TWENTY EIGHTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 25 April 2015

First Paper

Time allowed: **Three hours.**

1. In the triangle ABC, the length of the altitude from A to BC is equal to 1. D is the midpoint of AC. What are the possible lengths of BD?

2. A regular polygon with $n \geq 3$ sides is given. Each vertex is coloured either red, green or blue, and no two adjacent vertices of the polygon are the same colour. There is at least one vertex of each colour.

   Prove that it is possible to draw certain diagonals of the polygon in such a way that they intersect only at the vertices of the polygon and they divide the polygon into triangles so that each such triangle has vertices of three different colours.

3. Find all positive integers n for which both $837 + n$ and $837 - n$ are cubes of positive integers.

4. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$, with centres at D and E respectively, touch at B. The circle having DE as diameter intersects the circle $\mathcal{C}_1$ at H and the circle $\mathcal{C}_2$ at K. The points H and K both lie on the same side of the line DE. HK extended in both directions meets the circle $\mathcal{C}_1$ at L and meets the circle $\mathcal{C}_2$ at M. Prove that

   (a) $|LH| = |KM|$;

   (b) the line through B perpendicular to DE bisects HK.

5. Suppose a doubly infinite sequence of real numbers

   $$\ldots,\ a_{-2},\ a_{-1},\ a_0,\ a_1,\ a_2,\ \ldots$$

   has the property that

   $$a_{n+3} = \frac{a_n + a_{n+1} + a_{n+2}}{3}, \qquad \text{for all integers } n.$$

   Show that if this sequence is bounded (i.e., if there exists a number R such that $|a_n| \leq R$ for all n), then a_n has the same value for all n.