European Girls' Mathematical Olympiad
Irish Team Selection Test, 14 February 2015

[Note: $n!$ denotes the *factorial* of n, which is the product of the integers from 1 to n inclusive.]

1. (a) Which is the larger number: $A = 200!$ or $B = 100^{200}$? Justify your answer.
   (b) Which is the larger number: $A = 2000!$ or $B = 100^{2000}$? Justify your answer.

2. Show that for all positive integers $n \geq 2$ we have
$$n! < \left(\frac{n+1}{2}\right)^n .$$

3. In triangle ABC we denote by A', B', C' the midpoints of sides BC, CA and AB respectively. We extend AA' beyond A' with $A'M = AA'$. We extend BB' beyond B' with $B'N = BB'$ and extend CC' beyond C' with $C'P = CC'$. Denote by G_1, G_2 and G_3 the centroids of triangles MBC, NAC and PAB. Prove that triangles ABC and $G_1G_2G_3$ have the same area.

4. Let x, y, z, w be positive real numbers, and suppose that $xyzw = 16$. Show that
$$\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} \geq 4$$
with equality only when $x = y = z = w = 2$.

5. For any positive integer k define
$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} .$$
Prove that for $n \geq 1$,
$$1 + \frac{1}{n+1}\left(H_1 + H_2 + \cdots + H_n\right) = H_{n+1} .$$

6. We have a deck of 10,000 cards, numbered from 1 to 10,000. A step consists of removing every card which has a perfect square on it, and then renumbering the remaining cards, starting from 1, in a consecutive way (i.e., numbering them 1, 2, 3, etc.)
   Find, with proof, the number of steps needed to remove all but one card.

7. Determine all triples (a, b, c) of positive integers satisfying both of the following properties:
   (i) We have $a < b < c$, and a, b and c are three consecutive odd integers;
   (ii) The number $a^2 + b^2 + c^2$ consists of four equal digits.

8. (a) Find with proof all integers x, y such that
$$\frac{x^4 + x^2y^2 + y^4}{3}$$
is a prime number.
   (b) Prove that if x and y are integers, then
$$\frac{x^4 + x^2y^2 + y^4}{5}$$
is not a prime number.

9. A triangle has angles of $36^\circ, 72^\circ$ and 72°. Prove that it has at least one side whose length is not an integer.

10. Find with proof all positive integers k such that, for $n = 2^k$, every prime number which divides $n! + 1$ also divides $n + 1$.