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**Integral Equations
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Toeplitz Operators Associated to Unimodular Algebras

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Abstract. We introduce a class of function algebras, that we call unimodular, and study Toeplitz operators on the Hardy spaces associated to representing measures on these algebras. We show that our class of function algebras is very extensive and that a number of important results for Toeplitz operators and their associated C*-algebras extend to the very general setting we consider.

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1. Introduction

If A is a function algebra on a compact Hausdorff space G—that is, A is a closed subalgebra of $C(G)$ containing the constants and separating the points of G—and τ is a character of A, then the Hahn–Banach theorem coupled with the Riesz–Kakutani theorem guarantees the existence of a regular probability measure m on G such that $\tau(\varphi) = \int \varphi \, dm$, for all $\varphi \in A$. Any such measure is a called a *representing measure* for τ. Given a regular probability measure m on G, it is clearly a representing measure for a character on A if, and only if, it is *multiplicative* on A; that is, $\int \varphi\psi \, dm = \int \varphi \, dm \int \psi \, dm$, for all $\varphi, \psi \in A$.

If m is a representing measure for a character of A, denote by $H^p(A, m)$ the closure of A in $L^p(m)$, for $1 \leq p < \infty$. Denote by $H^\infty(A, m)$ the norm-closed unital subalgebra of $L^\infty(m)$ consisting of the functions φ for which $\varphi H^2(A, m) \subseteq H^2(A, m)$. Under suitable hypothesis on m, substantial portions of the classical Hardy space theory of the circle can be extended to these generalised Hardy spaces $H^p(A, m)$ [3, 5]. In this setting the author has shown that many results of the classical Toeplitz operator theory of the circle can be extended to the context of Toeplitz operators on these Hardy spaces [8, 9].

The hypotheses that we have in mind under which the Hardy space theory can be extended is that in which m is the *unique* representing measure for a character on A and m is not a point mass. This unique-representing-measure hypothesis is one that applies in great generality. For instance, if A is a Dirichlet algebra or logmodular algebra on G, then every character of A admits a unique representing measure.

The Toeplitz operator theory derived by the author in [8, 9] builds upon the powerful Hardy space theory available in the unique-representing-measure setting. This Toeplitz theory is at its most definitive in the case that G is a compact abelian group and m is the Haar measure of G. It is therefore natural to attempt to extend the results obtained in this setting to the case where the group is non-abelian. However, serious difficulties arise when one tries to do so. We briefly discuss these difficulties now, in order to justify the quite different approach to Toeplitz operator theory we have had to take in this paper.

Let G be a non-abelian compact group and let A be a function algebra on G. Suppose also that the Haar measure m of G is a unique representing measure for a character of A. Then A cannot be translation-invariant, by [10, Corollary 5]. This poses a serious difficulty, since many (perhaps most) of the interesting examples of function algebras that arise in practice in the setting of compact groups are translation invariant. This forces one to abandon the theory of [8] in the non-abelian group setting, if one wants to include translation-invariant function algebras in the framework. In order to include such algebras we work within the setting of unimodular function algebras. (See below for the definition of unimodularity.) Thus, if (A, m) is a pair consisting of a function algebra A and a representing measure m for a character of A, we impose the condition that A is a unimodular algebra in place of the condition that m is the *unique* representing measure for a character of A. There appears to have been no substantial Hardy space function theory developed in the context we are now considering and it seems likely to the author that no such theory is possible. This necessitates an approach to the corresponding Toeplitz operator theory that is very different from the approach used in the unique-representing-measure situation of [8]. Nevertheless, ingredients of some of the proofs in this paper occur in earlier papers of the author on Toeplitz theory. However, the mixture is different and there are a number of subtle and important differences in the proofs, and therefore full details are given here.

We indicate now how the paper is organized. In Section 2 we introduce the concept of a unimodular algebra and give a large number of examples. In Section 3 we develop some aspects of the Toeplitz operator and Toeplitz algebra theory related to unimodular algebras. Our principal results are spectral inclusion results relating the spectrum of a Toeplitz operator and the spectrum (range) of its symbol, and the proof of the existence of a canonical homomorphism from the Toeplitz algebra to the algebra of continuous functions. In Section 4, we specialize to the case of analytic Toeplitz operators. Here our principal results are a commutant characterization of these operators and the proof of connectivity of their spectra.

I should like to thank the referee for the suggestion that bounded symmetric domains be included in the considerations of this paper.

2. Unimodular function algebras

Let G be a compact Hausdorff space and let A be a closed subalgebra of $C(G)$ containing the constants. We say that A is a *unimodular algebra* on G if every element of $C(G)$ can be uniformly approximated by elements of the form $\bar{\theta} f$, where θ, f are functions in A and θ is unimodular (that is, $|\theta| = 1$). Obviously, in this case, A is a function algebra on G.

Suppose now that G is a closed subgroup of the group of unitaries $\mathbf{U}(n)$ in the matrix algebra $M_n(\mathbf{C})$. For $1 \leq i, j \leq n$, define the coordinate function Z_{ij} on G by letting $Z_{ij}(u)$ be the ij-th matrix entry of the element u of G. If α is an element of $M_n(\mathbf{C})$ all of whose entries consist of non-negative integers, denote by Z^α the product of all the functions $Z_{ij}^{\alpha_{ij}}$. Also, write $|\alpha|$ for the sum of the entries of α. We denote by $P_0(G)$ the linear span of all the elements Z^α and by $P(G)$ the closure of $P_0(G)$ in $C(G)$. The elements of $P_0(G)$ are called the *polynomial functions* on G. It is clear that $P_0(G)$ is closed under multiplication and that $P(G)$ is a function algebra on G. We write Δ_G for the determinant function restricted to G. Obviously, Δ_G belongs to $P_0(G)$ and $|\Delta_G| = 1$. The determinant function plays a crucial role in the study of the algebra $P_0(G)$.

