Two Enrichment Classes on Inequalities given in 2011

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LECTURE I

1 Introduction

Inequalities or inequations are the lifeblood of the system of real numbers $\mathbb{R}$, which contains a subset called the set of *positive numbers*, which is closed under addition and multiplication, by which is meant that both the sum and product of two positive numbers is positive. This enables us to *order* the real numbers: given $a, b \in \mathbb{R}$, we say a is bigger than b if $a - b$ is positive.

The theory of inequalities has to do with deciding when one real number is bigger than or equal to another real number, i.e., when their difference is positive or zero. In these lecture notes we introduce some symbolism, build up some simple propositions, illustrate these and introduce special named inequalities, which keep on recurring.

2 Language of Inequalities

The *Law of Trichotomy* tells us that every real number is either positive, zero or negative. More precisely, if $x \in \mathbb{R}$, then one and only one of the following possibilities holds; (i) x is positive; (ii) x is zero; (iii) x is negative. In symbols, either (i) $x > 0$ (read 'x is greater than zero'); (ii) $x = 0$ (read 'x is zero'; (iii) $x < 0$ (read 'x is less than zero'). We say x is *non-negative* and write, $x \geq 0$ to mean that x is either greater than or equal to zero. If $a, b \in \mathbb{R}$, we write $a \geq b$ to mean that $a - b \geq 0$. If $a, b, x \in \mathbb{R}$, we say x

is *between* a and b if either (i) $x > a$ and $x < b$ or (ii) $x < a$ and $x > b$. But note that, at most one of these possibilities can occur, and neither may occur.

If $a < b$, we denote the set of points between a and b by (a, b), and call it the *open interval* with endpoints a, b. Thus

$$(a, b) = \{x \in \mathbb{R} : a < x < b\};$$

Similarly, we define

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}; \ \ (a, b] = \{x \in \mathbb{R} : a < x \leq b\}; \ \ [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

The set $[a, b]$ stands for the *closed interval* with endpoints a, b. Hence, for instance,

$$(a, b) \subset [a, b) \subset [a, b], \ \ (a, b) \subset (a, b] \subset [a, b] \subset \mathbb{R},$$

and all the inclusions are proper containments. Sometimes it's convenient to introduce the symbols $\pm\infty$—but that's all they are: they are *not* real numbers—and use them to denote the negative and positive numbers like so:

$$(-\infty, 0) = \{x \in \mathbb{R} : x < 0\}, \ \ (0, \infty) = \{x \in \mathbb{R} : 0 < x\}.$$

Likewise, $\mathbb{R} = (-\infty, \infty)$, and

$$(-\infty, 0] = \{x \in \mathbb{R} : x \leq 0\}, \ \ [0, \infty) = \{x \in \mathbb{R} : 0 \leq x\}.$$

3 Elementary facts

The real numbers carry two binary combining operations, called addition and multiplication. It's important to know how these relate to the order structure imposed above. The basic fact is axiomatic:

Axiom 1 *Suppose $a, b \in \mathbb{R}$, $a > 0$ and $b > 0$. Then $a + b > 0$ and $ab > 0$.*

Proposition 1 *Suppose $a, b, c \in \mathbb{R}$. If $a < b$, then $a + c < b + c$. Conversely, if $a + c < b + c$, then $a < b$.*

Proof. Suppose $a < b$, so that $b - a > 0$. Then $(b + c) - (a + c) = b + c - a - c = b - a > 0$, which means that $b + c > a + c$. Conversely, if the latter occurs, then $b - a = (b + c) - (a + c) > 0$, i.e., $a < b$. □

Exercise 1 *Suppose $x \in \mathbb{R}$. Prove that $x < 0 \Leftrightarrow -x > 0$.*

Proposition 2 *Suppose $a, b, c \in \mathbb{R}$, $a < b$ and $b < c$. Then $a < c$*

