-1} k = n(n-1)/2$ we get

$$\sum_{k=1}^{n-1}\left(1 - \frac{2k}{n}\right) = (n-1) - \frac{2}{n}\cdot\frac{n(n-1)}{2} = (n-1) - (n-1) = 0.$$

This could also be seen directly from $1 - \frac{2(n-k)}{n} = \frac{2k}{n} - 1$. In any case, the desired result follows.

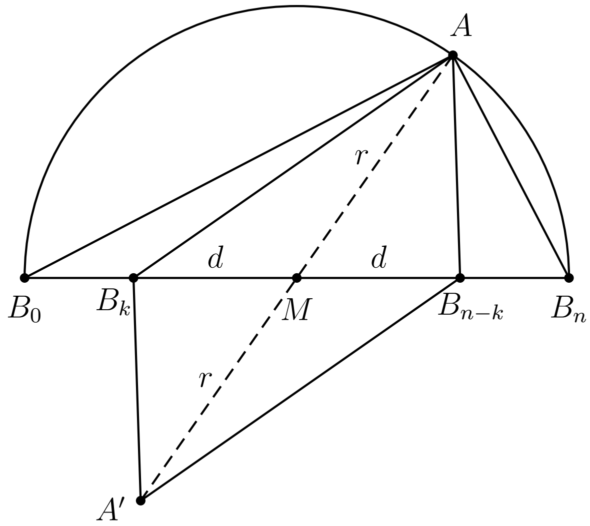
Solution 2

We are going to prove the stronger equations

$$|AB_k|^2 + |AB_{n-k}|^2 = |B_0B_k|^2 + |B_0B_{n-k}|^2 \qquad \text{for } k = 0, 1, \ldots, n.$$

Let M be the mid-point of B_0B_n. Let $r = |AM|$ and $d = |MB_k| = |MB_{n-k}|$.

If $n = 2m$ is even, then $M = B_m$, otherwise M is not among the points B_i. Because $\angle B_0AB_n = 90°$, A is on the circle with diameter B_0B_n and so $r = |B_0M| = |MB_n|$. If $n = 2m$ and $k = m$, the claimed equation now follows: $2|AM|^2 = 2r^2 = 2|B_0M|^2$.



Suppose now that $2k \neq n$, then $B_k \neq B_{n-k}$. Let A' be the point for which $AB_kA'B_{n-k}$ is a parallelogram. The diagonals of this parallelogram intersect at M and $|A'M| = |AM| = r$. The Parallelogram Law

$$2|AB_k|^2 + 2|AB_{n-k}|^2 = |B_kB_{n-k}|^2 + |AA'|^2$$

now gives us

$$|AB_k|^2 + |AB_{n-k}|^2 = 2d^2 + 2r^2.$$

On the other hand, $|B_0B_k|^2 + |B_0B_{n-k}|^2 = (r-d)^2 + (r+d)^2 = 2r^2 + 2d^2$ which establishes that

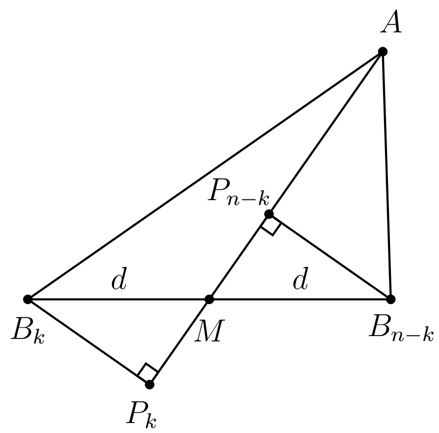
$$|AB_k|^2 + |AB_{n-k}|^2 = |B_0B_k|^2 + |B_0B_{n-k}|^2 \qquad \text{for } k = 0, 1, \ldots, n.$$

Summation over k followed by a division by 2 establishes the required equality.

Remark. Instead of using the Parallelogram Law, one could establish

$$|AB_k|^2 + |AB_{n-k}|^2 = 2d^2 + 2r^2$$

with the aid of the Theorem of Pythagoras directly as follows. Let P_k and P_{n-k} be the orthogonal projections of B_k and B_{n-k}, respectively, onto the line AM.



Note that the two right angled triangles B_kP_kM and $B_{n-k}P_{n-k}M$, both with hypotenuse of length d, are congruent. Recalling $|AM| = r$ and setting $e = |MP_k| = |MP_{n-k}|$, we obtain from Pythagoras' Theorem

$$\begin{aligned}
|AB_k|^2 + |AB_{n-k}|^2 &= |B_kP_k|^2 + (r+e)^2 + |B_{n-k}P_{n-k}|^2 + (r-e)^2 \\
&= |B_kP_k|^2 + e^2 + |B_{n-k}P_{n-k}|^2 + e^2 + 2r^2 \\
&= 2d^2 + 2r^2.
\end{aligned}$$

9. Proposed by Steve Buckley.

Show that for all non-negative numbers a, b,

$$1 + a^{2017} + b^{2017} \geq a^{10}b^7 + a^7b^{2000} + a^{2000}b^{10}\,.$$

When is equality attained?