Proposition 2.1. *Let G be a closed subgroup of $\mathbf{U}(n)$. Then $P(G)$ is a unimodular algebra on G. Moreover, $\Delta_G \bar{Z}_{ij}$ belongs to $P_0(G)$, for all indices i and j, where $1 \leq i, j \leq n$.*

Proof. Proof If u is an invertible matrix in $M_n(\mathbf{C})$, then $u^{-1} = \Delta(u)^{-1} v$, where Δu is the determinant of u and v is the adjugate matrix of u. In particular, if $u \in G$, then $u^* = u^{-1} = \Delta(u)^- v$. Equating corresponding matrix entries, we get $\bar{u}_{ij} = \Delta(u)^- v_{ji}$. Using the fact that v_{ji} is a cofactor of u, it is clear that $\bar{Z}_{ij} = \Delta_G^- f_{ij}$, for some polynomial function f_{ij} in $P_0(G)$.

Now let L be the linear span of all elements $Z^\alpha \bar{Z}^\beta$, where α and β belong to $M_n(\mathbf{C})$ and have non-negative integer entries. Clearly L is a self-adjoint subalgebra of $C(G)$ separating the points of G and therefore, by the Stone–Weierstrass theorem, L is dense in $C(G)$. Let $\varphi \in L$ and write it as a sum $\varphi = \sum_{k=1}^N \varphi_k$, of terms $\varphi_k = c_k Z^{\alpha(k)} \bar{Z}^{\beta(k)}$, where $c_k \in \mathbf{C}$. Since $\Delta_G \bar{Z}_{ij}$ belongs to $P_0(G)$ for all indices i and j, we may clearly choose a positive integer M such that all the functions $\Delta_G^M \bar{Z}^{\beta(k)}$ belong to $P_0(G)$ (for instance, take $M = |\beta(1)| + \cdots + |\beta(N)| + 1$). It follows that $\Delta_G^M \varphi$ belongs to $P_0(G)$. The proposition follows. □

As discussed in the introduction, we shall be particularly interested in the case where the Haar measure on a compact group is multiplicative on a function algebra on the group. The following simple proposition is useful in this context.

We denote the circle group in the plane by $\mathbf{T}$, so $\mathbf{T} = \mathbf{U}(1)$.

Proposition 2.2. *Let G be closed subgroup of $\mathbf{U}(n)$ that contains all scalar unitary matrices $\lambda 1_n$ ($\lambda \in \mathbf{T}$). Then the normalized Haar measure m of G is multiplicative on the algebra $P(G)$.*

Proof. Proof To see this it clearly suffices to show that $\int Z^\alpha\,dm = 0$, if $|\alpha| > 0$. Let $\lambda \in \mathbf{T}$ and let $v = \lambda 1_n$. Then, by invariance of m, we have $\int Z^\alpha\,dm = \int Z^\alpha(uv)\,dm(u) = \int Z^\alpha(u)\lambda^{|\alpha|}\,dm(u) = \lambda^{|\alpha|}\int Z^\alpha\,dm$. Consequently, if $\int Z^\alpha\,dm \neq 0$, then $\lambda^{|\alpha|} = 1$ for all $\lambda \in \mathbf{T}$, which is impossible. Hence, $\int Z^\alpha\,dm = 0$, as required. ☐

Corollary 2.3. *Normalized Haar measure on* $\mathbf{U}(n)$ *is multiplicative on* $P(\mathbf{U}(n))$.

Recall that a function algebra A is *antisymmetric* if whenever φ and $\bar{\varphi}$ belong to A, φ must be a constant. If A is a translation-invariant function algebra on a compact group G, then it is antisymmetric if, and only if, the normalized Haar measure m of G is multiplicative on A, by [10, Theorem 3]. In particular, this shows that $P(\mathbf{U}(n))$ is antisymmetric, since $P(\mathbf{U}(n))$ is translation-invariant.

If $G = \mathbf{SU}(2)$, then $\Delta_G = 1$ and therefore, by Proposition 2.1, and the Stone–Weierstrass theorem, $P(G) = C(G)$. Hence, in this case $P(G)$ is not antisymmetric and the normalized Haar measure on G is not multiplicative on $P(G)$.

Recall that a function algebra A on a compact Hausdorff space G is a *Dirichlet algebra* if every real-valued continuous function φ on G can be uniformly approximated by functions of the form $\mathrm{Re}\,(f)$, where f belongs to A; A is a *logmodular algebra* if every such function φ can be uniformly approximated by functions of the form $\log|f|$, where f is an invertible element of A. Since $\mathrm{Re}\,(f) = \log|e^f|$, every Dirichlet algebra is clearly a logmodular algebra; the converse is false, as is well known. Vast numbers of Dirichlet and logmodular algebras exist [3, 5]. The important point about these algebras for us is that all their characters admit unique representing measures.

We shall return to such algebras below in the context of extending our classes of examples of unimodular algebras. First, an observation concerning the algebras $P(G)$ and the logmodularity condition.

Let $G = \mathbf{U}(n)$. By Proposition 2.1 and Corollary 2.3, $P(G)$ is a unimodular algebra on G and normalized Haar measure on G is multiplicative on $P(G)$. We have already observed that $P(G)$ is translation invariant. Therefore, if G is non-abelian, that is, if $n > 1$, then m cannot be the unique representing measure for a character of $P(G)$, by [10, Corollary 5] and consequently, $P(G)$ is not a logmodular algebra in this case, by [10, Theorem 4]. Since m is not the unique representing measure for a character of $P(G)$, the Toeplitz theory derived in [8] does not apply in the context of the Hardy spaces $H^2(P(\mathbf{U}(n)), m)$, for $n > 1$.

Of course, if $n = 1$, then $\mathbf{U}(1) = \mathbf{T}$ and $P(\mathbf{T})$ is the disc algebra on the circle. In this case, $P(\mathbf{T})$ is a Dirichlet algebra on $\mathbf{T}$ and the Haar measure m is the unique representing measure for a character of $P(\mathbf{T})$. The Hardy spaces associated to $P(\mathbf{T})$ and m are the classical Hardy spaces and the associated Toeplitz operators are the classical ones upon which the general theory is modeled.