Proof. By hypothesis, $b - a > 0$ and $c - b > 0$. Hence, by Axiom 1, $(b - a) + (c - b) > 0$. But

$$c - a = c - b + (b - a) = (c - a) + (b - a) > 0.$$

In other words, $c > a$. □

Exercise 2 *Suppose $a, b, c \in \mathbb{R}$, $a \leq b$ and $b \leq c$. Prove that $a \leq c$.*

Proposition 3 *Suppose $a, b, c \in \mathbb{R}$ and $a < b$. (i)If $c > 0$, then $ac < bc$. (ii)If $c = 0$, then $ac = bc$. (iii) If $c < 0$, then $ac > bc$.*

Proof. By hypothesis, $b - a > 0$. Consider (i). If $c > 0$, then, by Axiom 1, $(b - a)c > 0$, i.e., $bc - ac > 0 \Leftrightarrow bc > ac$. Thus, (i) holds. Clearly, (ii) holds. If $c < 0$, then $-c > 0$ and so, again, by Axiom 1, $(-c)(b - a) > 0$, equivalently, $-cb + ca > 0$, i.e., $ac > bc$, which implies (iii). □

Theorem 1 *Suppose $x \in \mathbb{R}$. Then $x^2 \geq 0$. The inequality is strict unless $x = 0$.*

Proof. By Trichotomy, either (i) $x > 0$; or (ii) $x = 0$; or (iii) $x < 0$. Consider each of these in turn. If (i) holds, then, by Axiom 1, $x^2 = xx > 0$. If (ii) holds, then $x^2 = 0x = 0$. Finally, if $x < 0$, then $-x > 0$, whence, again by Axiom 1, $x^2 = (-x)(-x) > 0$. In summary, $x^2 \geq 0$.

Next, if $x^2 = 0$, and $x \neq 0$, then, by (i) or (iii), $x^2 > 0$, a contradiction. Thus, $x^2 = 0 \Leftrightarrow x = 0$. □

Theorem 2 *Suppose $a, b \in \mathbb{R}$. If $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$. Moreover, $a + b > 0$ unless $a = b = 0$.*

Proof. The first part is an easy consequence of Axiom 1. If $a + b = 0$, and a is positive, say, then $a + b > b \geq 0$, by Proposition 1, which is absurd. □

Theorem 3 *Suppose $a, b \in \mathbb{R}$. Then $ab \geq 0$, if and only if either (i) $a \geq 0$ and $b \geq 0$ or (ii) $a \leq 0$ and $b \leq 0$.*

Proof. This is an easy consequence of Axiom 1. □

Exercise 3 *Suppose $a, b \in \mathbb{R}$. Prove that $ab < 0$ if and only if either (i) $a > 0$ and $b < 0$ or (ii) $a < 0$ and $b > 0$.*

Proposition 4 *Suppose $a, b, c \in \mathbb{R}$, and $a > 0$. Then the minimum value of the quadratic $ax^2 + bx + c$, $x \in \mathbb{R}$, is equal to $(4ac - b^2)/4a$.*

Proof. Complete the square on x: $\forall x \in \mathbb{R}$

$$ax^2+bx+c = a(x+\frac{b}{2a})^2+c-a\frac{b^2}{4a^2} = a(x+\frac{b}{2a})^2+c-\frac{b^2}{4a} = a(x+\frac{b}{2a})^2+\frac{4ac-b^2}{4a}.$$

By Theorem 1, $(x + \frac{b}{2a})^2 \geq 0$, and $a > 0$ by assumption. Hence, by Theorem 2, $a(x + \frac{b}{2a})^2 \geq 0$. Therefore,

$$a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a} \geq \frac{4ac - b^2}{4a},$$

with equality iff $x = -b/2a$. Hence, the minimum value of the quadratic is $(4ac - b^2)/4a$, as stated. □

Exercise 4 *Suppose $a, b, c \in \mathbb{R}$, and $a < 0$. Prove that the maximum value of the quadratic $ax^2 + bx + c$, $x \in \mathbb{R}$, is equal to $(4ac - b^2)/4a$.*

