MULTIPlicativity Via A Hat Trick

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A short proof of the homomorphic property for Fock-adapted regular Markovian cocycles is given for two cases. The first is new, and facilitates dilation of quantum dynamical semigroups on a separable $C^*$-algebra.

0 Introduction

Let $j$ be a Fock-adapted regular Markovian cocycle on $A$ with noise dimension space $k$, where $A$ is a unital $C^*$-algebra acting on a Hilbert space $h$, and $k$ is a separable Hilbert space. If $j$ is completely bounded (CB) then the cocycle relation it enjoys may be expressed simply

$$j_{s+t} = j_s \circ \sigma_s \circ j_t \quad (0.1)$$

where $\sigma$ is the Fock space shift semigroup and $j_s$ is the map between matrix spaces $M(F_s; A)_h$ and $M(F_s; A)_h$ (defined below) induced by $j_s$. Regularity for the cocycle means that its associated semigroups are norm continuous. If $j$ is completely positive and contractive then it has a CB stochastic generator $\theta$ in the following sense: there is a completely bounded operator $\theta$ from $A$ into the matrix space $M(\hat{k}; A)_h$ such that $j$ satisfies the Evans-Hudson equation

$$dj_t = j_t \circ \theta_\alpha^\beta d\Lambda^\beta_\alpha(t),$$

in which $[\theta_\alpha^\beta]$ is the matrix of components of $\theta$ with respect to the basis of $k$ used for defining the matrix of quantum stochastic integrators $[\Lambda^\alpha_\beta]$. Furthermore, if $j$ is $^*$-homomorphic then the quantum It\"o formula implies that $\theta$ satisfies

$$\theta(a^*a) = \theta(a)^* (a \otimes I_k) + (a^* \otimes I_k) \theta(a) + \theta(a)^*(I_h \otimes P) \theta(a), \quad (0.2)$$

where $\hat{k} := C \oplus k$ and $P$ is the orthogonal projection in $B(\hat{k})$ with range $k \subset \hat{k}$. Conversely, (0.2) entails complete boundedness of $\theta$, which implies that $\theta$ generates a regular Markovian cocycle $j$ on $A^{10}$ in turn (0.2) also implies that $j$ is completely positive and contractive (Lemma 2.1 below). The question we seek to answer is when is $j$ multiplicative too, and so $^*$-homomorphic?

When $k$ has finite dimension $d$, Evans$^3$ proved multiplicativity by showing that the difference $J_t(a, b, \zeta, \eta) := \langle j_t(a)\zeta, j_t(b)\eta \rangle - \langle \zeta, j_t(a^*b)\eta \rangle$ satisfies

$$J_t(a, b, \zeta, \eta) = \sum_{i=1}^N \int_0^t \alpha_i(s) J_s(\phi_i(a), \psi_i(b), \zeta, \eta) ds, \quad (0.3)$$

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where \( N = (1 + d)^2(2 + d) \), each \( \phi_i \) and \( \psi_i \) is one of the bounded operators \( \theta^\beta \) and each \( \alpha_i \) is locally square integrable, which may be iterated and then estimated to be shown to be vanishingly small. The challenge in the case of infinite dimensional \( k \) was first taken up by Mohari and Sinha.\(^{14}\) Their essential idea was to make the following judiciously chosen relative boundedness assumption on \( \theta \):

For all \( \beta \) there is \((K, D)\) s.t. \( \forall a, u \parallel \theta(a)E(\beta)u \parallel \leq \parallel (a \otimes I_K)Du \parallel \). \(^{(0.4)}\)

(The operator \( E(\beta) \) is defined below.) With this they were able to tame the plague of infinitely varying indices arising from iteration of \((0.3)\), and again show that the result becomes vanishingly small.

The Mohari-Sinha assumption is perfectly tailored to the case where \( A \) is a von Neumann algebra. The reason for this is that (due to the necessary complete boundedness of \( \theta \)) their assumption is equivalent to the ultraweak continuity of \( \theta \). On the other hand, in the \( C^* \)-context their assumption now appears ad hoc.

Here we exploit the compete boundedness of both generator and cocycle to give a simple and short proof of multiplicativity under two alternative hypotheses. The first of these, which is new, is natural in the \( C^* \)-context. We apply it to give a direct proof of a dilation theorem due to Goswami, Pal and Sinha.\(^{4}\) The second is natural in the von Neumann algebra case. The first hypothesis is satisfied when the noise dimension space is finite dimensional; both hypotheses are satisfied when the algebra is finite dimensional.

A deeper study of the problem has been undertaken.\(^{12}\) There necessary and sufficient conditions on \( \theta \) are found by means of a product formula for iterated quantum stochastic integrals (cf. the work of Hudson and co-authors\(^{5}\)). Alternative additional hypotheses on \( \theta \) are then given for \((0.2)\) to entail these conditions. The question of whether \((0.2)\) alone implies multiplicativity without additional hypotheses remains open.