Let D be a bounded symmetric domain, more particularly, a symmetric ball in $\mathbf{C}^n$ and let G be its Shilov boundary. If K is the group of all invertible matrices T in $GL(n)$ such that $T(D) = D$, then K acts transitively on G and there is a unique (regular) Borel probability measure m on G that is invariant for this action. Moreover, the support of m is G itself.

Now let $P(G)$ be the closure of the polynomials on G, so that $P(G)$ is a function algebra on G. Since $\mathbf{T}1_n \subseteq K$, the same argument we used in the proof of Proposition 2.2 shows that m is multiplicative on $P(G)$. Toeplitz operators over the Hardy spaces $H^2(P(G), m)$ have been extensively studied, especially by H. Upmeier. His excellent book [12] should be consulted for the theory of bounded symmetric domains and for the theory of Toeplitz operators in this context.

There is a vast number of examples of symmetric balls. We mention just two and refer to [12] for others:

1. The open unit ball D of $\mathbf{C}^n$ for the 2-norm is a symmetric ball. In this case the Shilov boundary G is equal to the topological boundary $\mathbf{S}^{2n-1}$. The invariant probability measure on G is the restriction of the usual Lebesgue measure, normalized to 1.

2. The open unit ball D of $M_n(\mathbf{C})$, endowed with the operator norm, is a symmetric ball. In this case the Shilov boundary is equal to $\mathbf{U}(n)$ and the invariant measure is the Haar measure. Thus, this falls within the scope of our earlier considerations. It should be noted that this example was considered by C.A. Berger and L.A. Coburn in [1] in the case $n = 2$, where they obtained some beautiful results in the corresponding Toeplitz operator theory. This theory was subsequently generalized to the case $n > 2$ by Upmeier, see [11].

Returning to the general situation, suppose now the symmetric ball D in $\mathbf{C}^n$ is a tube-type Cartan domain. In this case its Shilov boundary G carries a polynomial Δ (its *norm function*) that is of modulus one on G and for which we have $\bar{Z}_i\Delta$ is a polynomial, for $i = 1, \ldots, n$. (Of course , Z_i denotes the ith coordinate function on $\mathbf{C}^n$.) Using this fact, the same argument that we gave in the proof of Proposition 2.1 can be used to show that $P(G)$ is a unimodular algebra on G.

Another class of examples of unimodular algebras—a very large class—is available in the setting of abelian, connected, compact groups. If G is such a group, the continuous character group Γ of G is torsion free and therefore admits a total ordering $\leq$ that is translation-invariant; that is, if $\gamma_1, \gamma_2, \gamma_3$ belong to Γ and $\gamma_1 \leq \gamma_2$, then $\gamma_1\gamma_3 \leq \gamma_2\gamma_3$. In general, Γ admits many such total orderings; we choose one and denote by Γ^+ the set of characters γ of Γ for which $1 \leq \gamma$, where 1 denotes the constant character of G. Of course, Γ is a subset of $C(G)$, which is why we write the operation on Γ multiplicatively. We denote by $A = A(G, \Gamma^+)$ the closed linear span of Γ^+ in $C(G)$. This is a Dirichlet algebra on G and normalized Haar measure m of G is the unique representing measure for a character on A, as is easily seen. It is well known that the linear span L of Γ is dense in $C(G)$. If $\varphi \in L$, we may write $\varphi = \sum_{k=1}^{N} c_k\gamma_k$, where $c_k \in \mathbf{C}$ and $\gamma_k \in \Gamma$. Choose $\gamma \in \Gamma^+$ such that $\gamma \geq \bar{\gamma}_k$, for $k = 1, \ldots, N$. Then $\gamma\varphi \in A$. It follows that A is a unimodular algebra on G.

The algebra $A(G, \Gamma^+)$ and its associated Hardy spaces play a very prominent role in the Toeplitz operator theory developed by the author in [6, 9].

Suppose now that A is a function algebra on an arbitrary compact, Hausdorff space G and that m is the unique representing measure for a character of A. Suppose also, to avoid trivialities, that m is not a point mass. Let $L^\infty = L^\infty(G, m)$ and $H^\infty = H^\infty(A, m)$. Let $\tilde{G}$ be the character space of the C*-algebra L^∞, and let the map, $L^\infty \to C(\tilde{G})$, $\varphi \mapsto \hat{\varphi}$, denote the Gelfand representation. Of course, this is a $*$-isomorphism. It is shown in [2], using the strong modularity of the algebra H^∞ in L^∞, that the Gelfand transforms of the unimodular functions of H^∞ separate the points of $\tilde{G}$. Since the closed linear span of the functions $\varphi\bar{\psi}$, where φ, ψ are transforms of unimodular functions of H^∞, form a C*-subalgebra of $C(\tilde{G})$, the Stone–Weierstrass theorem implies that this C*-subalgebra is equal to $C(\tilde{G})$. Hence, the transform algebra $\tilde{A} = \hat{H}^\infty$ is a unimodular algebra on $\tilde{G}$.

Note that a Dirichlet algebra is not, in general, a unimodular algebra. Indeed, A. Browder and J. Wermer [3, pp. 232–5] have constructed an example of a Dirichlet algebra A on the circle group $\mathbf{T}$ that admits no non-scalar unimodular functions. Hence, A cannot be a unimodular algebra on $\mathbf{T}$.

We look now at a tensor product construction that is another fruitful source of new examples of unimodular function algebras.

Suppose that $G = G_1 \times G_2$, where G_1 and G_2 are compact, Hausdorff spaces. If $f_1 \in C(G_1)$ and $f_2 \in C(G_2)$, define $f_1 \otimes f_2 \in C(G)$ by $(f_1 \otimes f_2)(u_1, u_2) = f_1(u_1)f_2(u_2)$. Let A_1 and A_2 be function algebras on G_1 and G_2, respectively. We denote by $A = A_1 \otimes A_2$ the closed linear span of all tensors $f_1 \otimes f_2$, where $f_1 \in A_1$ and $f_2 \in A_2$. It is clear that A is a function algebra on G, called the *tensor product algebra* of A_1 and A_2.

