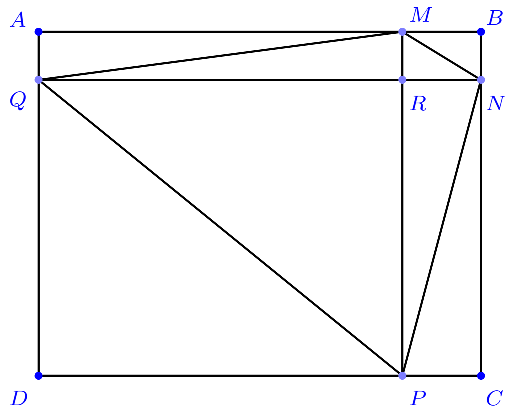
Local Selection Test 2014 Problem Proposals

1. In the diagram to the right, $ABCD$ is a rectangle, MP is parallel to AD and BC while QN is parallel to AB and CD. Assume $|MN| = 1$ cm, $|NP| = 5$ cm and $|PQ| = 7$ cm.



a) Calculate $|MQ|$.

b) The circle passing through A, M, Q intersects the circle passing through C, N, P at R and another point O. Prove that O is the centre the rectangle $ABCD$.

Solution:

1. a) Let $|AQ| = x, |QD| = y, |DP| = z$ and $|PC| = t$. Then by Pythagoras' theorem

$$\begin{aligned} x^2 + t^2 &= 1, \\ y^2 + t^2 &= 25, \\ y^2 + z^2 &= 49, \end{aligned}$$

Hence combining we obtain

$$x^2 + z^2 = x^2 + t^2 - (y^2 + t^2) + y^2 + z^2 = 1 - 25 + 49 = 25$$

and $|MQ| = \sqrt{x^2 + z^2} = 5$ cm.

b) Both the angles $\widehat{AOR}$ and $\widehat{COR}$ are right angles since AR and CR are diameters in the two given circles. Hence $\widehat{AOR} + \widehat{COR} = 180°$ so A, O and C are on the same line. Moreover, OR is height in the isosceles triangle $\triangle RAC$, so O is the midpoint of AC (to prove that, apply Pythagoras' theorem in $\triangle ROA$ and $\triangle ROC$.

2. Find all possible values of the positive integer $n < 1000$ such that the number

$$n^2 + 2015n - 5$$

is divisible by 105.

Solution I: If $n^2+2015n-5$ is a multiple of 105, then we can add or subtract any multiple of 105 and the result will still be a multiple of 105. We can use this to replace 2015 by 20. Indeed, by dividing 105 into 2015 we get $2015 = 105 \times 19 + 20$. Since $1995n = 105 \times 19n$ is a multiple of 105, it follows that

$$n^2 + 2015n - 5 - 1995n = n^2 + 20n - 5$$

is a multiple of 105. Note that n^2+20n can be completed to a square $n^2+20n+100 = (n+10)^2$ and moreover,

$$n^2 + 20n - 5 + 105 = n^2 + 20n + 100 = (n + 10)^2$$

must be a multiple of 105. Since $105 = 3 \times 5 \times 7$, it follows that $(n + 10)^2$ has 3, 5, and 7 as prime factors. This is only possible if $(n + 10)$ has 3, 5, and 7 as prime factors, or in other words, $n + 10$ is a multiple of 105.

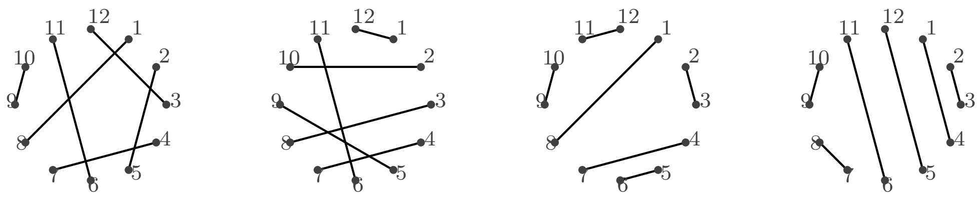
So $n + 10$ can be $105, 310, 415, 520, 625, 730, 835, 940$. Subtract 10 to find all possible values of n namely $95, 200, 305, 410, 515, 620, 725, 830, 935$.

Solution II: If 105 divides the given number, then it also divides

$$n^2 + 20n - 5$$

since $105|(2015 - 20)$. Hence $3, 5$ and 7 divide the number above. Hence $3|(n^2 + 20n - 5 - 18n + 6) = (n + 1)^2$ so $3|(n + 1)$. Also, $5|n$ and $7|(n^2 + 20n - 5 - 14n + 14) = (n + 3)^2$ so $7|(n + 3)$. We require $n = 5M = 3K - 1 = 7L - 3$ hence $3K - 5M = 1 = 3 \cdot 2 - 5 \cdot 1$ and $7L - 3K = 2 = 7 \cdot 2 - 3 \cdot 4$. Thus $3(K - 2) = 5(M - 1)$ and $3(K - 4) = 7(L - 2)$ hence $K = 5A + 2 = 7B + 4$ for $5A - 7B = 2 = 5 \cdot 6 - 7 \cdot 4$. Finally, $5(A - 6) = 7(B + 4)$ so $A = 7C + 6$, hence $K = 35C + 32$, hence $n = 105C + 95$. Values of n less than 1000 are $95, 200, 305, 410, 515, 620, 725, 830, 935$.

3. A clock pairing is a pattern in which the twelve hours of a clock are paired and the pairs are connected by line segments. Below you have 4 examples of how this can be done:



a) In how many different ways can a clock pairing be done?

b) We say that a clock pairing is good if the line segments connecting the pairs do not intersect. For example the first two clock pairings on the left above are not good, while the two clock pairings on the right are good. How many good clock pairings can be done?

Solution:

a) 1 can be paired with any one of 11 numbers. For each such pair, it remains to construct pairings for the remaining 10 numbers. Continuing the argument inductively, we get $11 \times 9 \times 7 \times 5 \times 3 = 99 \times 105 = 10395$.

b) By analyzing when two pairs cross each others we note that pairing two even numbers (or two odd numbers) will always lead to crossings. Indeed, connecting two even numbers leaves an odd number of numbers of each side and so one cannot pair the numbers on each side among themselves. Hence in a good pairings, all pairs must consist of an even and an odd, though not all such pairings are good.

Let $p(n)$ be the number of good pairings one can construct with n consecutive numbers. We can find $p(12)$ by an inductive process:
If we pair 1 with 2, then it remains to pair 10 other numbers so we can form $p(10)$ such good pairings. Similarly for the pair $(1, 12)$.
The pair $(1, 4)$ also requires $1 = p(2)$ pair $(2, 3)$ and it remains to pair 8 other numbers so $p(8)$

such good pairings exist. Similarly for the pair $(1, 10)$.
The pair $(1, 6)$ allows $p(4)$ good pairings on one side (for 2,3,4,5) and $p(6)$ on the other side (for 7, 8, ..., 12), thus a total of $p(4) \times p(6)$ good pairings. Similarly for the pair $(1, 8)$. This exhausts all pairs involving 1, so we have

$$p(12) = 2p(10) + 2p(2)p(8) + 2p(4)p(6)$$

where similarly $p(2) = 1, p(4) = 2$, then $p(6) = 2p(4) + p(2)p(2) = 5$, then $p(8) = 2p(6) + 2p(2)p(4) = 14$ and $p(10) = 2p(8) + 2p(2)p(6) + p(4)p(4) = 42$. Finally, $p(12) = 84 + 28 + 20 = 132$.