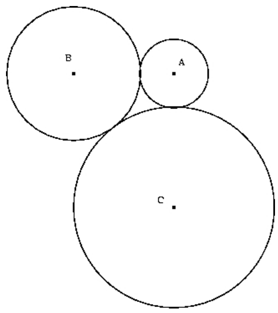
# EGMO Selection Test, 25 February 2012

**Instructions:**
**Do as many questions as you can.**
**Complete answers to a few questions are worth more than bits of all of them.**
**The use of calculators is not allowed.**

1. Suppose 251 numbers are chosen from 1,2,3,...,499,500. Show that, no matter how the numbers are chosen, there must be two that are consecutive.

2. Prove or disprove: For every positive integer $n$, the greatest common divisor of $5n + 4$ and $9n - 7$ is 1.

3. Three circles of radii 1, 2, 3 and centres at A, B, C are mutually tangent (as shown in diagram below). Find the area of the triangle ABC.




4. Prove that for all positive real numbers $x$ and $y$ satifying $x+y = 1$ the following inequality holds
$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 \geq \frac{25}{2}.$$

5. Suppose that $a$ and $m$ are integers larger than 1. Prove that the greatest common divisor of the pair $a - 1$ and $m$ is equal to the greatest common divisor of the pair of integers $a - 1$ and $(a^m - 1)/(a - 1)$.

6. We say that a positive integer is *triangular* if it is the sum of some positive consecutive integers starting from 1(thus, 1=1, 3=1+2, 6=1+2+3, 10=1+2+3+4 are triangular numbers). We denote by
$$t_1 < t_2 < \cdots < t_n < \cdots$$
the sequence of all triangular numbers. Prove that for all $n \geq 1$ we have
$$1 \cdot t_1 + 2 \cdot t_2 + \ldots n \cdot t_n = \frac{n(n+1)(n+2)(3n+1)}{24}.$$

7. Seven darts are thrown at a circular dartboard of radius 10cm. Show that there will always be two darts that are at most 10cm apart.

8. Let $k$ be a positive integer and $p_1 = 2 < p_2 < \cdots < p_k$ be the first $k$ prime numbers and let

$$M = 1 + p_1 p_2 \cdots p_k.$$

Prove the following:

(i) $M$ is not the square of an integer;

(ii) $M$ is not the cube of an integer;

(iii) $M$ is not the $q^{th}$ power of an integer for any $q \in \{p_1, p_2, \ldots, p_k\}$.

9. A set $\mathcal{A}$ consists of 7 consecutive positive integers less than 50, while another set $\mathcal{B}$ consists of 11 consecutive positive integers. If the sum of the numbers in $\mathcal{A}$ is equal to the sum of the numbers in $\mathcal{B}$, what is the maximum possible element which could be contained in $\mathcal{A}$?

10. (a) Prove that there exist infinitely many primes of the form $6n - 1$, with $n$ an integer.

(b) Let $S$ be the set of all integers of the form $a^2 + ab + b^2$, where $a$ and $b$ are integers.

Prove the following:

(i) If $x$ and $y$ are in $S$, then $xy$ is in $S$;

(ii) There exist infinitely many primes which are not in S.