Suppose now that m_1 and m_2 are regular probability measures on G_1 and G_2, respectively, and let m be their product measure, $m = m_1 \times m_2$. If m_1 is multiplicative on A_1 and m_2 is multiplicative on A_2, then m is multiplicative on A. To see this, one need only show that $\int fg\,dm = \int f\,dm \int g\,dm$, for f of the form $f_1 \otimes f_2$ and g of the form $g_1 \otimes g_2$, where $f_1, g_1 \in A_1$ and $f_2, g_2 \in A_2$. But in this case, $\int fg\,dm = \int f_1 g_1\,dm_1 \int f_2 g_2\,dm_2 = \int f_1\,dm_1 \int g_1\,dm_1 \int f_2\,dm_2 \int g_2\,dm_2 = \int f_1 \otimes f_2\,dm \int g_1 \otimes g_2\,dm = \int f\,dm \int g\,dm$. Hence, m is multiplicative on A, as claimed.

Proposition 2.4. *Let A_1 and A_2 be unimodular algebras on compact, Hausdorff spaces G_1 and G_2, respectively. Then the tensor product algebra $A = A_1 \otimes A_2$ is a unimodular algebra on $G_1 \times G_1$.*

Proof. Proof Choose dense linear subspaces L_1 and L_2 in $C(G_1)$ and $C(G_2)$, respectively, having the property that each function belonging to L_i is a product of a function of A_i and the conjugate of a unimodular function of A_i. If L denotes the space of all linear combinations of functions of the form $\varphi_1 \otimes \varphi_2$, where $\varphi_1 \in L_1$ and $\varphi_2 \in L_2$, then L is obviously a dense linear subspace of $C(G)$ (since, as is well known, $C(G) = C(G_1) \otimes C(G_2)$, as a C*-algebra tensor product). Let $\varphi \in L$. Then $\varphi = \sum_{k=1}^{N} \varphi_k^1 \otimes \varphi_k^2$, for elements $\varphi_1^1, \ldots, \varphi_N^1$ of L_1 and $\varphi_1^2, \ldots, \varphi_N^2$

of L_2. Hence, there exist sequences of unimodular functions $\theta_1^1, \ldots, \theta_N^1$ in A_1 and $\theta_1^2, \ldots, \theta_N^2$ in A_2 such that $\theta_k^j \varphi_k^j$ belongs to A_j, for $j = 1, 2$ and $k = 1, \ldots, N$. Let θ be the product of the functions $\theta_1^1 \otimes \theta_1^2, \ldots, \theta_N^1 \otimes \theta_N^2$. Then θ is a unimodular function of A and clearly $\theta\varphi$ belongs to A. This proves that A is unimodular algebra on $G_1 \times G_2$, as required. □

It is obvious that we have now a means to considerably increase our stock of examples of unimodular algebras on compact groups. For instance, it follows from the preceding considerations that for every pair of positive integers n and l, the function algebra $P(\mathbf{U}(n)) \otimes P(\mathbf{U}(l))$ on the compact group $\mathbf{U}(n) \times \mathbf{U}(l)$ is a unimodular algebra and that the normalized Haar measure on $\mathbf{U}(n) \times \mathbf{U}(l)$ is multiplicative on $P(\mathbf{U}(n)) \otimes P(\mathbf{U}(l))$ (the normalized Haar measure on the product group $\mathbf{U}(n) \times \mathbf{U}(l)$ is, of course, the product measure obtained from the corresponding normalized Haar measures on $\mathbf{U}(n)$ and $\mathbf{U}(l)$).

Another class of examples of the tensor product construction is obtained by taking G to be the product $\mathbf{U}(n) \times G_2$ of the compact group $\mathbf{U}(n)$ and an abelian, connected, compact group G_2, and A to be the tensor product algebra of $P(\mathbf{U}(n))$ and $A(G_2, \Gamma^+)$. Here Γ is the group of continuous characters of G_2 and Γ^+ is the positive cone for some translation-invariant total ordering of Γ. Normalized Haar measure on G is multiplicative on A, since it is the product of the normalized Haar measures on $\mathbf{U}(n)$ and G_2, and these Haar measures are multiplicative on $P(\mathbf{U}(n))$ and $A(G_2, \Gamma^+)$, respectively. Since these two algebras are unimodular, A is a unimodular algebra.

As the examples given in this section indicate, there is a very large variety of unimodular algebras on compact groups and, more generally, on compact, Hausdorff spaces. We turn now, in the next section, to a study of the theory of Toeplitz operators on the Hardy spaces associated to unimodular algebras.

3. Toeplitz operators with continuous symbols

If a is an element of a unital Banach algebra A, we denote by $\sigma(a)$, $r(a)$ and $W(a)$ its spectrum, spectral radius and numerical range, respectively. Recall that $W(a)$ is the set of all numbers $\tau(a)$, where τ is a unital, linear functional on A of norm equal to one.

To avoid repetition of hypotheses, we make the following notational conventions and standing assumptions for the sequel:

Henceforth, A denotes a unimodular algebra on a compact, Hausdorff space G and m a representing measure for a character of A that has has full support; that is, the support of m is equal to G. We write $L^p = L^p(G, m)$ and $H^p = H^p(A, m)$ for $1 \leq p \leq \infty$. We denote by $\mathbf{A}$ the C-subalgebra of $B(H^2)$ generated by the Toeplitz operators T_φ, where $\varphi \in C(G)$, and by $\mathbf{K}$ the closed commutator ideal of $\mathbf{A}$.*

If $\varphi \in L^\infty$, we denote by T_φ the compression to H^2 of the multiplication operator M_φ on L^2. We call T_φ a *Toeplitz operator* with *symbol* φ. Note the easily-verified, but important fact that, for all $\varphi \in L^\infty$ and $\psi \in H^\infty$, we have $T_{\bar{\psi}\varphi} = T_\psi^* T_\varphi$ and $T_{\varphi\psi} = T_\varphi T_\psi$. Note also that $T_\varphi^* = T_{\bar{\varphi}}$.

If $T \in B(H^2)$, we denote the numerical range of T relative to $B(H^2)$ by $W(T)$.

