EGMO and UCD Enrichment Programme in Mathematics

Selection Test 4 February 2017

1. Triangle ABC has area S. Denote by M, N and P the midpoints of BC, CA and AB respectively. Prove that

$$2S\left(\frac{1}{AB}+\frac{1}{BC}+\frac{1}{CA}\right) \leq AM+BN+CP < \frac{3}{2}(AB+BC+CA).$$

2. A positive integer is said to be *near-square* if it is a product of two positive integers differing by 1. For example, 20 is a near-square because $20 = 4 \times 5$. Prove that every near-square positive integer can be expressed as the ratio of two other near-square positive integers.

3. Alice and Bob play a game with a string of 2017 pearls. In each move, one player cuts the string between two pearls and the other player chooses one of the resulting parts of the string while the other part is discarded.

   In the first move, Alice cuts the string. Thereafter, the players take turns. A player loses if he or she obtains a string with a single pearl such that no more cuts are possible.

   Which of the two players has a winning strategy?

4. The diagonals of the convex quadrilateral $ABCD$ of area 1 intersect at O. If $\frac{BO}{DO} = \frac{1}{2}$ and $\frac{AO}{CO} = \frac{3}{4}$, find the area of triangles AOB, BOC, COD and DOA.

5. Determine with proof all prime numbers p for which $7p+4$ is the square of an integer.

6. (a) Simplify $(x^2-1)^2+(x^2+2x)^2-(x^2+x+1)^2$ and then factor the result as far as possible.

   (b) Show that there are infinitely many pairs of integers m, n for which m^2+n^2-mn is the square of an integer.

7. ABC is an acute triangle. The bisector AL, the altitude BH and the median CM are such that $\angle CAL = \angle ABH = \angle ACM$. Find the angles of triangle ABC.

8. Let p, q, r be prime numbers with

$$p < q < r < q + p^4 \quad \text{and} \quad pq^2 = r^2 + 1.$$

   Find, with proof, all possible values for p, q and r.

9. The positive real numbers a, b, c satisfy the double inequality

$$\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{c+a} \geq \frac{c^2}{a+b}+\frac{a^2}{b+c}+\frac{b^2}{c+a} \geq \frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}.$$

   Prove that $a = b = c$.

10. The function μ is defined on the set of positive integers as follows:

    - $\mu(1) = 1$ and $\mu(p) = -1$ for any prime number p;
    - $\mu(ab) = \mu(a)\mu(b)$ for any positive integers with $\gcd(a, b) = 1$;
    - $\mu(n) = 0$ if n is a positive integer which is divisible with a square of a prime number.

    (For instance $\mu(15) = \mu(3)\mu(5) = 1$ and $\mu(12) = 0$, because 12 is divisible with 2^2).

    Prove that for any positive integer $n > 1$, we have $\sum_{d|n} \mu(d) = 0$.