**General notation**

The algebraic tensor product is denoted \( \otimes \), the Hilbert space tensor product \( \otimes \), and the spatial and ultraweak tensor products of concrete operator spaces (closure in the norm and ultraweak topologies respectively) are denoted by \( \otimes_{sp} \) and \( \otimes_{uw} \). As usual tensor symbols between Hilbert space vectors will be dropped. If \( H \) and \( h \) are Hilbert spaces and \( x \in h \), the map \( u \mapsto ux \) from \( H \) to \( H \otimes h \) will be denoted \( E_x \), with the particular choice of \( H \) determined by the context; the adjoint of \( E_x \) is denoted \( E^x \), and when an orthonormal basis \( (e_i)_{i \in I} \) for \( h \) is understood, \( E_{(i)} \) denotes \( E_{x} \) for \( x = e_i \), and similarly for
When \( H = \mathbb{C} \) the notations \(|x\rangle\) and \( \langle x|\) are also used, and the (operator) spaces consisting of these operators are denoted \(|h\rangle\) and \(\langle h|\) respectively.

1 Preliminaries

Throughout this note \( \mathcal{A} \) is a unital \( C^* \)-algebra acting on the Hilbert space \( \mathfrak{h} \), \( k \) is a separable Hilbert space, and \( \mathcal{F} \) denotes the symmetric Fock space over \( L^2(\mathbb{R}_+; k) \). The Hilbert space \( \hat{k} \) is defined to be \( \mathbb{C} \oplus k \), the noise space augmented by a copy of \( \mathbb{C} \).

**Operator spaces: matrix, row and column spaces**

Let \( V \) be a concrete operator space on a Hilbert space \( H \), that is a closed linear subspace of \( B(H) \). For any other Hilbert space \( h \) we define the \( h \)-matrix space over \( V \) to be

\[
M(h; V)_h := \{ A \in B(H \otimes h) : E^dAE_e \in V \ \forall d, e \in h \}.
\]

Since the map \( e \mapsto E_e \) is (completely) isometric, to check that an operator \( A \) belongs to \( M(h; V)_h \) it is enough to check that \( E^dAE_e \in V \) for all \( d \) and \( e \) from some total subset \( S \) of \( h \), in particular an orthonormal basis \( (e_i)_{i \in I} \).

Such a choice of basis leads to the identifications

\[
H \otimes h \cong \bigoplus_{i \in I} H, \quad A \mapsto [A^i_j], \quad A^i_j := E^{(i)}AE^{(j)},
\]

which explains our terminology. The matrix space sits between more familiar tensor products with \( B(h) \):

\[
V \otimes_{sp} B(h) \subset M(h; V)_h \subset V \otimes_{uw} B(h),
\]

where the first inclusion is an equality if \( V \) or \( h \) is finite dimensional, and the second is an equality if and only if \( V \) is ultraweakly closed.

In the same way we can define row and column spaces constructed from \( V \) and \( h \):

\[
R(h; V)_h := \{ A \in B(H \otimes h; H) : AE_e \in V \ \forall e \in h \},
\]

\[
C(h; V)_h := \{ A \in B(H; H \otimes h) : E^dA \in V \ \forall d \in h \},
\]

and now the inclusion (1.1) reads

\[
V \otimes_{sp} |h\rangle \subset C(h; V)_h \subset V \otimes_{uw} |h\rangle
\]

where \( V \otimes_{sp} |h\rangle = \overline{\text{lin}}\{a \otimes |x\rangle : a \in V, x \in h\} = \overline{\text{lin}}\{E_xa\} \). If \( V \) is taken to be the \( C^* \)-algebra \( \mathcal{A} \) then we obtain a familiar Hilbert \( C^* \)-module.
Let $W$ be another operator space, acting on the Hilbert space $K$ say. Then, given $h$ as above, any CB map $\phi : V \to W$ induces a unique CB map $\phi^{(h)} : M(h; V)_b \to M(h; W)_b$ satisfying

$$E^{d}\phi^{(h)}(A)E_x = \phi(E^{d}AE_x) \quad \forall d, a \in h, A \in M(h; V)_b,$$

(1.2)

and moreover, $\|\phi^{(h)}\|_{cb} = \|\phi\|_{cb}$. The same thing happens for row and column spaces. The map $\hat{j}_s$ appearing in (0.1) is an example of such a $\phi^{(h)}$, where $V = A$, $j_s$ is a CB map $A \to M(F_s; A)_b$, and $h = F^*$, where $F_s$ and $F^*$ are the symmetric Fock spaces over $L^2([0, s]; k)$ and $L^2([s, \infty]; k)$ respectively.