Theorem 3.1. *Let φ be an element of $C(G)$.*

(1) $\|T_\varphi\| = r(T_\varphi) = \|\varphi\|_\infty$;

(2) T_φ is a positive operator if, and only if, φ is a positive element of $C(G)$;

(3) If T_φ is left or right invertible, then φ is invertible in $C(G)$;

(4) $\sigma(\varphi) \subseteq \sigma(T_\varphi) \subseteq W(T_\varphi) = \operatorname{co}\sigma(\varphi)$, where co denotes the convex hull in $\mathbf{C}$.

Proof. Proof It is clear that $\|T_\varphi\| \le \|\varphi\|_\infty$. To show the reverse inequality, we may invoke the unimodularity hypothesis on A to reduce to the case where $\varphi = \bar{\theta} f$, where θ and f belong to A and θ is unimodular. Since $\|T_{\bar{\theta}f}\| = \|T_\theta^* T_f\| \le \|T_f\| = \|T_{\bar{\theta}f\theta}\| = \|T_{\bar{\theta}f} T_\theta\| \le \|T_{\bar{\theta}f}\|$, we have $\|T_{\bar{\theta}f}\| = \|T_f\|$. Using this condition and the fact that $\|\bar{\theta} f\|_\infty = \|f\|_\infty$, we may suppose that $\varphi = f$; that is, we may suppose that $\varphi \in A$. In this case we have, for each positive integer n, $(\int |\varphi^n|\, dm)^{1/n} = (\int |T_\varphi^n(1)|\, dm)^{1/n} \le (\int |T_\varphi^n(1)|^2\, dm)^{1/2n} = \|T_\varphi^n(1)\|_2^{1/n} \le \|T_\varphi^n\|^{1/n} \|1\|_2^{1/n}$. Hence, $\|\varphi\|_\infty = \lim(\int |\varphi^n|\, dm)^{1/n} \le \lim \|T_\varphi^n\|^{1/n} = r(T_\varphi) \le \|T_\varphi\|$. Therefore, $\|T_\varphi\| = \|\varphi\|_\infty$, for all $\varphi \in C(G)$, as required.

It is obvious that T_φ is positive, if φ is. Suppose then that T_φ is a positive operator and we shall show that φ is positive. Let $t = \|T_\varphi\| = \|\varphi\|_\infty$. Then $\|T_\varphi - t\| \le t$, by [7, Lemma 2.2.2]. Hence, $\|\varphi - t\|_\infty = \|T_\varphi - t\| \le t$. Also, since $T_{\bar{\varphi}} = T_\varphi$, we have $\bar{\varphi} = \varphi$. Applying [7, Lemma 2.2.2] again, we deduce that φ is positive. This proves (2).

Suppose now that T_φ is left invertible. Then $T_\varphi^* T_\varphi$ is invertible and therefore, $T_\varphi^* T_\varphi \ge \delta$, for some positive number δ. If P is the projection of L^2 onto H^2, then $PM_{\bar{\varphi}} PM_\varphi P \le PM_{\bar{\varphi}\varphi} P$, where M_ψ is the multiplication operator associated to a function ψ in L^∞, and therefore $T_{\bar{\varphi}} T_\varphi \le T_{\bar{\varphi}\varphi}$. Hence, $T_{\bar{\varphi}\varphi} \ge \delta$, and consequently, $\bar{\varphi}\varphi \ge \delta$, by Condition (2). It follows that φ is invertible. By taking adjoints, we see that φ is also invertible if T_φ is right invertible. This proves (3).

The inclusion $\sigma(\varphi) \subseteq \sigma(T_\varphi)$ is immediate from (3). Hence, $\|\varphi\|_\infty \le r(T_\varphi)$ and it is clear now that Condition (1) holds. The containment of $\sigma(T_\varphi)$ in $W(T_\varphi)$ holds because the numerical range always contains the spectrum. Since the map, $\varphi \mapsto T_\varphi$, is a unital isometry, the inclusion $W(T_\varphi) \subseteq W(\varphi)$ is obvious. However, since φ is normal, we have $W(\varphi) = \operatorname{co}\sigma(\varphi)$, by [4, p. 53]. Hence, $W(T_\varphi) \subseteq \operatorname{co}\sigma(\varphi)$ and, since $\sigma(\varphi) \subseteq W(T_\varphi)$ and the numerical range is always a convex set, we have $W(T_\varphi) = \operatorname{co}\sigma(\varphi)$. Therefore, (4) holds. □

If θ is a unimodular function in A, set $P_\theta = 1 - T_\theta {T_\theta}^*$. Because T_θ is an isometry, P_θ is a projection belonging to $\mathbf{K}$. It is easily checked that $(P_\theta)_\theta$ is an increasing net, where we define $\theta_1 \le \theta_2$ to mean that $\theta_2 \bar{\theta}_1$ belongs to A.

Theorem 3.2. *The net (P_θ) is an increasing approximate unit for $\mathbf{K}$.*

Proof. Proof Let L denote the linear space consisting of the products $\bar{\theta}f$, where $\theta, f \in A$ and θ is unimodular. Let $\mathbf{A}_0$ denote the linear span of all products $T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_n}$, where $\varphi_1, \ldots, \varphi_n$ belong to L, and let $\mathbf{K}_0$ denote the linear span of the corresponding operators $T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_n} - T_{\varphi_1\varphi_2\cdots\varphi_n}$. If $\varphi \in L$, then the equation $T_\varphi(T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_n} - T_{\varphi_1\varphi_2\cdots\varphi_n}) = T_\varphi T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_n} - T_{\varphi\varphi_1\cdots\varphi_n} + T_{\varphi(\varphi_1\cdots\varphi_n)} - T_\varphi T_{\varphi_1\varphi_2\cdots\varphi_n}$ shows that $\mathbf{K}_0$ is a left ideal of the algebra $\mathbf{A}_0$. Similarly, $\mathbf{K}_0$ is a right ideal of $\mathbf{A}_0$. It follows that the closure $\overline{\mathbf{K}_0}$ is an ideal in the algebra $\overline{\mathbf{A}}_0$ and clearly, $\overline{\mathbf{A}}_0 = \mathbf{A}$, since L has norm-closure equal to $C(G)$ because A is unimodular. It is obvious that the quotient algebra $\mathbf{A}/\overline{\mathbf{K}_0}$ is commutative, so $\overline{\mathbf{K}_0}$ contains $\mathbf{K}$. We now show that the net (P_θ) is an approximate unit for $\overline{\mathbf{K}_0}$ and this will imply that $\overline{\mathbf{K}_0} = \mathbf{K}$, since the projections P_θ belong to $\mathbf{K}$. This will then prove the theorem.