4 Examples

Example 1 *Describe the set of all real x such that $x^2 + x \leq 0$.*

Solution. Let $S = \{x \in \mathbb{R} : x^2 + x \leq 0\}$. Suppose $x \in S$. Then $x(x + 1) \leq 0$. Hence, either (i) $x \leq 0$ and $x + 1 \geq 0$ or (ii) $x \geq 0$ and $x + 1 \leq 0$. Now (i) holds iff $x \leq 0$ and $x \geq -1$, i.e., $-1 \leq x \leq 0$, equivalently, $x \in [-1, 0]$. In addition, (ii) holds iff $x \geq 0$ and $x \leq -1 < 0$. But no real number can be simultaneously positive and negative. Thus, (ii) can't happen. It follows from (i) that $S \subset [-1, 0]$. On the other hand, if $x \in [-1, 0]$, then $-1 \leq x \leq 0$, i.e., $x + 1 \geq 0$ and $x \leq 0$, so that $x(x + 1) \leq 0$, i.e., $x \in S$. In other words, $[-1, 0] \subset S$. It follows that, $S = [-1, 0]$.

Example 2 *$\forall x \in \mathbb{R}$, $x^2 - x + 1 > 0$.*

Solution. Complete the square on x:

$$x^2 - x + 1 = (x - \frac{1}{2})^2 + 1 - \frac{1}{4} = (x - \frac{1}{2})^2 + \frac{3}{4} \geq \frac{3}{4},$$

with equality iff $x = 1/2$. □

Example 3 *There is no real x such that $x^2 + 2 \le x$.*

Solution. By definition, $x^2 + 2 \le x \Leftrightarrow x^2 - x + 2 \le 0$. But, completing the square on x, we see that

$$x^2 - x + 2 = (x - \frac{1}{2})^2 + 2 - \frac{1}{4} = (x - \frac{1}{2})^2 + \frac{7}{4} \ge \frac{7}{4}.$$

In other words, $x^2 - x + 2$ is never negative or zero, i.e., the solution set of the inequality $x^2 + 2 \le x$ is empty, equivalently, $\{x \in \mathbb{R} : x^2 + 2 \le x\} = \emptyset$. $\square$

Example 4 *Solve the inequality*

$$x^3 + 2 \le x^2, \; x \in \mathbb{R}.$$

Solution. If x satisfies the inequality, then $x^3 - x^2 + 2 \le 0$. Now (?) $x^3 - x^2 + 2 = (x + 1)(x^2 - 2x + 2)$, and (?) $x^2 - 2x + 2 > 0$. Hence, $x \le -1$. Conversely, if $x \le -1$, then $x+1 \le 0$. But, as already noted, $x^2-2x+2 > 0$, whence

$$x^3 - x^2 + 2 = (x + 1)(x^2 - 2x + 2) \le 0.$$

It follows that

$$\{x \in \mathbb{R} : x^3 + 2 \le x^2\} = (-\infty, -1].$$

$\square$

Example 5 *Suppose $a, b \in \mathbb{R}$. Prove that*

$$T = \{x \in \mathbb{R} : x^2 + (a + b)x + ab \ge 0\} = (-\infty, \infty) \setminus (c, d),$$

where $c = \max(-a, -b) = -\min(a, b)$, $d = \min(-a, -b) = -\max(a, b)$.

Proof. Clearly, (?) $x \in T \Leftrightarrow (x + a)(x + b) \ge 0$. Hence, by Theorem 3,

$$x \in T \Leftrightarrow \; (i)\, x + a \ge 0 \,\&\, x + b \ge 0 \text{ or } (ii)\, x + a \le 0 \,\&\, x + b \le 0.$$

But (i) holds iff $x \ge \max(-a, -b) = c$, while (ii) holds iff $x \le \min(-a, -b) = d$. Thus, $T = [c, \infty) \cup (-\infty, d] = \mathbb{R} \setminus (d, c)$. $\square$

Example 6 *Suppose $a, b, c, d \in \mathbb{R}$. Then*

$$ab + cd \begin{cases} < ad + bc, & \text{if } a < c \text{ and } b > d, \\ > ad + bc, & \text{if } a < c \text{ and } b < d. \end{cases}$$