This construction will be used repeatedly below in a tweaked version when the CB map in question has a matrix space as its range: if $\phi : V \to M(h'; V)_b$ is a CB map then

$$\phi^{h} : M(h; V)_b \to M(h'; M(h; V)_b)_b$$

is the map obtained by composing the map $\phi^{(h)}$ together with the natural completely isometric isomorphism

$$M(h; M(h'; V)_b)_b \to M(h'; M(h; V)_b)_b$$

(1.3)

that is obtained by restriction of the flip automorphism $B(H \otimes h' \otimes h) \to B(H \otimes h \otimes k')$. In particular we will write $\phi^{\circ n}$ when $h = \hat{k}^{\otimes n}$, and simply $\phi^{\circ}$ when $n = 1$. Thus $\hat{j}_s$ is a map

$$M(\hat{k}; A)_b \to M(F_s; M(\hat{k}; A)_b)_b$$

and not the map $\hat{j}_s$ discussed previously. The doffing of the hat is not some sleight of hand designed to pull the wool over the reader’s eyes.

The following result gives a useful property of those CB maps $\phi : V \to M(h'; V)_b$ whose range is restricted to lie in a particular subspace.

**Lemma 1.1.** Let $h$ and $h'$ be Hilbert spaces and $V$ an operator space. If $\phi : V \to M(h'; V)_b$ is a CB map such that $\phi(a)E_{x'} \in V \otimes_{sp} |h'\rangle$ for all $x' \in h'$, and $A \in M(h; V)_b$ satisfies $AE_x \in V \otimes |h\rangle$ for all $x \in h$ then

$$\phi^{h}(A)E_{\xi} \in V \otimes_{sp} |h \otimes h'\rangle \quad \forall \xi \in h \otimes h'.$$

**Proof.** Since the map $\xi \mapsto E_{\xi}$ is isometric it suffices to prove that

$$\phi^{(h)}(A)E_{x'x} \in V \otimes_{sp} |h' \otimes h\rangle \quad \forall x \in h, x' \in h'.$$

If $\phi^{(h)}$ is the CB map $C(h; V)_b \to C(h; M(\hat{k}; V)_b)_b$ induced by $\phi$ then

$$\phi^{(h)}(A)E_{x'x} = \phi^{(h)}(AE_x)E_{x'}.$$
Moreover, for any $a \in V$ and $y \in h$,
\[ \phi^{(b)}(a \otimes |y\rangle)E_{x'} = \phi(a)E_{x'} \otimes |y\rangle. \]
Therefore, by the assumptions on $\phi$ and $A$,
\[ \phi^{(b)}(A)E_{x'} = \phi^{(b)}(AE_{x'}) \in V \otimes_{sp} |h\rangle \otimes_{sp} |h'\rangle = V \otimes_{sp} |h \otimes h'\rangle. \]

Markovian cocycles

Following our recent philosophy,\(^{11}\) we concern ourselves here with \textit{completely bounded processes}, that is time-indexed families $j = (j_t)_{t \geq 0}$ of CB maps from an operator space $V$ on the \textit{initial space} $h$ into $B(h \otimes F)$. Such processes are required to be \textit{measurable}:
\[ t \mapsto \langle \xi, j_t(a)\zeta \rangle \text{ is measurable } \forall a \in V, \xi, \zeta \in h \otimes F, \]
and \textit{adapted}:
\[ j_t(V) \subset M(F_{[0,t]}; V)_{tb} \subset M(F; V)_{tb}, \]

Although our process will be everywhere defined we will still make use of the following exponential domain:
\[ \mathcal{E} := \text{Lin}\{e(f) : f \in S\}, \text{ where } S = \text{Lin}\{e\mathbf{1}_{[0,t]} : c \in k, t > 0\}. \]

A CB process $j$ is a \textit{Markovian cocycle} on $V$ if it satisfies (0.1), and furthermore is a \textit{regular cocycle} if the semigroup of maps $(E^{(0)}t_j(\cdot)E^{(0)})_{t \geq 0}$ is norm continuous. If the cocycle acts on the $C^*$-algebra $A$ and is both regular and composed of CP contractions then there is a CB map $\theta : A \rightarrow M(\hat{k}; V)_{tb}$ such that $j$ strongly satisfies the Evans-Hudson equation
\[ dj_t = j_t \circ \theta_D dA^D(t), \quad (1.4) \]
where the matrix of maps $[\theta^a_D]$ are the components of $\theta$ with respect to any orthonormal basis $\eta = (e_i)_{i \geq 0}$, extended to a basis $\hat{\eta} = (e_\alpha)_{\alpha \geq 0}$ of $\hat{k}$ by putting $e_0 = 1 \in \mathbb{C}$. Conversely any CB map $\theta : V \rightarrow M(\hat{k}; V)_{tb}$ gives rise to a strong solution of (1.4) that is a Markovian cocycle (although the resulting cocycle need not consist of bounded maps, let alone CB maps, and so the definition of cocycle must be extended appropriately).\(^{9}\)