To show that (P_θ) is an approximate unit as claimed, it suffices to show that for any $T \in \mathbf{K}_0$, there exists an unimodular function θ in A such that $T = TP_\theta$—in this case, for any unimodular function θ' in A such that $\theta' \geq \theta$, we clearly also have $T = TP_{\theta'}$. We may suppose that T is of the form $T = T_{\varphi_n}\cdots T_{\varphi_1} - T_{\varphi_n\cdots\varphi_1}$, where $\varphi_1, \ldots, \varphi_n$ belong to L. Clearly, since $\varphi_1, \varphi_1\varphi_2, \ldots, \varphi_1\varphi_2\cdots\varphi_n$ belong to L, we can find a unimodular function θ in A such that $\theta\varphi_1$, $\theta\varphi_1\varphi_2$,…,$\theta\varphi_1\cdots\varphi_n$ belong to A. Then $T_{\varphi_n}\cdots T_{\varphi_1}T_\theta = T_{\varphi_n}\cdots T_{\varphi_1\theta} = T_{\varphi_n}\cdots T_{\varphi_2\varphi_1\theta} = \cdots = T_{\varphi_n\cdots\varphi_1\theta}$, so $TT_\theta = 0$ and therefore, $T = TP_\theta$, as required. □

Corollary 3.3. *The only Toeplitz operator with continuous symbol belonging to* $\mathbf{K}$ *is the zero operator.*

Proof. Proof Let $\varphi \in C(G)$ and suppose that $T_\varphi \in \mathbf{K}$. Since the net $(T_\varphi P_\theta)_\theta$ converges to T_φ, the net $(T_\varphi T_\theta T_\theta{}^*)_\theta$ converges to zero. But $\|T_\varphi\| = \|T_\theta{}^* T_\varphi T_\theta T_\theta{}^* T_\theta\| \leq \|T_\varphi T_\theta T_\theta{}^*\|$, so $T_\varphi = 0$. □

Theorem 3.4. *There is a unique $*$-homomorphism π from* $\mathbf{A}$ *onto $C(G)$ such that $\pi(T_\varphi) = \varphi$, for all $\varphi \in C(G)$. Moreover, the kernel of π is equal to* $\mathbf{K}$.

Proof. Proof It suffices to show that the map ρ from $C(G)$ to $\mathbf{A}/\mathbf{K}$ that maps φ onto $T_\varphi + \mathbf{K}$ is a $*$-isomorphism. Clearly, ρ is linear and preserves adjoints. Moreover, it is multiplicative, since $\mathbf{K}$ contains all operators of the form $T_\varphi T_\psi - T_{\varphi\psi}$, as we saw in the the proof of Theorem 3.2. Injectivity of ρ is immediate from Corollary 3.3 and Theorem 3.1(1). Finally, surjectivity of ρ follows from the observation that the range of the map is a C*-algebra containing the elements $T_\varphi + \mathbf{K}$, where $\varphi \in C(G)$, and these generate the quotient algebra $\mathbf{A}/\mathbf{K}$. □

Theorem 3.5. $\mathbf{A}$ *acts irreducibly on H^2.*

Proof. Proof Let Q be a projection in $B(H^2)$ commuting with all the T_φ, where $\varphi \in C(G)$. We shall show that $Q = 0$ or $Q = 1$. Let $\psi = Q(1)$. If $f \in A$, then $Q(f) = QT_f(1) = T_f Q(1) = f\psi$. Hence, if f_1 and f_2 belong to A, then $\int \psi f_1 \bar{f}_2 \, dm = < Q(f_1), f_2 > = < Q(f_1), Q(f_2) > = \int |\psi|^2 f_1 \bar{f}_2 \, dm$. It follows that $\int \psi\varphi \, dm = \int |\psi|^2 \varphi \, dm$, for all φ in the linear span L of all products $\varphi_1\bar{\varphi}_2$, where $\varphi_1, \varphi_2 \in A$. Since L is a self-adjoint subalgebra of $C(G)$ separating the points of G, the Stone–Weierstrass theorem implies that L is norm-dense in $C(G)$. It follows that $\int \psi\varphi\, dm = \int |\psi|^2 \varphi\, dm$, for all φ in $C(G)$. Hence, $\psi = |\psi|^2$ (as elements of L^1). In particular, $\psi = \bar{\psi}$. Let $c = \int \psi\, dm$. Since m is multiplicative on A, it is also multiplicative on H^2. Hence, since $\psi \in H^2$, we have $\int (\psi - c)^2\, dm = (\int \psi\, dm - c)^2 = 0$, so $\psi = c$. Because ψ is an idempotent, this implies that $c = 0$ or $c = 1$. Consequently, $Q(\varphi) = 0$ or $(1-Q)(\varphi) = 0$, for all $\varphi \in A$ and therefore, by norm-density of A in H^2, for all $\varphi \in H^2$. Thus, $Q = 0$ or $Q = 1$, as required. □

We are now in a position to determine the precise conditions under which the theory is trivial.

Theorem 3.6. *The following are equivalent conditions:*

(1) For some non-zero element φ of $C(G)$, the operator T_φ is compact;
(2) $H^2 = L^2$;
(3) $\dim(H^2) < \infty$;
(4) $A = C(G)$;
(5) G is a singleton set.

