EGMO Selection Test, 23 February 2013

Instructions:
Time allowed: 3 hours.
Do as many questions as you can.
Complete answers to a few questions are worth more than bits of all of them.

1. Ciara and Emma are two of the ten students from which a team of five is to be selected for the next mathematical talent competition. In how many ways can a team of five students be formed so that Emma is on the team but Ciara is not?

2. We say that a positive integer is *triangular* if it is the sum of some positive consecutive integers starting from 1 (thus, $1 = 1$, $3 = 1 + 2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$ are triangular numbers).

   Prove that if n is triangular, then so is $25n + 3$.

3. Let $n \geq 1$ be a positive integer. Evaluate in terms of n the sum

$$1 \cdot 2 \cdot 4 + 2 \cdot 3 \cdot 5 + 3 \cdot 4 \cdot 6 + \ldots + n(n + 1)(n + 3).$$

4. Let ABC be an isosceles triangle with $|AB| = |AC|$. The bisector of the angle ABC meets the side AC at the point D.

   Prove that if the triangle BCD is isosceles, the triangle ABD must also be isosceles.

5. We are given a set X containing 100 integers, none of which is divisible by 3. We are asked to carry out the following task: choose 7 integers from this set so that for any pair of integers x and y we choose, the difference $x - y$ is *not* divisible by 9.

   (a) Prove that this task is impossible.

   (b) If we are instead asked to choose 6 integers from X, is the task always possible?

6. Let x, y be *positive integers* with

$$3x + 4y + xy = 2012.$$

   (a) Prove that $x + y \geq 83$.

   (b) Prove also that the same inequality is valid if 2012 is replaced by 2013.

7. In a deck of 52 cards, on each card there is a letter (one of A, B, C, D) and a number (between 1 and 13 inclusive). A poker hand consists of 5 cards from the deck (the ordering of the cards does not matter). Answer the following questions, justifying carefully your answer in each case. You may use the notation $\binom{n}{r}$ in your answers.

   (a) How many poker hands are there?

   (b) How many poker hands do not have a card with a 7?

   (c) How many poker hands have (at least) two cards with the same letter?

   (d) In how many poker hands does every letter appear?

8. We would like to place stamps worth exactly n cents on an envelope. However, there are only 5-cent and 12-cent stamps available to us (although we have an unlimited supply of both of these stamps). Prove that we can perform the task provided $n \geq 44$.

9. There are several people at a party. Any two people who are not friends with each other have exactly two friends in common. Anna and Brian are friends with each other, but don't have any friends in common. Show that Anna and Brian have the same number of friends at the party.

10. (a) Determine **with proof** the largest and smallest of the three numbers

$$\sqrt{7},\ 1+\sqrt[3]{3},\ \sqrt[4]{67}.$$

(b) Let x_1, x_2, x_3, x_4 be positive real numbers with $x_1 + x_2 + x_3 + x_4 = 1$.

Prove that

$$\frac{x_1x_2}{x_1+x_2} + \frac{x_1x_3}{x_1+x_3} + \frac{x_1x_4}{x_1+x_4} + \frac{x_2x_3}{x_2+x_3} + \frac{x_2x_4}{x_2+x_4} + \frac{x_3x_4}{x_3+x_4} \leq \frac{3}{4}.$$