The happy situation we are dealing with here is one in which we are given a CB generator $\theta : V \rightarrow M(\hat{k}; V)_{tb}$ that generates a CB cocycle $(j_t : V \rightarrow M(F_t; V)_{tb})$. In this case the first fundamental formula of quantum stochastic calculus can be written in the following basis independent form:
\[ \langle u\mathbb{E}(f), [j_t(a) - a \otimes I_F]v\mathbb{E}(g) \rangle = \int_0^T \langle u\widehat{f}(s)\mathbb{E}(f), j_s(\theta(a))v\widehat{g}(s)\mathbb{E}(g) \rangle \, ds \quad (1.5) \]
for all \( u, v \in \mathfrak{h}, f, g \in L^2(\mathbb{R}_+; k), a \in \mathcal{V} \) and \( t \geq 0 \). Here \( \hat{f}(s) \) and \( \hat{g}(s) \) are the images of \( f(s) \) and \( g(s) \) under the map

\[
k \ni d \mapsto \hat{d} := (1, d) \in \hat{k}.
\]

Similarly the second fundamental formula, or quantum Itô formula, is

\[
\langle j_t(a)u_\varepsilon(f), j_t(b)v_\varepsilon(g) \rangle = \int_0^t \left\{ \langle j_s(\varepsilon(a))\xi(s), j_s(\varepsilon(b))\zeta(s) \rangle + \langle j_s(\varepsilon(a))\xi(s), j_s(\varepsilon(b))\zeta(s) \rangle + \langle j_s(\varepsilon(a))\xi(s), j_s(\varepsilon(b))\zeta(s) \rangle \right\} ds
\]

(1.6)

where \( \xi(s) = u\hat{f}(s)\varepsilon(f), \zeta(s) = v\hat{g}(s)\varepsilon(g), \varepsilon(a) = a \otimes I_k \), for any Hilbert space \( H, \Delta_H := I_h \otimes P \), and \( P \) is the orthogonal projection of \( \hat{k} \) onto \( k \).

Not only do we have the above formulae, but everything also lifts up to arbitrary matrix spaces.

**Proposition 1.2.** Let \( \theta : \mathcal{V} \to \mathcal{M}(\hat{k}; \mathcal{V}) \) be a CB map that generates a CB cocycle \( j \). Then for any Hilbert space \( h \) the CB map \( \theta^h \) generates the cocycle \( j^h \) on \( \mathcal{M}(h; \mathcal{V})_h \).

**Proof.** The following identity is easily verified

\[
\langle \xi\varepsilon(f), j^h_s(A) - A \otimes I_\mathcal{H} \varepsilon(g) \rangle = \int_0^t \langle \xi\hat{f}(s)\varepsilon(f), j^h_s(\varepsilon(A))\eta\hat{g}(s)\varepsilon(g) \rangle ds,
\]

for \( \xi, \eta \in H \otimes h \), and consequently for all vectors in \( H \otimes h \) and \( H \otimes h \). In view of (1.5) and the relation \( j^h \otimes k = (j^h) \), the result follows from the uniqueness of weakly regular weak solutions of quantum stochastic differential equations.\( \square \)

The above result is summarised by the commutative diagram

\[
\begin{array}{ccc}
\theta & \longrightarrow & j \\
\downarrow & & \downarrow \\
\theta^h & \longrightarrow & j^h
\end{array}
\]

in which horizontal arrows denote generation of Markovian cocycles.

## 2 The Homomorphic Property

As explained in the introduction, the identity (0.2) is necessarily satisfied by any generator \( \theta \) of a regular \(*\)-homomorphic cocycle. Previous work on regular CP contraction cocycles,\( ^7,^8 \) summarised in the following result, implies that such a map \( \theta \) is both CB and the generator of a CP contraction cocycle.
Proposition 2.1. Let \( \theta : A \to M(\hat{k};A)_b \) be a linear map satisfying (0.2). Then \( \theta \) is CB and generates a CP contraction cocycle.

Proof. The identity (0.2) implies that \( \theta \) has the block matrix form

\[
\theta(a) = \begin{bmatrix} \tau(a) & \delta^1(a) \\ \delta(a) & \sigma(a) \end{bmatrix} - a \otimes P
\]

(2.1)

where \( \sigma \) is a \( * \)-homomorphism \( A \to M(k;A)_b \), \( \delta \) a \( \sigma \)-derivation \( A \to C(k;A)_b \), and \( \delta^1(a) := \delta(\sigma^*)a \). Furthermore, a number of applications of the well known cohomological result of Christensen and Evans\(^1\) leads to the decompositions

\[
\delta(a) = \sigma(a)L - aL, \quad \tau(a) = L^*\sigma(a)L - \frac{1}{2}\{a, L^*L\} + i[a,h]
\]

for some \( L \in C(k;A'' \) and \( h = h* \in A'' \), and so \( \theta \) is clearly CB.