Proof. Proof The implications (5) $\Rightarrow$ (4) $\Rightarrow$ (2) are obvious. If (2) holds, then the map, $\varphi \mapsto T_\varphi$, is clearly multiplicative and therefore, $\mathbf{A}$ is abelian. Since $\mathbf{A}$ acts irreducibly on H^2, by Theorem 3.5, H^2 must therefore be one-dimensional. Hence, (2) $\Rightarrow$ (3). If (3) holds, then T_1 is a non-zero compact Toeplitz operator, so (3) $\Rightarrow$ (1). Finally, suppose that (1) holds, that is, suppose there exists T_φ a compact operator for which $\varphi \neq 0$, and we shall show that this implies that (5) holds and thereby prove the theorem. Since $\mathbf{A}$ acts irreducibly on H^2, our assumption implies that $\mathbf{A}$ contains the ideal $K(H^2)$ of compact operators on H^2. Now T_φ does not belong to $\mathbf{K}$, by Corollary 3.3, so $K(H^2) \not\subseteq \mathbf{K}$. Consequently, $K(H^2) \cap \mathbf{K} = 0$, by simplicity of $K(H^2)$, and therefore, since $\mathbf{A}$ is primitive, $\mathbf{K} = 0$. Hence, $\mathbf{A}$ is abelian and therefore, $\dim(H^2) = 1$. It follows that $\dim(A) = 1$ and therefore, since A is a function algebra in $C(G)$, $\dim(C(G)) = 1$. Hence, G is a singleton set and (5) holds, as required. □

Henceforth, to avoid trivialities, we shall assume that G is not a singleton set.

Theorem 3.7. *The only compact Toeplitz operator on H^2 is the zero operator.*

Since $\dim(H^2) = \infty$, we may speak of the essential spectrum $\sigma_e(T)$ of an operator T on H^2. Of course, this is the spectrum of the image of T in the Calkin algebra $B(H^2)/K(H^2)$.

Theorem 3.8. *If $\varphi \in C(G)$, then $\sigma(\varphi) \subseteq \sigma_e(T_\varphi)$.*

Proof. Proof It suffices to show that φ is invertible in $C(G)$, if T_φ is a Fredholm operator. Now the ideal of compact operators $K(H^2)$ is either contained in $\mathbf{K}$, or its intersection with $\mathbf{K}$ is the zero space. In either case, this implies that $T_\varphi + \mathbf{K}$ is invertible in $\mathbf{A}/\mathbf{K}$, if T_φ is Fredholm. Hence, by Theorem 3.4, φ is invertible in $C(G)$. □

If A is a Dirichlet algebra, and $\varphi \in C(G)$, then T_{e^φ} is necessarily invertible, by [8, Corollary 7.2]. This is no longer true in our present setting. Example 8.2 of [8] provides a counterexample. An inspection of this example shows that it is obtained by the transition from a triple (G, A, m) to $(\tilde{G}, \tilde{A}, \tilde{m})$ as explained in Section 2, where $\tilde{m}$ is the unique representing measure for a character on $\tilde{A}$ and $\tilde{m}$ has full support; therefore, the function algebra $\tilde{A}$ involved is a unimodular algebra, and all our assumptions are satisfied. Nevertheless, there exists $\varphi \in C(\tilde{G})$ such that T_{e^φ} is not invertible.

4. Analytic Toeplitz operators

A Toeplitz operator T_φ is *analytic* if the symbol φ belongs to H^∞. Clearly, the map, $H^\infty \to B(H^2)$, $\varphi \mapsto T_\varphi$, is an algebra homomorphism. Our first result of this section shows that this map is in fact an isometry.

Theorem 4.1. *If $\varphi \in H^\infty$, then $\|T_\varphi\| = r(T_\varphi) = \|\varphi\|_\infty$.*

Proof. Proof Reasoning as in the proof of Theorem 3.1, we have, for each positive integer n, $(\int |\varphi^n|\, dm)^{1/n} = (\int |T_\varphi^n(1)|\, dm)^{1/n} \le (\int |T_\varphi^n(1)|^2\, dm)^{1/2n} \le \|T_\varphi^n\|^{1/n}$. Hence, $\|\varphi\|_\infty = \lim(\int |\varphi^n|\, dm)^{1/n} \le \lim \|T_\varphi^n\|^{1/n} = r(T_\varphi)$. The result follows. □

Theorem 4.2. *Let $T \in B(H^2)$. Then T is an analytic Toeplitz operator if, and only if, $TT_f = T_f T$, for all $f \in A$.*

Proof. Proof We suppose that $TT_f = T_f T$ for all $f \in A$ and show that T is an analytic Toeplitz operator (obviously, all analytic Toeplitz operators commute). Let L be the linear space of functions of the form $\bar{\theta} g$, where θ and g belong to A and θ is unimodular. Since A is unimodular, L is norm dense in $C(G)$ and therefore norm dense in L^2 also. Let P be the projection of L^2 onto H^2. For each unimodular function θ in A, define S_θ in $B(L^2)$ by setting $S_\theta(h) = \bar{\theta} TP(\theta h)$ and note that $\|S_\theta\| \le \|T\|$. We shall show that for each pair h_1 and h_2 in L^2, the net $(< S_\theta(h_1), h_2 >)_\theta$ converges. It will then follow that $(S_\theta)_\theta$ converges in the weak operator topology to a bounded linear operator on L^2. By density of L in L^2, we may reduce to the case where $h_i = \bar{\theta}_i g_i$, with $\theta_i, g_i \in A$ and θ_i are unimodular, for $i = 1, 2$. In this case, let θ be a unimodular function in A for which $\theta \ge \theta_1, \theta_2$; that is, $\theta\bar{\theta}_1, \theta\bar{\theta}_2 \in A$. Then

$$\begin{aligned} &< S_\theta(h_1), h_2 > = < \bar{\theta} TP(\theta\bar{\theta}_1 g_1), \bar{\theta}_2 g_2 > \\ &= < P(\bar{\theta}\theta_2 TP(\theta\bar{\theta}_1 g_1)), g_2 > = < T_{\bar{\theta}\theta_2} T T_{\theta\bar{\theta}_1}(g_1), g_2 > \\ &= < T_{\bar{\theta}\theta_2} T_{\theta\bar{\theta}_1} T(g_1), g_2 > = < T_{\theta_2\bar{\theta}_1} T(g_1), g_2 >. \end{aligned}$$

Hence, the net $(< S_\theta(h_1), h_2 >)_\theta$ converges, as required.

