EGMO selection test
February 2014

Problem 1: In an old maths book Mary found the problem asking to determine three integers a, b, c such that

$$\frac{1}{14}\left(a^2+b^2+c^2\right)=ab+ac+bc\,.$$

(a) Someone scribbled a solution in this book: $b=-1, c=2$, but a is not legible. Find all possibilities for a in this case.

(b) Prove that for each solution (a,b,c) to the given equation, both sides of this equation are the square of an integer.

(c) Prove that for each square number S there exists a solution (a,b,c) to the given equation such that both sides of the equation are equal to S.

Solution by Bernd Kreussler:

(a) Substituting $b=-1$ and $c=2$ yields the quadratic equation

$$a^2-14a+33=0\,.$$

The two solutions $a=3$ and $a=11$ are easily found.

(b) Multiplying both sides of the given equation by 14 and then adding $2(ab+ac+bc)$ gives

$$(a+b+c)^2=16(ab+ac+bc)\,.$$

As 16 and $(a+b+c)^2$ are perfect squares, $ab+ac+bc$ is the square of an integer as well.

(c) From part (a) we know that $(a,b,c)=(3,-1,2)$ is a solution to the given equation and $\frac{1}{14}(3^2+(-1)^2+2^2)=3\cdot(-1)+3\cdot 2+(-1)\cdot 2=1$. Therefore, for any square number $S=n^2$ we have a solution $(a,b,c)=(3n,-n,2n)$, as both sides of the equation are homogeneous of degree two.

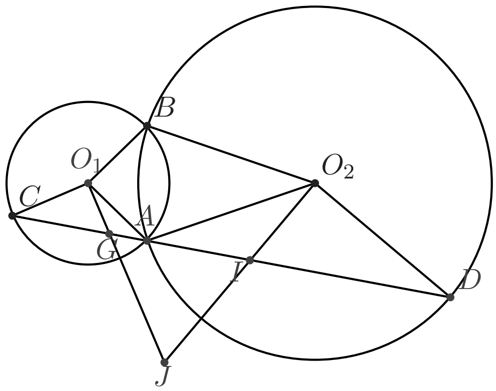
Problem 2: Let $A = \{1, 2, 3, 4, ..., 55\}$. Write A in the form

$$B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6 \cup B_7 \cup B_8 \cup B_9 \cup B_{10}$$

where each set B_i has exactly i elements, and i divides exactly the sum of the elements of B_i.

Solution by Gordon Lessells: Define $C_1 = \{1\}, C_2 = \{2, 3\}$,$C_3 = \{4, 5, 6\}$, $C_4 = \{7, 8, 9, 10\}$, $C_5 = \{11, 12, 13, 14, 15\}$,$C_6 = \{16, 17, ..., 21\}$, $C_7 = \{22, 23, ..., 28\}$, $C_8 = \{29, 30, ..., 36\}$, $C_9 = \{37, 38, ..., 45\}$, $C_{10} = \{46, 47, ..., 55\}$. The number of elements divides the sum of the elements for each of the sets C_1, C_3, C_5, C_7, C_9 but not for the others. We proceed to alter the other sets so that the divisibility condition holds. We remove one element of the set C_{2k} and replace it with the current element in C_1 for $k = 1, 2, 3, 4, 5$.The removed element becomes the new element of C_1.
At the end of this process $C_2 = \{1, 3\}$, $C_4 = \{7, 2, 9, 10\}$, $C_6 = \{16, 8, 18, 19, 20, 21\}$, $C_8 = \{17, 30, 31, 32, 33, 34, 35, 36\}$ and $C_{10} = \{46, 47, 48, 49, 50, 51, 52, 53, 29, 55\}$. $C_1 = \{54\}$. Mod arguments can be used to show that one element is always available to swap with the current element of C_1.The required sets B_i can be taken to be the current C_i for $i = 1, 2, 3, ..., 10$.
There are other approaches that can be used. For example, applying a greedy algorithm approach you come up with the following decomposition $B_{10} = \{55, 54, 53, 52, 51, 50, 49, 48, 47, 41\}$,$B_9 = \{46, 45, 44, 43, 42, 40, 39, 38, 32\}$,$B_8 = \{37, 36, 35, 34, 33, 31, 30, 28\}$, $B_7 = \{29, 27, 26, 25, 24, 23, 19\}$, $B_6 = \{22, 21, 20, 18, 17, 16\}$, $B_5 = \{15, 14, 13, 12, 11\}$, $B_4 = \{10, 9, 8, 5\}$, $B_3 = \{7, 6, 2\}$, $B_2 = \{3, 1\}$ and $B_1 = \{4\}$.
The most popular method in the actual exam was to try and fit the multiples of i inside B_i and then try to complete the set with other numbers whose sum mod i is null. This worked well together with the observation that the elements in B_1 and B_2 can be chosen last, since there is a large degree of freedom in their choice.

