Combinatorics with Repetitions and Conditions

Note: The questions with Hints written in red have not yet been discussed in class. These hints are a new addition to the original hand-out.

Theorem. ***(Permutations Review)*** *The number of ways to choose a **sequence** of k elements from among n distinct elements (where $n \geq r$) is*

$$P_k^n = n(n-1)(n-2)\cdots(n-k+1).$$

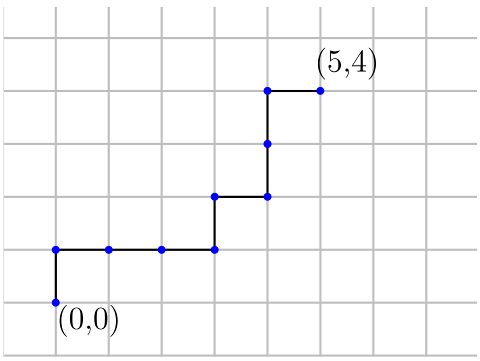
*Note: **"Sequence"** means that the order in which we choose the elements is important.*

Theorem. ***(Combinations Review)*** *The number of ways to choose a **set** of k distinct elements from among n distinct elements (where $n \geq k$) is*

$$C_k^n = \frac{n(n-1)...(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

*Note: **"Set"** means that the order in which we choose the elements is unimportant.*

1. How many routes are there from the point $(0, 0)$ to the point $(5, 4)$, if each step in the path is either one grid unit to the right or one unit upwards? The example below shows such a path.



Hint: Describe the route to your friend by a sequence of instructions.

Theorem. *The number of routes from the point $(0, 0)$ to the point (m, n) in a grid, where each step in the path is either one unit to the right or one unit upwards, is:*

$$C_n^{n+m} = C_m^{n+m}.$$

2. In how many ways can 6 identical candy canes be split among Anna, Devon and Micah?

Hint: Draw all the candies in a row, then separate them into 3 compartments by 2 inner walls. The 6 candy canes and the 2 walls occupy 8 places in a string, and choosing where to place the walls decides how many candy canes there will be in each compartment. We can represent the candy canes by stars and the walls by bars

and call this method "the stars and bars method".

Theorem. *The number of ways to split n indistinguishable objects into k distinct categories is:*

$$C_n^{n+k-1} = C_{k-1}^{n+k-1}.$$

3. In how many ways can we choose (x_1, x_2, x_3, x_4) non-negative integers such that

$$x_1 + x_2 + x_3 + x_4 = 10?$$

[Hint: This is still a stars and bars problem: You should distribute 10 stars into 4 compartments, with x_i= how many stars to put in the i-th compartment. As before, you need 3 bars to separate the compartments.]

4. In how many ways can you choose 12 doughnuts from 5 different types
a) If you wish to have at least one of each type?

[Hint: You have 5 compartments separated by 4 walls. You know that there's one doughnut in each compartment from the start, so the problem is really to distribute the 7 remaining doughnuts.]

b) If you can have no more than 4 chocolate ones?

[Hint: either split the problem into 5 cases depending on whether you choose 0, 1, 2, 3 or 4 chocolate doughnuts; you now have a remaining number of doughnuts to be split into 4 compartments. As an alternate, shorter solution, consider the problem with 12 doughnuts from 5 different types and no restrictions, and then subtract the case when you choose at least 5 chocolate doughnuts. In this last case, 5 doughnuts are guaranteed to be chocolate and it remains to distribute 7 remaining doughnuts into 5 categories.]

5. (**Dealing with indistinguishable objects**) You have 10 large bags of coins and a digital scale that gives an exact reading of the weight. All of the bags but one contain genuine coins weighing 10 grams each. The remaining bag contains fake coins weighing 9 grams each. How can you identify the bag with the fakes using only 1 weighing?

Answer: put together one coin from the first bag, two from the second...

6.

Theorem. *For fixed natural numbers n and k,*
a) The number of ways to choose an increasing sequence of k integer numbers:

$1 \leq x_1 < x_2 < ... < x_k \leq n$ is

$$C_k^n$$

[Proof: You choose sets of k numbers, then arrange them in increasing order.]

b) The number of sequences of integer numbers $1 \leq x_1 \leq x_2 \leq ... \leq x_k \leq n$ is

$$C_k^{n+k-1}$$

[Proof: the problem is equivalent to choosing an increasing sequence of k numbers: $1 \leq x_1 < x_2 + 1 < ... < x_k + k - 1 \leq n + k - 1$. Alternatively, we can formulate this as a stars and bars problem: we consider strings of length n, and place bars after the x_i-th place of the string, for each i, and stars everywhere else. But note that since $x_1 \geq 1$, the first place in the string will always be occupied by a star, so we need not count it at all.]

c) The number of ways to choose a set of k integer numbers from the set $\{1, 2, ..., n\}$, such that no two of the chosen numbers are consecutive:

$$\begin{array}{lll} 0 & \text{if} & k > \frac{n+1}{2}; \\ C_k^{n-k+1} & \text{if} & k \leq \frac{n+1}{2}; \end{array}$$

[Hint: the problem is equivalent to choosing an increasing sequence $1 \leq x_1 < x_2 - 1 < x_3 - 2 < ... < x_k - (k-1) \leq n - (k-1)$. Alternatively, we can formulate this as a stars and bars problem: we consider strings of length n, and place bars in the x_i-th place of the string, for each i, and stars everywhere else. We know that each bar except possibly the last one is followed by a star, so $k - 1$ stars have their guaranteed place. It remains to fill in $n - (k - 1)$ places with k bars and the remaining stars.]