Proof. This follows from the following identity.

$$ab + cd = ad + bc + (a - c)(b - d).$$

□

Example 7 *Suppose $a, b, c, d \in \mathbb{R}$. Then*

$$(a+b)(c+d) \begin{cases} < (a+d)(b+c), & \text{if } a < c \text{ and } b < d, \\ > (a+d)(b+c), & \text{if } a < c \text{ and } b > d. \end{cases}$$

Proof. This follows from the following identity.

$$(a+b)(c+d) = (a+d)(b+c) - (a-c)(b-d).$$

□

5 Additional Exercises

1. For what real numbers x is it true that $x^3 + 2x \leq x^2$?
2. For what real numbers x is it true that
$$x + \frac{1}{x} \geq 2?$$
3. Suppose a, b are positive real numbers. Prove that
$$\frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \frac{a+b}{2},$$
and determine when the equality is strict.
4. Suppose $a, b, c \in \mathbb{R}$. Determine the minimum value of $(x-a)^2 + (x-b)^2 + (x-c)^2$.
5. Suppose $a, b, c \in \mathbb{R}$. Prove that
$$8a^2b^2c^2 \leq (a^2+b^2)(b^2+c^2)(c^2+a^2).$$
When does the equality hold?
6. Suppose $a, b, c \in \mathbb{R}$. Prove that
$$ab + bc + ca \leq a^2 + b^2 + c^2.$$
When does the equality hold?

6 The absolute value function

The absolute value of a real number is its *numerical* value. Thus, the absolute value of -2.147 is 2.147; the absolute value of $\sqrt{2}$ is $\sqrt{2}$. Generally, we define the absolute value of any real number x, which we denote by $|x|$, as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x \geq 0. \end{cases}$$

In other words, $|x|$ is the distance between 0 and x on the number line. More generally, if $a, b \in \mathbb{R}$, then $|a - b|$ is their distance apart.

Since in this way we're assigning a unique real number $|x|$ to any real number x, we're actually defining a function on $\mathbb{R}$, which we call the absolute value function. Sometimes we denote this by *abs*, for short, so that $abs(x) = |x|, \ \forall x \in \mathbb{R}$.

Note that $|-x| = |x| \geq 0$, $\forall x \in \mathbb{R}$, and that $|x| = 0 \Leftrightarrow x = 0$. Hence, the graph of *abs* lies above the horizontal axis, and is the union of two straight lines that emanate from the origin, one with slope 1 and the other with slope -1, to form a V-shaped curve, that is symmetric with respect to the vertical axis. Notice that this graph is *convex*, i.e., the line segment joining any two points on it lies above the corresponding arc of the curve joining them. Also,

$$-|x| \leq x \leq |x|, \ \forall x \in \mathbb{R}.$$

How is *abs* related to addition and multiplication?

Theorem 4 *The function abs is multiplicative and sub-additive on* $\mathbb{R}$*, by which we mean the following: If* x, y *are two real numbers, then*

$$|xy| = |x|\,|y|, \ \ |x + y| \leq |x| + |y|.$$

Proof hint: Consider separately the cases $xy \geq 0$ and $xy \leq 0$. □

Exercise 5 *Sketch the graphs of the functions* $y = |x - 1|$, $y = |x + 2|$ *and* $y = |x-1|+|x+2|$*. Describe the solution set of the inequality* $|x-1| \leq |x+2|$*.*

Exercise 6 *Write out a proof of Theorem 4.*

Exercise 7 *Prove that*

$$|a - b| \leq |x - a| + |x - b|, \ \forall a, b, x \in \mathbb{R}.$$

Exercise 8 *Prove that*

$$|\lambda x + (1 - \lambda)y| \leq \lambda|x| + (1 - \lambda)|y|, \ \forall x, y \in \mathbb{R}, \ \forall \lambda \in [0, 1].$$