Let S denote the weak operator topology limit of the net (S_θ), $S = \lim S_\theta$. If $\psi \in C(G)$, we claim that S and the multiplication operator M_ψ commute. Using density of L in $C(G)$, we may suppose that $\psi = \bar{\theta} f$, for elements θ, f of A with θ unimodular. Using the fact that multiplication operators are normal, and

Fuglede's theorem, we clearly need only show that $SM_f = M_f S$. Hence, it suffices to show that, for h_1 and h_2 in L^2, we have $\lim < (S_\theta M_f - M_f S_\theta)(h_1), h_2 > = 0$. Again using density of L in L^2, we may reduce to the case where $h_i = \bar{\theta}_i g_i$, with $\theta_i, g_i \in A$ and θ_i are unimodular, for $i = 1, 2$. As before, let θ be a unimodular function in A for which $\theta \geq \theta_1, \theta_2$. Then

$$\begin{aligned} < (S_\theta M_f - M_f S_\theta)(h_1), h_2 > &= < \bar{\theta} T P(\theta f \bar{\theta}_1 g_1), \bar{\theta}_2 g_2 > - < f \bar{\theta} T P(\theta \bar{\theta}_1 g_1), \bar{\theta}_2 g_2 > \\ &= < P(\theta_2 \bar{\theta} T P(\theta f \bar{\theta}_1 g_1)), g_2 > - < P(\theta_2 f \bar{\theta} T P(\theta \bar{\theta}_1 g_1)), g_2 > \\ &= < T_{\theta_2 \bar{\theta}} T T_{\theta \bar{\theta}_1 f}(g_1), g_2 > - < T_{f \theta_2 \bar{\theta}} T T_{\theta \bar{\theta}_1}(g_1), g_2 > \\ &= < T_{\theta_2 \bar{\theta}} T_{\theta \bar{\theta}_1 f} T(g_1), g_2 > - < T_{f \theta_2 \bar{\theta}} T_{\bar{\theta}_1 \theta} T(g_1), g_2 > \\ &= < T_{\theta_2 \bar{\theta}_1 f} T(g_1), g_2 > - < T_{f \theta_2 \bar{\theta}_1} T(g_1), g_2 > = 0. \end{aligned}$$

Hence, as claimed, S commutes with all the multiplication operators M_ψ, where ψ belongs to $C(G)$. Therefore, $S = M_\varphi$, for some function $\varphi \in L^\infty$. Since the compression of all the operators S_θ to H^2 is equal to T, it follows that the compression of S to H^2 is also equal to T; that is, $T_\varphi = T$.

Finally, we show that $\varphi \in H^\infty$. If $f \in H^2$, then $M_\varphi(f) = T(f) \in H^2$. Hence, $\varphi H^2 \subseteq H^2$ and therefore $\varphi \in H^\infty$. ☐

Corollary 4.3. *The analytic Toeplitz operators form a maximal commutative subalgebra of* $B(H^2)$.

Corollary 4.4. *If* $\varphi \in H^\infty$, *then* $\sigma(T_\varphi) = \sigma_{H^\infty}(\varphi)$*—the spectrum of* φ *as an element of the algebra* H^∞.

Proof. Proof Let B be the algebra of analytic Toeplitz operators. Then $\sigma(T_\varphi) = \sigma_B(T_\varphi)$, since B is a maximal, commutative subalgebra of $B(H^2)$. Since the map, $\varphi \mapsto T_\varphi$, is an isomorphism of H^∞ onto B (injectivity follows from Theorem 4.1), we have $\sigma_B(T_\varphi) = \sigma_{H^\infty}(\varphi)$. ☐

Corollary 4.5. *The spectrum of an analytic Toeplitz operator is connected.*

Proof. Proof By Corollary 4.4, we need only show that $\sigma_{H^\infty}(\varphi)$ is connected for every $\varphi \in H^\infty$. Hence, by Shilov's idempotent theorem, we need only show that the commutative Banach algebra H^∞ contains no idempotents except the trivial ones 0 and 1. Suppose then φ is an idempotent in H^∞. Then $\varphi = \bar{\varphi}$ and if $c = \int \varphi \, dm$, then $\int |\varphi - c|^2 \, dm = (\int \varphi \, dm - c)^2 = 0$. Hence, $\varphi = c$ and therefore, $c = 0$ or $c = 1$. Thus, $\varphi = 0$ or $\varphi = 1$. ☐

References

[1] C.A. Berger and L.A. Coburn, *Wiener–Hopf operators on* U_2. Integr. Equat. Oper. Th. **2** (1979), 139–173.

[2] A. Bernard, J.B. Garnett and D.E. Marshall, *Algebras generated by inner functions.* J. Fuct. Anal. **25** (1977), 275–285.

[3] A. Browder, *Introduction to Function Algebras.* Benjamin, New York–Amsterdam, 1969.

[4] K.R. Goodearl, *Notes on Real and Complex C*-Algebras.* Shiva, Nantwich, 1982.

[5] G.M. Leibowitz, *Lectures on Complex Function Algebras.* Scott–Foresman, Illinois, 1970.

[6] G.J. Murphy, *Ordered groups and Toeplitz algebras.* J. Operator Theory **18** (1987), 303–326.

[7] G.J. Murphy, *C*-Algebras and Operator Theory.* Academic Press, Boston–San Diego, 1990.

[8] G.J. Murphy, *Toeplitz operators on generalised H^2 spaces.* Integr. Equat. Oper. Th. **15** (1992), 825–852.

[9] G.J. Murphy, *An index theorem for Toeplitz operators.* J. Operator Theory **29** (1993), 97–114.

[10] G.J. Murphy, *Translation-invariant function algebras on compact groups.* Adv. Stud. Comtemp. Math. **3** (2001), 39–42.

[11] H. Upmeier, *Toeplitz C*-algebras in bounded symmetric domains.* Ann. of Math. **119** (1984), 549–576.

[12] H. Upmeier, *Toeplitz Operators and Index Theory in Several Complex Variables.* Birkhaüser, Basel, 1996.

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