TWENTY NINTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 23 April 2016

Second Paper

Solutions

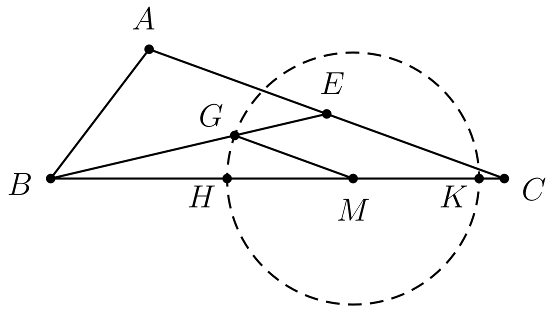
6. Proposed by Jim Leahy.

Triangle ABC has sides $a = |BC| > b = |AC|$. The points K and H on the segment BC satisfy $|CH| = (a+b)/3$ and $|CK| = (a-b)/3$. If G is the centroid of triangle ABC, prove that $\angle KGH = 90°$.

Solution

Because $|CH| > |CK|$ and both, H and K, are between C and B, we have $|KH| = |CH| - |CK| = \frac{a+b}{3} - \frac{a-b}{3} = \frac{2}{3}b$. If M is the midpoint of KH, then $|KM| = |HM| = \frac{1}{3}b$. Next, we show that $|GM| = \frac{1}{3}b$ as well. Here are two ways to show this.

Version 1. Let E be the midpoint of AC.

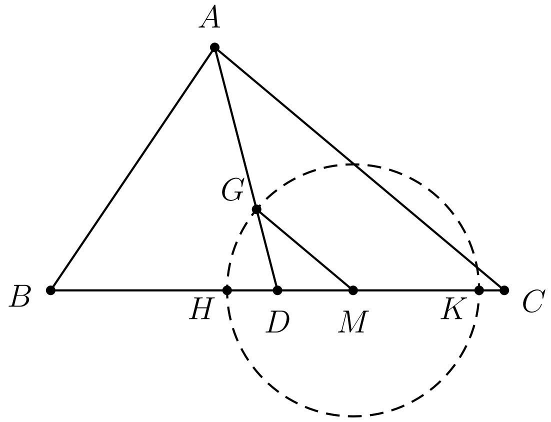


Because $|CM| = |CK| + |KM| = \frac{a-b}{3} + \frac{b}{3} = \frac{a}{3} = \frac{1}{3}|BC|$ and $|EG| = \frac{1}{3}|BE|$, the lines GM and EC are parallel. This implies that

$$\frac{|GM|}{|EC|} = \frac{|BM|}{|BC|} = \frac{|BC| - |CM|}{|BC|} = 1 - \frac{|CM|}{|BC|} = \frac{2}{3}.$$

Hence, using that $|EC| = b/2$, we obtain $|GM| = \frac{2}{3}|EC| = \frac{1}{3}b$, as desired.

Version 2. Let D be the midpoint of BC.



Because

$$|CM| = |CK| + |KM| = \frac{a-b}{3} + \frac{b}{3} = \frac{a}{3} < \frac{a}{2} = |CD|,$$

M is between C and D and we have

$$|DM| = |CD| - |CM| = \frac{a}{2} - \frac{a}{3} = \frac{a}{6} = \frac{1}{3}|DC|.$$

Because $|DG| = \frac{1}{3}|DA|$, the lines GM and AC are parallel, hence

$$\frac{|AC|}{|GM|} = \frac{|DA|}{|DG|} = 3 \quad \text{and so} \quad |GM| = \frac{b}{3}.$$

After establishing that $|GM| = \frac{b}{3}$, we obtain $|GM| = |KM| = |HM|$, thus G is on the circle with diameter KH which implies that $\angle KGH = 90°$.

7. Proposed by Rachel Quinlan.

A rectangular array of positive integers has four rows. The sum of the entries in each column is 20. Within each row, all entries are distinct. What is the maximum possible number of columns?

Solution

Let the number of columns be k. Then the sum of all the numbers in the array is $20k$. Since each row consists of k distinct positive integers, the least possible value of the sum of the entries in a single row is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Thus

$$20k \geq 4 \times \frac{k(k+1)}{2}.$$

After dividing both sides by $2k$ we get $10 \geq k + 1$, and so $k \leq 9$. This shows that the number of columns cannot exceed 9.

On the other hand, the bound of 9 can be attained as the following example shows.

1	2	3	4	5	6	7	8	9
2	1	4	3	5	7	6	9	8
8	9	6	7	5	3	4	1	2
9	8	7	6	5	4	3	2	1

8. Proposed by Finbarr Holland.

Suppose a, b, c are real numbers such that $abc \neq 0$. Determine x, y, z in terms of a, b, c such that

$$bz + cy = a, \quad cx + az = b, \quad ay + bx = c.$$

Prove also that

$$\frac{1-x^2}{a^2} = \frac{1-y^2}{b^2} = \frac{1-z^2}{c^2}.$$

Solution

Multiply the first equation by a, the second by b and the third by c. Add the difference between the first and second of the resulting equations to the third one, thereby obtaining the following one:

$$2cay = c^2 + a^2 - b^2.$$

Since $ca \neq 0$, it follows that

$$y = \frac{c^2 + a^2 - b^2}{2ca}.$$

Likewise, by cyclicity,

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \quad z = \frac{a^2 + b^2 - c^2}{2ab}.$$