So now let \( j \) denote the Markovian cocycle generated by \( \theta \), and let \( \hat{\theta}(a) = \theta(a) + a \otimes P \). Then for any \( (u_1, f_1), \ldots, (u_n, f_n) \in \mathfrak{h} \times S \) and \( A = [a_j^i] \in M_n(A) \) the semigroup representation of Markovian cocycles\(^2,8,9\) gives

\[
\sum_{i,j=1}^n \langle u_i \varepsilon(f_i 1_{[0,t_i]}), j_i(a_j^i)u_j \varepsilon(f_j 1_{[0,t_j]}) \rangle_{\mathfrak{h} \otimes \mathfrak{x}} = (u, \mathcal{P}^1_{t_{i_1} - t_0} \ast \cdots \ast \mathcal{P}^m_{t_{n-t_{n-1}}}(A)u)_{\mathfrak{h}^n} \quad (2.2)
\]

where the discontinuities of the \( f_i 1_{[0,t_i]} \) are contained in \( \{0 = t_0 \leq \cdots \leq t_m = t\} \), and \( \mathcal{P}^k \) is the semigroup on \( M_n(A) \) with generator \( \Phi_k := Y_k^* \hat{\theta}^{(n)}(\cdot)Y_k \) where

\[
Y_k = \text{diag}[E_{f_1(t_k)}, \ldots, E_{f_n(t_k)}] \in M_n(B(\mathfrak{h};\mathfrak{h} \otimes \hat{k})).
\]

From (0.2) we have

\[
\hat{\theta}^{(n)}(A^*A) = \hat{\theta}^{(n)}(A)^*(I_{\mathfrak{h}} \otimes P^\perp \otimes I_{C_n}) \hat{\theta}^{(n)}(A)
\]

\[
+ \hat{\theta}^{(n)}(A^*A \otimes P^\perp) + (A \otimes P^\perp)^* \hat{\theta}^{(n)}(A) \quad (2.3)
\]

for all \( n \geq 1 \) and \( A \in M_n(A) \). Moreover since \( P^\perp \tilde{f}_i(t_k) = (1,0) \in \hat{k} \), we have \( (A \otimes P^\perp)Y_k = \text{diag}[E_{(1,0)}, \ldots, E_{(1,0)}]A \), and so it follows that each generator \( \Phi_k \) is conditionally positive. Thus each semigroup \( \mathcal{P}^k \) is positive, and the quantity (2.2) is nonnegative whenever \( A \in M_n(A)_+ \). A simple renumbering trick is now sufficient to conclude that \( j \) is in fact CP.

Finally note that from (0.2) that \( \theta(1) = -\theta(1)^*(I_{\mathfrak{h}} \otimes P)^* \theta(1) \leq 0 \). But if we put \( \varphi = \sum_{i=1}^n u_i \varepsilon(f_i) \) and let \( \varphi \in (\mathfrak{h} \otimes \mathfrak{E})^n \) be the vector with ith component \( u_i \varepsilon(f_i) \), then

\[
\langle \varphi, [j(1) - 1 \otimes I_{\mathfrak{x}}] \varphi \rangle = \int_0^t \langle \varphi, j_s^{(n)}(Z(s)^* \theta(1)Z(s)) \varphi \rangle ds,
\]
in which \( Z(s) = [E_{j_n(s)} \cdots E_{j_1(s)}] \in B(\mathfrak{h}^n; \mathfrak{h} \otimes \hat{k}) \). Thus \( j_n(1) \) is contractive, implying that \( j \) is a CP contraction cocycle as required.

Thus any map \( \theta : A \to \text{M}(\hat{k}; A)_h \) satisfying (0.2) is CB and generates a CP contraction cocycle, and so we can apply our first result, Proposition 1.2.

Before reaching our goal we require a third and final device, namely that the polarised form of (0.2),

\[
\theta(ab) = \theta(a)\iota(b) + \iota(a)\theta(b) + \theta(a)\Delta\theta(b) \quad \forall a, b \in A, \tag{2.4}
\]

remains valid (most of the time) when we pass to the \( h \)-matrix space over \( A \).

Such an observation has already made a veiled appearance in (2.3).

**Proposition 2.2.** Let \( \theta : A \to \text{M}(\hat{k}; A)_h \) be CB map satisfying (2.4).

(a) Suppose that \( \theta(a)E_{\chi} \in A \otimes_{sp} \hat{k} \) for all \( a \in A \) and \( \chi \in \hat{k} \). If \( B \in \text{M}(h; A)_h \) satisfies \( BE_x \in A \otimes_{sp} h \) for all \( x \in h \) then \( AB \in \text{M}(h; A)_h \) for all \( A \in \text{M}(h; A)_h \), and moreover

\[
\theta^h(AB) = \theta^h(A)\iota^h(B) + \iota^h(A)\theta^h(B) + \theta^h(A)\Delta_{h \otimes h} \theta^h(B). \tag{2.5}
\]

(b) If \( A \) is a von Neumann algebra and \( \theta \) is normal then \( \theta^h \) satisfies (2.5) on all of \( \text{M}(h; A)_h = A \otimes_{uw} B(h) \).