Solution

We use AM-GM three times, taking means of 2017 items, each of which equals 1^{2017} or a^{2017} or b^{2017}:

$$\frac{2000 \cdot 1^{2017} + 10\,a^{2017} + 7\,b^{2017}}{2017} \geq 1^{2000} \cdot a^{10} \cdot b^7$$

$$\frac{10 \cdot 1^{2017} + 7\,a^{2017} + 2000\,b^{2017}}{2017} \geq 1^{10} \cdot a^7 \cdot b^{2000}$$

$$\frac{7 \cdot 1^{2017} + 2000\,a^{2017} + 10\,b^{2017}}{2017} \geq 1^7 \cdot a^{2000} \cdot b^{10}$$

Adding these three inequalities gives the desired inequality.

For equality, all of these inequalities must hold with equality. By the conditions for equality in AM-GM, this means that $a = b = 1$.

10. Proposed by Steve Buckley.

Given a positive integer m, a sequence of real numbers $a = (a_1, a_2, a_3, \ldots)$ is called m-powerful if it satisfies

$$\left(\sum_{k=1}^{n} a_k\right)^m = \sum_{k=1}^{n} a_k^m \qquad \text{for all positive integers } n.$$

(a) Show that a sequence is 30-powerful if and only if at most one of its terms is non-zero.

(b) Find a sequence none of whose terms is zero but which is 2017-powerful.

Solution

If at most one term in a sequence a is non-zero, it is immediate that it is m-powerful for all $m \in \mathbb{N}$. We will see that there are no other 30-powerful sequences (or indeed, m-powerful for any even m), but that there are many other sequences that are m-powerful for all odd m.

We begin our analysis with a simple lemma.

Lemma 1. *For $m \in \mathbb{N}$, let $P_m(x,y) = (x+y)^m - x^m - y^m$, $x, y \in \mathbb{R}$. Then $P_m(x,y) = 0$ whenever $xy = 0$. Moreover, P_m has no other roots if m is even, but $P_m(x,-x) = 0$ if m is odd.*

Proof. Let $p(t) = (1+t)^m - 1 - t^m$, $t \in \mathbb{R}$. By expansion of $(1+t)^m$, we see that p is a polynomial with non-negative coefficients, and so $p(t) > 0$ for $t > 0$. Suppose additionally that m is even. If $-1 \le t < 0$, then $(1+t)^m - 1 \le 0$, so $p(t) \le -t^m < 0$. If $t < -1$, then $0 < (1+t)^m < t^m$, so $p(t) < -1 < 0$. We conclude that 0 is the only root of p when m is even.

If $xy \neq 0$, then $P_m(x,y) = x^m p(t)$, where p is as above, and $t = y/x \neq 0$. It follows that $P_m(x,y) \neq 0$ if m is even and $xy \neq 0$.

The statements that $P_m(x,y) = 0$ when $xy = 0$ (regardless of the parity of m) and $P_m(x,-x) = 0$ when m is odd, both follow immediately. □

Fix an arbitrary $m \in \mathbb{N}$ and an arbitrary sequence $a = (a_n)$. For $m, n \in \mathbb{N}$, let

$$S(m,n) = \left(\sum_{k=1}^{n} a_k\right)^m - \sum_{k=1}^{n} a_k^m,$$

and let $D(m,n) = S(m,n+1) - S(m,n)$. The condition "$a$ is m-powerful" says that $S(m,n) = 0$, $n \in \mathbb{N}$, and so we also have $D(m,n) = 0$.

Define $x_n = a_{n+1}$ and $y_n = \sum_{k=1}^{n} a_n$. The equation $S(m,n) = 0$ can be written as

$$y_n^m = \sum_{k=1}^{n} a_k^m,$$

so if this equation holds, then the equation $D(m,n) = 0$ can be written as

$$(x_n + y_n)^m - x_n^m - y_n^m = 0.$$

Suppose now that m is even and a is m-powerful. By Lemma 1, we conclude that $x_n y_n = 0$ for all $n \in \mathbb{N}$. We will use these last equations to prove by induction on $i \in \mathbb{N}$, that at most one of the terms $a_1, \ldots, a_i$ is non-zero; we call this property A_i.

A_1 is trivially true, so suppose that A_i is true for a specific $i = n \in \mathbb{N}$. If $a_1, \ldots, a_n$ are all zero, then A_{n+1} follows immediately, so we may assume that exactly one of these terms is non-zero. But now $y_n \neq 0$, so the equation $x_n y_n = 0$ implies that $x_n = a_{n+1} = 0$, and we again deduce A_{n+1}. This finishes the proof that if m is even, then the m-powerful sequences are those with at most one non-zero term. Part (a) follows.

The analysis is similar for m odd, but now P_m has other roots in Lemma 1. By considering these roots, our analysis readily leads us to see that $a = ((-1)^n c)_{n=1}^{\infty}$ is m-powerful for every $c \in \mathbb{R}$. Taking any non-zero c, we get an example with the properties required in (b).