Next, focusing on z,

$$\begin{aligned}
1 - z^2 &= (1-z)(1+z) \\
&= \left(\frac{2ab - (a^2 + b^2 - c^2)}{2ab}\right)\left(\frac{2ab + (a^2 + b^2 - c^2)}{2ab}\right) \\
&= \left(\frac{2ab - a^2 - b^2 + c^2}{2ab}\right)\left(\frac{2ab + a^2 + b^2 - c^2}{2ab}\right) \\
&= \left(\frac{c^2 - (a-b)^2}{2ab}\right)\left(\frac{(a+b)^2 - c^2}{2ab}\right) \\
&= \frac{\big(c - (a-b)\big)\big(c + (a-b)\big)\big((a+b) - c\big)\big(a+b+c\big)}{4a^2b^2} \\
&= \frac{(b+c-a)(c+a-b)(a+b-c)(a+b+c)}{4a^2b^2},
\end{aligned}$$

whence

$$\frac{1-z^2}{c^2} = \frac{4s(s-a)(s-b)(s-c)}{a^2b^2c^2},$$

where $2s = a + b + c$. Since the expression on the right side of this equation is symmetric in a, b, and c, we may conclude as well that

$$\frac{1-x^2}{a^2} = \frac{1-y^2}{b^2} = \frac{(b+c-a)(c+a-b)(a+b-c)(a+b+c)}{4a^2b^2c^2},$$

whence the result.

9. Proposed by Gordon Lessells.

Show that the number

$$\left(\frac{251}{\frac{1}{\sqrt[3]{252}-5\sqrt[3]{2}}-10\sqrt[3]{63}}+\frac{1}{\frac{251}{\sqrt[3]{252}+5\sqrt[3]{2}}+10\sqrt[3]{63}}\right)^3$$

is an integer and find its value.

Solution

Let $a^3 = 252$ and $b^3 = 250$, then $ab = \sqrt[3]{252}\sqrt[3]{250} = 10\sqrt[3]{63}$. Using that $251 = \frac{a^3+b^3}{2}$ and $1 = \frac{a^3-b^3}{2}$, the given expression inside the bracket becomes

$$\frac{\frac{a^3+b^3}{2}}{\frac{\frac{a^3-b^3}{2}}{a-b}-ab}+\frac{\frac{a^3-b^3}{2}}{\frac{\frac{a^3+b^3}{2}}{a+b}+ab}.$$

The second term is obtained from the first by replacing b with $-b$. Therefore, we consider only one of the two terms. The first term is equal to

$$\frac{a^3+b^3}{\frac{a^3-b^3}{a-b}-2ab}=\frac{a^3+b^3}{a^2+ab+b^2-2ab}=\frac{a^3+b^3}{a^2-ab+b^2}=a+b.$$

Hence, the second term is equal to $a-b$ and the given number is equal to

$$\big((a+b)+(a-b)\big)^3=(2a)^3=8\cdot 252=2016.$$

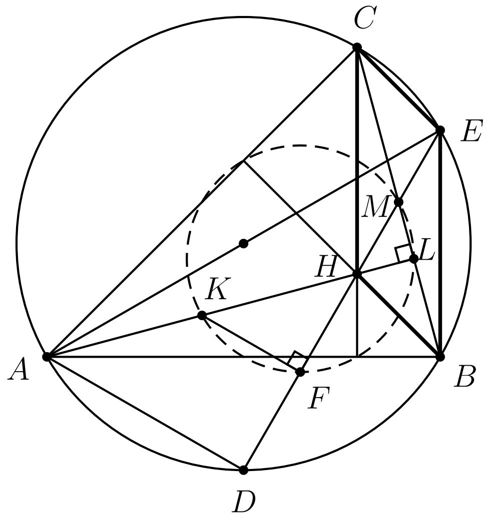
10. Proposed by Jim Leahy.

Let AE be a diameter of the circumcircle of triangle ABC. Join E to the orthocentre, H, of $\triangle ABC$ and extend EH to meet the circle again at D. Prove that the *nine point circle* of $\triangle ABC$ passes through the midpoint of HD.

NOTE. The *nine point circle* of a triangle is a circle that passes through the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments that join the orthocentre to the vertices.

Solution

Let F be the mid pint of DH, K be the mid point of AH, L on BC the foot of the altitude from A, and let M be the intersection point of BC and DE.



Because AE is a diameter of the circumcircle, $\angle ACE = \angle ABE = 90°$, so CE is perpendicular to AC and BE is perpendicular to AB. Because BH is on the altitude which is perpendicular to AC, we get $BH \| CE$. Similarly, CH is perpendicular to AB and so $CH \| BE$. This shows that $CHBE$ is a parallelogram and M is the intersection point of its diagonals. Hence M is the mid point of BC and thus is on the nine point circle.

Because AE is a diameter of the circle, $\angle ADE = 90°$. As F, K are the mid points of DH and AH, respectively, we obtain $\angle KFM = 90°$. Because $\angle KLM = 90°$ as well, the quadrilateral $KFLM$ is cyclic.

As K, L and M are on the nine point circle, F is on that circle as well.