**Remark.** The map \( \iota^h \) is of course nothing but the ampliation \( A \mapsto A \otimes I_\hat{k} \).

**Proof.** (a) Let \( x, y \in h \) and \( \chi, \delta \in \hat{k} \), then

\[
E^xAE^y = (A^xE^y)^*BE^y \in R(h; A)_h(A \otimes_{sp} h) = A,
\]

and so \( AB \in \text{M}(h; A)_h \). Let \( \{ f_\gamma \}_{\gamma \in J} \) be an orthonormal basis for \( h \), and for any finite subset \( J_0 \) let \( Q_0 \) denote the orthogonal projection onto \( \text{Lin}\{ f_\gamma : \gamma \in J_0 \} \). Then \( \sum_{\gamma \in J_0} E^y(\gamma)E^{(\gamma)} = I_H \otimes Q_0 \) for the relevant choice of \( H \). In particular since \( BE_y \in A \otimes_{sp} h \) for each \( y \in h \), we have

\[
\lim_{J_0}(I_H \otimes Q_0)BE_y = BE_y
\]

with convergence in the norm topology (check this on elementary tensors and extend by continuity). So now

\[
E^x\theta^h(AB)E^y = E^x\theta(E^xABE_y)E^y = \lim_{J_0} \sum_{\gamma \in J_0} E^x\theta(E^xAE(\gamma)E^{(\gamma)}BE_y)E^y,
\]

\( \square \)
since $\theta$ is certainly norm continuous. Now (2.4) can be used to break the above into the sum of three terms:

$$I_1 = \lim_{J_0} \sum_{\gamma \in J_0} E^x \theta(E^x A E_{(\gamma)}) \iota(E^{(\gamma)} B E_y) E_\delta$$

$$= \lim_{J_0} E^x E^x \theta^h(A) (I_0 \otimes Q_0 \otimes I_0) (B \otimes I_0) E_\delta E_y = E^x \theta(A) h(B) E_y \delta.$$

Similarly $I_2 = E^x \theta^h(A) \theta^h(B) E_y \delta$, with norm convergence assured by essentially the same argument as above, since $\theta^h(B) E_y \delta \in A \otimes_{sp} [h \otimes \hat{k}]$ by Lemma 1.1. Finally, Lemma 1.1 can be used again to show that

$$I_3 = \lim_{J_0} \sum_{\gamma \in J_0} E^x \theta(E^x A E_{(\gamma)}) \Delta \theta(E^{(\gamma)} B E_y) E_\delta$$

$$= \lim_{J_0} E^x \theta^h(A) (I_0 \otimes Q_0 \otimes P) \theta^h(B) E_y \delta = E^x \theta^h(A) \Delta h \otimes h \theta^h(B) E_y \delta.$$

Thus $\theta^h(AB)$ satisfies (2.5) as required.

(b) This case is much simpler since it is easily verified that $\theta$ satisfies (2.5) on $A \otimes B(h)$. Since both sides of the identity are separately ultraweakly continuous in $A$ and in $B$, the result follows.

Our trio of propositions can now be exploited in the main act.

**Theorem 2.3.** Let $\theta : A \to \mathcal{M}(\hat{k}; A)$ be a linear map satisfying (0.2). Then $\theta$ is completely bounded and generates a completely positive contraction cocycle $j$ which, under either of the following conditions, is $^*$-homomorphic:

(a) $\theta(a) E_{\chi} \in A \otimes_{sp} \hat{k}$ for all $a \in A$ and $\chi \in \hat{k}$.

(b) $A$ is a von Neumann algebra and $\theta$ is ultraweakly continuous.

**Proof.** In the spirit of the original proofs multiplicativity\textsuperscript{3,14} we shall obtain vanishing estimates through iteration for the difference

$$\langle u \varepsilon(f), [j_t(ab) - j_t(a)j_t(b)]v \varepsilon(g) \rangle, \quad (2.6)$$

although here we have already used Proposition 2.1 to ascertain that $\theta$ is completely bounded and $j$ is a CP contraction cocycle, so that the product $j_t(a)j_t(b)$ above is well defined. Also, by Proposition 1.2, the map $\theta^n$ generates the CB cocycle $\hat{j}^n$ on $\mathcal{M}(\hat{k} \otimes n; A)$ for each $n \geq 1$, and setting

$$\hat{j}^0 = j, \quad \theta^0 = \theta,$$

we obtain the identity

$$\langle \hat{j}_{t}^{n} \rangle = j_{t}^{(n+1)} \quad \forall n \geq 0. \quad (2.7)$$
The following CB maps \( \varphi_{n,j}, \psi_{n,j} : M(\mathbb{k}^\otimes n; A)_b \to M(\mathbb{k}^\otimes (n+1); A)_b \) \((n \geq 0, j \in \{1, 2, 3\})\) appear in the iteration:

\[
\varphi_{n,1} = \psi_{n,3} = \delta^n, \quad \varphi_{n,2} = \varphi_{n,3} = \psi_{n,1} = \theta^n, \quad \psi_{n,3} = \Delta_{b \oplus \mathbb{k}^\otimes n} \theta^n.
\]