Problem 3: A circle Γ_1 with centre O_1 intersects another circle Γ_2 with centre O_2 at the points A and B. Consider a point C on the circle Γ_1 and a point D on the circle Γ_2 such that the line CD passes through the point A. Let J be a point such that O_1J is perpendicular to O_1C and O_2J is perpendicular to O_2D. Prove that the point J lies on the circumcircle of $\triangle BO_1O_2$.



Solution by Anca Mustata: There are many ways to prove this, but the main idea is to translate the fact that $\widehat{CAD} = 180°$ into the fact that the sum of opposite angles in the quadrilateral O_1BO_2J is 180°. Thus all methods reduce to some angle chasing. This was a very popular method but unfortunately only very few answers involved the use of $\widehat{CAD} = 180°$.

Method I: Assume Γ_2 is the larger circle. Then $\widehat{BAC} = \frac{1}{2}\widehat{BO_1C}$ and $\widehat{BAD} = 180° - \frac{1}{2}\widehat{BO_2D}$. Since $\widehat{BAC} + \widehat{BAD} = 180°$, we obtain $\widehat{BO_1C} = \widehat{BO_2D}$. Now $\widehat{BO_1J} = 360° - \widehat{BO_1C} - 90°$ while $\widehat{BO_2J} = \widehat{BO_2D} - 90°$ and so $\widehat{BO_1J} + \widehat{BO_2J} = 180°$. Hence O_1BO_2J is cyclic.

Method II: Let G and I be the points of intersection of the lines O_1J and O_2J respectively with the line CD. Then $\widehat{O_1JO_2} = 180° - \widehat{JIA} - \widehat{JGA}$ and $\widehat{JIA} = \widehat{O_2ID} = 90° - \widehat{O_2DA} = 90° - \widehat{O_2AD}$ while by the same argument $\widehat{JGA} = 90° - \widehat{O_1AC}$. All in all,

$$\begin{aligned}\widehat{O_1JO_2} &= 180° - (90° - \widehat{O_2AD}) - (90° - \widehat{O_1AC}) \\ &= \widehat{O_2AD} + \widehat{O_1AC} = 180° - \widehat{O_1AO_2} = 180° - \widehat{O_1BO_2}\end{aligned}$$

so O_1BO_2J is cyclic. We used the fact that $\widehat{O_2AD} + \widehat{O_1AC} + \widehat{O_1AO_2} = 180°$ and that $\triangle O_1AO_2 \equiv \triangle O_1BO_2$ by S.S.S.

Method III: Let O_2D intersect O_1C at E. Like in Method I we prove $\widehat{BO_1C} = \widehat{BO_2D}$. Then $\widehat{EO_1B} = \widehat{EO_2B} = 180° - \widehat{BO_2D}$ and so EO_1O_2B is cyclic. Thus $\widehat{O_1EO_2} = \widehat{O_1BO_2}$. But the quadrilateral EO_1JO_2 is cyclic since

$\widehat{EO_1J} + \widehat{EO_2J} = 90^\circ + 90^\circ = 180^\circ$. Hence $\widehat{O_1BO_2} = \widehat{O_1EO_2} = 180^\circ - \widehat{O_1JO_2}$. Hence O_1BO_2J is cyclic.