7 The square function

After a linear function, the next simplest function is the quadratic, and the simplest one of these is the square of the identity function: $x \to x^2$. Clearly, $(-x)^2 = x^2 \geq 0, \ \forall x \in \mathbb{R}$, and $x^2 = 0 \Leftrightarrow x = 0$. In addition, the graph of the square function is similar to that of *abs*, but is smoother; it is U-shaped. Apart from that, it too lies above the horizontal axis, is symmetric with respect to the vertical axis, and convex. However, while the square function is multiplicative, so that $(xy)^2 = x^2 y^2, \ \forall x, y \in \mathbb{R}$, it is *not* sub-additive on $\mathbb{R}$, i.e., the inequality

$$(x + y)^2 \leq x^2 + y^2, \ \forall x, y \in \mathbb{R},$$

is false (?) in general. Moreover, the opposite inequality is also untrue. However, using convexity, a simple geometric argument (?) shows that

$$(\lambda x + (1 - \lambda)y)^2 \leq \lambda x^2 + (1 - \lambda)y^2, \ \forall x, y \in \mathbb{R}, \ \forall \lambda \in [0, 1].$$

Here's an analytical proof. Expand the LHS and subtract if from the RHS to get

$$\lambda x^2 + (1 - \lambda)y^2 - [\lambda^2 x^2 + 2\lambda(1 - \lambda)xy + (1 - \lambda)^2 y^2] = \lambda(1 - \lambda)(x - y)^2,$$

which is plainly non-negative. Moreover, if $0 < \lambda < 1$, the inequality is strict unless $x = y$. The case $\lambda = 1/2$ deserves special mention.

Proposition 5 *If $x, y \in \mathbb{R}$, then*

$$xy \leq \frac{x^2 + y^2}{2},$$

with equality iff $x = y$.

Proof. The stated result is equivalent to the statement that $(x-y)^2 \geq 0$, with equality iff $x = y$. $\square$

Corollary 1 *If $x, y \in \mathbb{R}$, then*

$$|xy| \leq \frac{x^2 + y^2}{2},$$

with equality iff $x = \pm y$.

Proof.

$$|xy| = |x|\,|y| \leq \frac{|x|^2 + |y|^2}{2} = \frac{x^2 + y^2}{2},$$

with equality iff $|x| = |y|$, and the latter holds (?) iff $x = \pm y$. $\square$

Corollary 2 *If $a > 0, b > 0$, then*

$$2ab = \min\{a^2t^2 + \frac{b^2}{t^2} : t > 0\}.$$

Proof. For all $t > 0$,

$$a^2t^2 + \frac{b^2}{t^2} = (at)^2 + \left(\frac{b}{t}\right)^2 \geq 2(at)(\frac{b}{t}) = 2ab,$$

with equality iff $at = b/t$, i.e., $t^2 = b/a$. $\square$

Example 8 *If a, b, c are real numbers, then*

$$8a^2b^2c^2 \leq (a^2 + b^2)(b^2 + c^2)(c^2 + a^2),$$

with equality iff $|a| = |b| = |c|$.

Proof. Since $2|ab| \leq a^2 + b^2$, with equality iff $|a| = |b|$, etc., and $|a|^2 = a^2$, etc., this is easy. $\square$

Example 9 *If a, b, c are real numbers, then*

$$ab + bc + ca \leq a^2 + b^2 + c^2,$$

with equality iff $a = b = c$.

Proof. Three applications of Proposition 5 tells us that

$$ab \le \frac{a^2+b^2}{2},\ bc \le \frac{b^2+c^2}{2},\ ca \le \frac{c^2+a^2}{2},$$

with equality holding when $a = b$, $b = c$, $c = a$, respectively. Add these inequalities to get

$$ab + bc + ca \le \frac{a^2+b^2+b^2+c^2+c^2+a^2}{2} = a^2+b^2+c^2,$$

as desired. Clearly, the equality holds if $a = b = c$.

Conversely, if, for some triple of real numbers a, b, c,

$$ab + bc + ca = a^2 + b^2 + c^2,$$

then (?)

$$0 = a^2+b^2-2ab+b^2+c^2-2bc+c^2+a^2-2ca = (a-b)^2+(b-c)^2+(c-a)^2,$$

a sum of three non-negative numbers equal to zero. This can only happen if each of them is zero, whence $a - b = b - c = c - a = 0$, i.e., $a = b = c$. □

8 The Cauchy-Schwarz inequality

This is the name given to an ubiquitous inequality of the following sort, but not the most general form of it.

