UCD Enrichment Programme in Mathematics
Selection Test 15 February 2014

1. A convex quadrilateral $ABCD$ is given. On the extended diagonal BD we consider two points D and E such that B lies between D and E, D lies between B and F and $BE = BD = DF$. Prove that the area of the quadrilateral $AECF$ is three times bigger than the area of $ABCD$.

2. (a) Prove that if n is a positive integer, then $n^n + (n+1)^n$ is not divisible by 2014.

   (b) Find a positive integer m for which $m^m + (m+2)^m$ is divisible by 2014.

   (c) Find with proof the least positive integer k for which $k^k(k+1)^k$ is divisible by 2014.

3. The length of each side of a triangle is an integer and is a divisor of the perimeter of the triangle. Prove that the triangle is equilateral.

4. Show that it is not possible to find 14 consecutive integers such that each of them is divisible by at least one of the numbers $2, 3, 5, 7, 11$.

5. Let G be the centroid of triangle ABC. Denote by G_1, G_2 and G_3 the centroids of triangles ABG, BCG and CAG. Prove that
$$[G_1G_2G_3] = \frac{1}{9}[ABC].$$

6. Show that for any integer $n \geq 1$ we have
$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

7. Show that
$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$
is not an integer for any $n > 1$.

8. Prove that if a and b are positive real numbers,
$$\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} \leq \sqrt[3]{2\,(a+b)\left(\frac{1}{a} + \frac{1}{b}\right)}\,.$$

9. Let $\{p_n\}$ be the increasing sequence of prime numbers, that is,
$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ \ldots$$
Prove that for all integers $k > 2$, we have

   (i) $p_{k+9} \geq 3k + 25$;

   (ii) $p_{k+1} \leq 1 + p_1p_2\cdots p_k$;

   (iii) $p_{k+1} < p_{k-1}\sqrt{p_1p_2\cdots p_k}$.

10. Let j, n be two integers such that $n \geq 1$ and $0 \leq j \leq n$. Prove that
$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$