(a) For each \( n \) define

\[
S_n = \{ A \in M(\mathbb{k}^\otimes n; A)_b : A E_{\delta} \in A \otimes \mathbb{k}^\otimes n \} \quad \forall \delta \in \mathbb{k}^\otimes n \}.
\]

Part (a) of Proposition 2.2 implies that if \( A \in M(\mathbb{k}^\otimes n; A)_b \) and \( B \in S_n \) then \( AB \in M(\mathbb{k}^\otimes n; A)_b \) and

\[
\theta^n(AB) = \sum_{j=1}^{\delta} \varphi_{n,j}(A) \psi_{n,j}(B).
\]

So applying the first and second fundamental formulae ((1.5) and (1.6)) and (2.7) to the difference

\[
\langle \xi \varepsilon(f), [j^n_1(AB) - j^n_1(A)j^n_1(B)] \xi \varepsilon(g) \rangle
\]

yields

\[
\sum_{j=1}^{\delta} \int_0^t \langle \hat{\xi} \varepsilon(f), [j_1^{(n+1)}(A_j B_j) - j_1^{(n+1)}(A_j) j_1^{N+1}(B_j)] \hat{\xi} \varepsilon(g) \rangle \, ds,
\]

where \( A_j = \varphi_{n,j}(A) \) and \( B_j = \psi_{N,j}(B) \).

But \( \theta \) and \( \varepsilon \) satisfy the hypothesis of Lemma 1.1, and so for any \( b \in A \), \( n \geq 0 \) and \( j_0, \ldots, j_n \in \{1, 2, 3\} \) we have

\[
\psi_{j_0, j_1} \circ \cdots \circ \psi_{j_{n-1}, j_n} (b) E_{\delta} \in A \otimes_{\mathrm{ap}} \mathbb{k}^\otimes (n+1) \quad \forall \delta \in \mathbb{k}^\otimes (n+1).
\]

Thus the difference (2.6) can be iterated and after \( n \) steps an upper bound for the modulus is made up of \( 3^n \) integrals over the simplex \( \{ s \in \mathbb{R}_+^N : t \geq s_1 \geq \cdots \geq s_N \geq 0 \} \) whose integrands are of the form

\[
| \langle u \hat{\psi}^{(n)}_n(s) \varepsilon(f), [j_n^{(n)}(AB) - j_n^{(n)}(A) j_n^{(n)}(B)] \hat{\psi}^{(n)}_n(s) \varepsilon(g) \rangle |
\]

where \( A \) is of the form \( \varphi_{n,j_0} \circ \cdots \circ \varphi_{n,j_n} (b) \) and similarly for \( B \), but now using the \( \psi_{n,j} \). But \( j^n \) is a contraction process and \( \theta \) is CB, so the modulus of (2.6) is bounded above by \( 3^n |n!|^{-1} C_1 C_2 \) where

\[
C_1 = \| \hat{u} \varepsilon(f) \| \| a \| \| b \| \| v \| \| \varepsilon(g) \|, \quad C_2 = \| \hat{f} 1_{[0,t]} \| \max \{ 1, \| \theta \|_{\infty} \} \| \hat{g} 1_{[0,t]} \|.
\]

Hence \( j \) is multiplicative, and thus \( * \)-homomorphic.

(b) The proof in this case is identical except that part (b) of Proposition 2.2 should be employed rather than part (a), and the subspace \( S_n \) no longer plays a role. since (2.8) is replaced by

\[
\psi_{n,j_0} \circ \cdots \circ \psi_{n,j_n} (b) \in A \otimes_{\mathrm{uw}} B(\mathbb{k}^\otimes (n+1)).
\]
3 A Dilation Theorem Revisited

Not only does part (a) of Theorem 2.3 cover Evans’ original result (since he assumes that $k$ is finite dimensional, and so $A \otimes_{sp} \hat{k} = C(k, A)_h$), but it also facilitates the following short proof of a dilation result of Goswami, Pal and Sinha\cite{goswami} for quantum dynamical semigroups. Moreover this new proof avoids having to pass to the universal enveloping algebra and subsequently showing that the (normal $^{*}$-homomorphic) process constructed there leaves the $C^*$-algebra “invariant”.