Theorem 5 *Suppose $a_1, a_2, a_3, b_1, b_2, b_3$ are real numbers. Then*

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

Let

$$A = a_1^2 + a_2^2 + a_3^2,\ B = b_1^2 + b_2^2 + b_3^2,\ C = a_1b_1 + a_2b_2 + a_3b_3.$$

The claim is that $C^2 \le AB$. If either A or B is zero, then $C = 0$ (?), and the result holds. So, suppose $AB > 0$.

First Proof. Consider the quadratic expression

$$(a_1 - tb_1)^2 + (a_2 - tb_2)^2 + (a_3 - tb_3)^2 = A - 2Ct + Bt^2.$$

As a function of the real number t, this is a quadratic that takes only nonnegative values. In particular, its minimum value is nonnegative. According to Proposition 4, therefore,

$$0 \leq A - \frac{(2C)^2}{4B} = \frac{AB - C^2}{B},$$

which proves the claim.
Second Proof. Since the square of every real number is nonnegative,

$$0 \leq (a_i b_j - b_i a_j)^2 = a_i^2 b_j^2 - 2(a_i b_i)(a_j b_j) + b_i^2 a_j^2, \ i, j = 1, 2, 3.$$

Add these nine inequalities, to get

$$0 \leq AB - 2C^2 + BA = 2(AB - C^2),$$

which is what we want.
Third Proof. If $t > 0$, then, by Corollary 2,

$$2a_i b_i \leq a_i^2 t + \frac{b_i^2}{t}, \ i = 1, 2, 3.$$

Add these inequalities to find that

$$2C \leq At + \frac{B}{t}, \ \forall t > 0.$$

Hence, squaring both sides,

$$4C^2 \leq A^2 t^2 + 2AB + \frac{B^2}{t^2},$$

whence, once more by Corollary 2 (?),

$$4C^2 \leq 2AB + 2Ab = 4AB, \ C^2 \leq AB.$$

□

Example 10 *Suppose a, b, c, x, y are real numbers. Then*

$$|a \cos x \cos y + b \cos x \sin y + c \sin x| \leq \sqrt{a^2 + b^2 + c^2}.$$

Proof. Apply Theorem 5 with $a_1 = a, a_2 = b, a_3 = c$, and

$$b_1 = \cos x \cos y, \ b_1 = \cos x \sin y, \ b_3 = \sin x,$$

and observe that

$$b_1^2 + b_2^2 + b_3^2 = \cos^2 x(\cos^2 y + \sin^2 y) + \sin^2 x = \cos^2 x + \sin^2 x = 1.$$

□

9 Arithmetic, Geometric and Harmonic Means

If a, b are positive real numbers, we define, their arithmetic mean, $A(a,b)$, geometric mean, $G(a,b)$, and harmonic mean, $H(a,b)$, respectively, by

$$A(a,b) = \frac{a+b}{2},\ G(a,b) = \sqrt{ab},\ H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

Note that $H(a,b)$ is the reciprocal of $A(1/a, 1/b)$. The corresponding means of a finite number of positive real numbers can be defined similarly: If $a, b, c > 0$, then, by definition,

$$A(a,b,c) = \frac{a+b+c}{3},\ G(a,b,c) = \sqrt[3]{abc},\ ,H(a,b,c) = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

In like manner, if $a, b, c, d > 0$,

$$A(a,b,c,d) = \frac{a+b+c+d}{4},\ G(a,b,c,d) = \sqrt[4]{abcd},\ H(a,b,c,d) = \frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}},$$

and so forth for any set of positive numbers.

Proposition 6

$$H(a,b) \leq G(a,b) \leq A(a,b),\ \forall a,b > 0.$$

There is equality throughout iff $a = b$.