7. In how many ways can Aisling and her two daughters, Brianna and Ciara, together with Brianna's two children and Ciara's three children, arrange themselves in a row in front of the ice-cream stand, each mother always giving priority to her own children over herself?

[The question is: in how many ways can we arrange the 8 people in a raw, subject to the priority rule above. Draw the family tree, find the number of options for placing the root (Aisling), then for choosing the set of places occupied by each branch in the queue, then for placing the people within each branch within their assigned set of places.]

8. A number of x girls and y boys played soccer until a total of n goals have been scored. You arrive at the end and are asked to guess who scored each goal in order. How many options do you have if:
a) You know nothing about the match or the players, except their names.
b) You know that exactly k of the goals have been scored by girls.

Theorem. ***(Binomial Theorem)*** *Let x and y be symbols that can take any value. Let n be a positive integer. Then*

$$(x+y)^n = \sum_{k=0}^{n} C_k^n x^k y^{n-k}.$$

9. **(Permutations, with conditions)** You have 8 different beads each of a different color; half of them are cubical and half are spheres.
a) In how many ways can they be arranged in a row?
b) How many different patterns of bracelets can you make with them?
c) How many different patterns of bracelets can be made if the cubes should alternate with spheres?
d) In how many ways can they be arranged in a row with cubes and spheres alternating?

10. **(Permutations, but with some indistinguishable objects)**
In how many ways can the letters in BANANA be scrambled (arranged in random order)?

[Hint: Give each letter a number label. There are $5! = 120$ ways of making strings from the symbols $B_1, A_2, N_3, A_4, N_5, A_5$. If we then forget the number labels, we note that $A_2N_3A_4N_5A_5B_1$ and $A_2N_4A_4N_3A_5B_1$ result in the same word $ANANAB$. Similarly for the $3! = 6$ ways of rearranging the A letters among themselves. Thus the 5! strings of symbols $B_1, A_2, N_3, A_4, N_5, A_5$ can be grouped into sets of 2!3! strings which result in the same word. We thus have $\frac{5!}{2!3!}$ words in all.

Practice exercises:

11. Find the number of ways of placing 4 marbles in 10 distinguishable boxes if:
a) The marbles are distinguishable, and no box can hold more than one marble.
b) The marbles are indistinguishable, and no box can hold more than one marble.
c) The marbles are distinguishable, and each box can hold any number of them.
d) The marbles are indistinguishable, and each box can hold any number of them.

12. How many 8 letter codes can you form with the letters A, B, C, D?

13. How many 5 letter codes can you form with the letters A, B, C, D, E if no letter is to be used more than once and B is never to immediately follow A?

[Hint: From the total number of codes, subtract those codes in which B immediately follows A. In these codes, AB can be considered as one symbol.]

14. In how many ways can the letters in MISSISSIPPI be scrambled?

15. In how many ways can you arrange the numbers $1, 2, 4, 5, 6, 8, 9, 10, 12, 15$ in a row if a number always has priority over its double and over its triple?

16. For fixed k, a and b such that $a < b$ and $bk \geq n$, find the number of solutions in nonnegative integers to

$$x_1 + x_2 + ... + x_k = n$$

under the condition that $a \leq x_i \leq b$ for each i.

17. Suppose your class has n students, and k of them are chosen randomly to receive free tickets to the premiere of the "Hunger Games" movie. Use this context to prove the following theorem:

Theorem. ***(Pascal's Identity)*** *Let n and k be positive integers with $n \geq k$. Then*

$$C^n_k = C^{n-1}_{k-1} + C^n_{k-1}.$$

18. Anna, Brienne, Clare and David ask you to guess which books has each of them read from a given list of 10 books. What are your chances of guessing correctly if the only thing you know is that no book was read by all of them.

[Hint: count the books they haven't read].

19. a) Prove that it is impossible to choose 16 numbers from among the numbers 1,2,...,30 so that no two of them differ by 3.
b) In how many ways can 15 numbers be chosen from among the numbers 1,2,...,30 so that no two of them differ by 3?
c) In how many ways can 14 numbers be chosen from among the numbers 1,2,...,30 so that no two of them differ by 3?

[Hint: First split the numbers 1, 2,...,30 into 3 groups, each obtained by counting by 3-s. For example, one group is $3, 6, 9, 12, 15, 18, 21, 24, 27, 30$. At most how many numbers can be chosen from this group so that no 2 of them differ by 3? Apply the reasoning from Theorem 6.c) to each group (you can divide by 3 first).]