**Theorem 3.1.** Let $\mathcal{P} = (\mathcal{P}_t)_{t \geq 0}$ be a norm continuous completely positive contraction semigroup on the $C^*$-algebra $A$, and suppose that $A$ is separable. Then $\mathcal{P}$ has a $^{*}$-homomorphic stochastic dilation: there is a separable Hilbert space $k$ and a regular $^{*}$-homomorphic cocycle $j$ with noise space $k$ such that $\mathcal{P}_t = E^z(0)j_t(\cdot)E^z(0)$.

**Proof.** Let $\tau$ be the generator of $\mathcal{P}$. The first part of the proof is by now standard:\cite{pal} since $\mathcal{P}$ is completely positive and contractive, $\tau(a^*) = \tau(a)^*$ and $\tau(1) \leq 0$, and

$$(a, b) \mapsto \tau(a^*b) - a^*\tau(b) - \tau(a)^*b + a^*\tau(1)b$$

defines a nonnegative definite kernel $k : A \times A \to A \subseteq B(\mathfrak{h})$. Let $(\mathcal{K}, \gamma)$ be its minimal Gelfand pair, thus $\gamma$ is a map $A \to B(\mathfrak{h}; \mathcal{K})$ satisfying

$$\gamma(a)^*\gamma(b) = k(a, b)$$

and $K = \overline{\text{lin}}\gamma(A)\mathfrak{h}$.

The sesquilinearity of $k$ and minimality of $(\mathcal{K}, \gamma)$ imply that $\gamma$ is a linear map, and thus a bounded operator, and that there is a unique unital representation $\pi$ of $A$ on $K$ satisfying $\pi(a)\gamma(b) = \gamma(ab) - \gamma(a)b$. That is, $\gamma$ is a $\pi$-derivation.

Next, let $F = \overline{\text{lin}}\{(a, b) : a, b \in A\} \subseteq B(\mathfrak{h}; \mathcal{K})$ and note that $F$ is a Hilbert $A$-module, with $A$-valued inner product defined by $(\phi, \psi) = \phi^*\psi$. Separability of $A$ implies that $F$ is countably generated, and so Kasparov’s Absorption or Stabilisation Theorem\cite{kasparov} implies that there is a separable Hilbert space $\mathfrak{h}$ and an adjointable isometry $\alpha : F \to A \otimes_{sp} \mathfrak{h} \subseteq B(\mathfrak{h}; \mathfrak{h} \otimes \mathfrak{h})$. The vectors $\{\phi u : \phi \in F, u \in \mathfrak{h}\}$ are total in $K$, and moreover $\langle \phi u, \phi' u' \rangle = \langle \alpha(\phi)u, \alpha(\phi')u' \rangle$. Thus there is an isometry $V : K \to \mathfrak{h} \otimes \mathfrak{h}$ satisfying

$$V\phi = \alpha(\phi), \quad V^*\psi = \alpha^*\psi \quad \forall \phi \in F, \psi \in A \otimes_{sp} \mathfrak{h}. $$

So now let $k = \mathfrak{h} \otimes \mathbb{C}$ and define $\sigma : A \to B(\mathfrak{h} \otimes k)$ and $\delta : A \to B(\mathfrak{h}; \mathfrak{h} \otimes k)$ by

$$\sigma(a) = \begin{bmatrix} V\pi(a)V^* & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \delta(a) = \begin{bmatrix} \alpha(\gamma(a)) \\ 0 \end{bmatrix}.$$
where $d = (-\tau(1))^{1/2}$. If we now define $\theta$ in terms of $\tau$, $\delta$ and $\sigma$ through (2.1) then we get a map $\mathcal{A} \to B(\mathfrak{h} \otimes \hat{k})$ that satisfies (0.2). Thus to apply Theorem 2.3 to this $\theta$ we must show not only that $\theta(\mathcal{A}) \subset M(\hat{k}; \mathcal{A})_b$, but also that the columns of each $\theta(a)$ lie in the module $\mathcal{A} \otimes_{sp} \mathfrak{h}$. The only difficult term is dealt with by appealing to (3.1), since for all $a \in \mathcal{A}$ and $x \in \mathfrak{h}$

$$V\pi(a)V^*E_x = V\pi(a)\alpha^*(E_x) = \alpha(\pi(a)\alpha^*(E_x)) \in \mathcal{A} \otimes_{sp} \mathfrak{h}.$$ 

Thus $\theta$ is the generator of a $^*$-homomorphic cocycle on $\mathcal{A}$ by Theorem 2.3 that dilates $\mathcal{P}$ since the semigroup $(E^{\tau(t)}_t)_{t \geq 0}$ has generator $\tau$.

Remarks. (i) A less pedestrian use of Hilbert modules would shorten the proof even further.

(ii) If the semigroup $\mathcal{P}$ is conservative, that is $\tau(1) = 0$, then we may dispense with one dimension of noise in the dilation and use $\mathfrak{h}$ rather than $\mathfrak{k}$.

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