Proof. We'll only deal with the statement that $G(a,b) \leq A(a,b)$. The claim is that

$$\sqrt{ab} \leq \frac{a+b}{2},$$

with equality iff $a = b$. Let $x = \sqrt{a}, y = \sqrt{b}$, so that $x, y > 0$ and $x^2 = a, y^2 = b$. Hence

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b} = xy \leq \frac{x^2+y^2}{2} = \frac{a+b}{2},$$

by Corollary 1, with equality iff $x = y$, i.e., $a = b$. □

We can extend this result first to four numbers and then to three.

Proposition 7

$$H(a,b,c,d) \leq G(a,b,c,d) \leq A(a,b,c,d),\ \forall a,b,c,d > 0.$$

There is equality throughout iff $a = b = c = d$.

Proof. Again, we'll only prove that $G(a,b,c,d) \leq A(a,b,c,d)$, with equality iff $a = b = c = d$. To see this, let $x = \sqrt{ab}, y = \sqrt{cd}$. Then (?)

$$\begin{aligned} G(a,b,c,d) &= \sqrt{\sqrt{ab}\sqrt{cd}} \\ &= G(x,y) \\ &\leq A(x,y) \\ &= \frac{x+y}{2} \text{ (with equality iff } x = y) \\ &= \frac{\sqrt{ab}+\sqrt{cd}}{2} \\ &\leq \frac{\frac{a+b}{2}+\frac{c+d}{2}}{2} \text{ (with equality iff } a = b \text{ and } c = d) \\ &= \frac{a+b+c+d}{4} \\ &= A(a,b,c,d), \end{aligned}$$

with equality throughout iff $x = y, \& a = b, \& c = d$. Equivalently (?), iff $a = b = c = d$. ☐

Proposition 8

$$G(a,b,c) \leq A(a,b,c),\ \forall a,b,c > 0.$$

There is equality throughout iff $a = b = c$.

Proof. Let $d = \sqrt[3]{abc}$ and apply the previous proposition, noting that

$$\sqrt[4]{abcd} = \sqrt[4]{d^3 d} = d.$$

Hence

$$d \leq \frac{a+b+c+d}{4} \Leftrightarrow 4d \leq a+b+c+d \Leftrightarrow 3d \leq a+b+c \Leftrightarrow G(a,b,c) \leq A(a,b,c),$$

as desired. Moreover, there is equality throughout iff $a = b = c = d$. ☐

It's clear from the proof of Proposition 7 that the geometric mean of eight numbers doesn't exceed their arithmetic mean. We leave it as an exercise to show that a proof similar to that used in Proposition 8 can be constructed to prove first that the geometric mean of seven numbers doesn't exceed their arithmetic mean, next that the geometric mean of six numbers doesn't exceed their arithmetic mean, and thence that the geometric mean of five numbers doesn't exceed their arithmetic mean. Indeed, a similar approach can be used to prove the following general statement, which tells us that the geometric mean of any set of positive numbers doesn't exceed their arithmetic mean, and that both means are equal iff the numbers are all equal to one another.

Theorem 6 *Let n be a natural number and let $a_1, a_2, \ldots, a_n$ be positive numbers. Then*

$$G(a_1, a_2, \ldots, a_n) = \sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} = A(a_1, a_2, \ldots, a_n),$$

with equality iff $a_1 = a_2 = \cdots = a_n$.

Example 11 *Suppose $a, b, c > 0$. Then*

$$3 \leq \frac{a}{b} + \frac{b}{c} + \frac{c}{a},$$

with equality iff $a = b = c$.

