EGMO and UCD Enrichment Programme in Mathematics

Selection Test 24 February 2018

1. In triangle ABC, P is a point on AB, Q is a point on AC and X is the point of intersection of the line segments PC and QB.

   The quadrilateral $APXQ$ has area 4. The triangles QXC and PXB have area 5 and 1 respectively. What is the area of the triangle ABC?

2. Are there any positive integers n and m such that the integer $32^n + 3125^m$ is a prime number?

   Prove your assertion.

3. How many different pairs of integers (x, y) satisfy the equation

   $$10x^2 + 29xy + 21y^2 = 15?$$

   Write down 3 such pairs.

4. Suppose n is a positive integer, such that all the digits of $72n$, written in decimal notation, are 0's and 1's.

   Find the smallest such n.

5. Emma writes the numbers $1, 2, 3, \ldots, 9$ into the cells of a 3×3 table (placing a different number in each cell). Then, she performs a series of *moves* as follows. In each move, she chooses an arbitrary 2×2 square of the table, and either increases by 1 or else decreases by 1 all four numbers of that square. After performing a sequence of moves, Emma notices that all 9 numbers in the table are equal to some number n.

   Find, with proof, all possible values of n.

6. The non-zero real numbers a, b, c, d satisfy the equalities

   $$a + b + c + d = 0, \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{abcd} = 0.$$

   Find, with proof, all possible values of the product $(ab - cd)(c + d)$.

7. Let $\mathcal{S}$ be a set of 2018 points in the plane, no three of which are collinear. Let P be a point not contained in any line segment that connects two points from $\mathcal{S}$. Prove that the number of triangles that contain P, with vertices at three different points in $\mathcal{S}$, is even.

8. (a) Prove that for any positive real numbers x, y we have $x^3 + y^3 \geq x^2y + xy^2$.

   (b) Prove that for any real numbers $0 \leq x, y, z \leq 1$ we have

   $$3 + x^3 + y^3 + z^3 \geq x^2 + y^2 + z^2 + x + y + z.$$

9. Let A, B, C, D be four distinct points on a circle (in this order). Let P be the intersection of AD and BC, and let Q be the intersection of AB and CD. Prove that the angle bisectors of $\angle DPC$ and AQD are perpendicular.

10. Let $\{S_n : n = 0, 1, 2...\}$ be a sequence defined by $S_0 = 1$ and $S_n = S_{n-1} + \frac{1}{\sqrt{n}}$ for $n \geq 1$.

    Show that $S_n \leq 2\sqrt{n}$, for all $n \geq 1$.