

TWENTY EIGHTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 25 April 2015

Second Paper

Time allowed: **Three hours.**

6. Suppose x, y are nonnegative real numbers such that $x + y \leq 1$. Prove that

$$8xy \leq 5x(1 - x) + 5y(1 - y),$$

and determine the cases of equality.

7. Let $n > 1$ be an integer and $\Omega := \{1, 2, \dots, 2n - 1, 2n\}$ the set of all positive integers that are not larger than $2n$.

A nonempty subset S of Ω is called *sum-free* if, for all elements x, y belonging to S , $x + y$ does not belong to S . We allow $x = y$ in this condition.

Prove that Ω has more than 2^n distinct sum-free subsets.

8. In triangle $\triangle ABC$, the angle $\angle BAC$ is less than 90° . The perpendiculars from C on AB and from B on AC intersect the circumcircle of $\triangle ABC$ again at D and E respectively. If $|DE| = |BC|$, find the measure of the angle $\angle BAC$.

9. Let $p(x)$ and $q(x)$ be non-constant polynomial functions with integer coefficients. It is known that the polynomial

$$p(x)q(x) - 2015$$

has at least 33 different integer roots. Prove that neither $p(x)$ nor $q(x)$ can be a polynomial of degree less than three.

10. Prove that, for all pairs of nonnegative integers, j, n ,

$$\sum_{k=0}^n k^j \binom{n}{k} \geq 2^{n-j} n^j.$$