Proof. The inequality is equivalent to the statement that $3abc \leq a^2bc + b^2ca + c^2ab$. But

$$(a^2bc)(b^2ca)(c^2ab) = a^3b^3c^3 = (abc)^3,$$

and so

$$abc = \sqrt[3]{(a^2bc)(b^2ca)(c^2ab)} \leq \frac{a^2bc + b^2ca + c^2ab}{3},$$

with equality iff $a^2bc = b^2ca = c^2ab$, i.e., $a = b = c$. □

Example 12 *Suppose $0 \leq x \leq 1$. Then $4/27$ is the maximum value of $x(1-x)^2$.*

Proof. The claim is that $x(1-x)^2 \leq 4/27$, whenever $0 \leq x \leq 1$, and that $x(1-x)^2 = 4/27$, for at least one value of x in this range. The inequality certainly holds if $x = 0$ or $x = 1$. So, suppose $0 < x < 1$, so that $x(1-x)^2$ is the product of three positive numbers. This suggests that it might be profitable to use $G(a,b,c) \leq A(a,b,c)$, with appropriate choices for a,b,c. Note that the choice $a = x, b = c = 1-x$ won't do. Why? However, we have better luck with $a = x, b = c = (1-x)/2$, because then $a+b+c = 1$, and we see that

$$\sqrt[3]{x(1-x)^2} = \sqrt[3]{abc}\sqrt[3]{4} \leq \sqrt[3]{4}\frac{x + \frac{1}{2}(1-x) + \frac{1}{2}(1-x)}{3} = \frac{\sqrt[3]{4}}{3},$$

and there is equality when and only when $x = \frac{1}{2}(1-x)$, i.e., $x = \frac{1}{3}$. Thus, $\forall x \in [0,1]$, $x(1-x)^2 \leq 4/27$, and equality occurs here iff $x = 1/3$. $\square$

Example 13 *Suppose* $0 < x$. *Then* $3/2$ *is the minimum value of* $2x + 1/8x^2$.

Proof. The claim is that

$$\frac{3}{2} \leq 2x + \frac{1}{8x^2}, \ \forall x > 0,$$

and that equality holds for some positive x. Observing that $2x + 1/8x^2$ is three times the arithmetic mean of the three positive numbers, $x, x, 1/8x^2$, we can apply $G(a,b,c) \leq A(a,b,c)$, to get that

$$2x + \frac{1}{8x^2} = 3\left(\frac{x + x + \frac{1}{8x^2}}{3}\right) \geq 3\sqrt[3]{x.x.\frac{1}{8x^2}} = 3\sqrt[3]{\frac{1}{8}} = \frac{3}{2}.$$

Moreover, equality holds here iff

$$x = \frac{1}{8x^2}, \ 8x^3 = 1, \ 2x = 1.$$

Thus, $\frac{3}{2} \leq 2x + \frac{1}{8x^2}$, $\forall x > 0$, and there is equality iff $x = 1/2$. $\square$

10 Exercises

1. Show that for all real a, b, x,

   $$|a\cos x + b\sin x| \leq \sqrt{a^2 + b^2}.$$

2. Suppose $a, b, c > 0$. Prove that

   $$abc(a+b+c) \leq a^4 + b^4 + c^4,$$

   with equality iff $a = b = c$. [Hint: Use Example 9 or Theorem 5.]

3. Suppose P, Q are two rectangles with dimensions a, b and c, d respectively. Form the rectangle R with dimensions $a+c, b+d$. Prove that

$$\sqrt{\text{area of } P} + \sqrt{\text{area of } Q} \leq \sqrt{\text{area of } R}.$$

4. Suppose $a, b, c > 0$, and $d, e, f > 0$. Prove that

$$\sqrt[3]{abc} + \sqrt[3]{def} \leq \sqrt[3]{(a+d)(b+e)(c+f)},$$

and that there is equality iff

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}.$$

5. Give a geometric interpretation of the previous exercise.

6. Suppose $a, b, c > 0$. Prove that $H(a, b, c) \leq G(a, b, c)$, and that the inequality is strict unless $a = b = c$.

7. Suppose $a, b, c > 0$. Prove that

$$9 \leq (a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}),$$

and that the inequality is strict unless $a = b = c$.

8. (IMO 1984) Prove that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27},$$

where $x, y, z > 0$ and $x + y + z = 1$.

9. (IMO 2001) Prove that if a, b, c are positive numbers, then

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

10. (IMO 2004) Suppose $a, b, c > 0$ and

$$(a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) < 10.$$

Prove that a, b, c are the side lengths of a triangle.