Topological and Analytical Indices in C*-Algebras

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Dedicated to the memory of my friend Gert K. Pedersen

1 Introduction

If φ is a continuous non-vanishing function on the closed unit circle $\mathbf{T}$ in the plane $\mathbf{C}$ and T_φ is the corresponding Toeplitz operator on the Hardy space H^2 of $\mathbf{T}$, then the well known theorem of I. Gohberg and M. Krein asserts that

$$\operatorname{index}(T_\varphi) = -\operatorname{wn}(\varphi), \tag{1}$$

where $\operatorname{index}(T_\varphi)$ is the Fredholm index of T_φ and $\operatorname{wn}(\varphi)$ is the winding number of φ around the origin. Our objective in this paper is to obtain a far-reaching generalization of this result.

The Gohberg–Krein index theorem has already inspired many significant generalizations, including the Atiyah–Singer index theorem. In this paper we restrict ourselves to generalizations involving Toeplitz operators over Hardy spaces other than the one on the circle. In this setting there has also been a large variety of extensions of the Gohberg–Krein theorem and it is clear from these that one is going to have to modify both sides of Equation 1 in order to extend the result appropriately. Examples show that if one replaces the Toeplitz operators on the circle by classes of Toeplitz operators on other Hardy spaces, then the usual integer-valued Fredholm index may have to be replaced by the real-valued index of M. Breuer [3, 4] or by some other analytic index. There is also a problem with generalizing the winding number. The idea is to replace it with some kind of topological index $\omega(\varphi)$ for a symbol function φ. Although many analogues of the winding number have been found in various different cases, such as the average winding number for an almost periodic function, and some beautiful index theorems have been obtained using these topological index functions, nevertheless the definition of the analogue of the winding number in each case has been made on a somewhat *ad hoc* basis. In this paper we present a systematic account of topological index functions for "symbol functions" in C*-algebras in terms of an appropriate trace. Our ideas make fundamental use of the concept of the quantized differential calculus introduced by Alain Connes [6].

The concept of Fredholm index also has to be extended in the more general setting. In this case we make use of a concept of analytic index of an operator that also is defined in terms of an appropriate trace; this is based on earlier work of the author [7, 9]. We then derive an index theorem that has not only the Gohberg–Krein index theorem as a special case, but a number of other index theorems, including a famous real-valued index theorem due to L.A. Coburn, R.G. Douglas, D.G. Schaeffer and I.M. Singer [5]. The trace approach to the index of a Toeplitz operator and the topological index of the corresponding symbol provides a unifying framework in which the proof of the index theorem finds a natural and elegant setting.

2 The topological index

In this section we construct and derive the basic properties of topological index functions associated to traces on C*-algebras. We then calculate formulas for a number of interesting examples. Before going into details of the construction we begin by defining some terms and discussing some key examples.

If C is a unital C*-algebra, a *topological index* on C is a locally constant function

$$\omega : \operatorname{Inv} C \to \mathbf{R}, \quad \varphi \mapsto \omega(\varphi),$$

from the topological group $\operatorname{Inv} C$ of invertible elements of C to the reals $\mathbf{R}$ such that

$$\omega(\varphi\psi) = \omega(\varphi) + \omega(\psi),$$

for all $\varphi, \psi \in \operatorname{Inv} C$. Locally constancy for ω means, of course, continuity relative to the discrete topology on $\mathbf{R}$.

Since ω is a homomorphism, $\omega(1) = 0$.

Now suppose φ is an arbitrary element of C. Then we claim that $\omega(e^{\varphi}) = 0$. For, the map

$$[0,1] \to \mathbf{R}, \quad t \mapsto \omega(\exp(t\varphi)),$$

is locally constant and therefore, by connectedness of $[0,1]$, it must be constant, from which we get $\omega(e^{\varphi}) = \omega(e^{0}) = \omega(1) = 0$.

Conversely, if ω is a homomorphism from $\operatorname{Inv} C$ to the additive group $\mathbf{R}$ for which $\omega(e^{\varphi}) = 0$, for all φ in C, then ω is locally constant and therefore a topological index on C. To see this we observe that if φ and ψ belong to $\operatorname{Inv} C$ and $\|\varphi - \psi\| < \|\psi^{-1}\|^{-1}$, then $\|\varphi\psi^{-1} - 1\| < 1$ and therefore $\varphi\psi^{-1} = e^{\theta}$, for some element $\theta \in C$. Hence $\omega(\varphi) = \omega(e^{\theta}\psi) = \omega(e^{\theta}) + \omega(\psi) = \omega(\psi)$.

If $\pi : \operatorname{Inv} C \to \operatorname{Inv} C/\operatorname{Exp} C$ is the quotient homomorphism, where $\operatorname{Exp} C$ is the subgroup of $\operatorname{Inv} C$ generated by the exponential elements e^{θ} ($\theta \in C$), then it is clear that the set of topological index functions on C is just the set of all functions of the form $\omega = \hat{\omega} \circ \pi$, where $\hat{\omega}$ is a homomorphism from the group $\operatorname{Inv} C/\operatorname{Exp} C$ to the additive group $\mathbf{R}$. However, this abstract formulation does not help to identify interesting topological index functions on C, nor to give formulas for their computation.

The prototypical example of a topological index is, of course, the winding number wn on $C(\mathbf{T})$, the C*-algebra of all continuous complex-valued functions on $\mathbf{T}$. Moreover, since every invertible function φ in $C(\mathbf{T})$ can be written in the form $\varphi = z^n e^{\psi}$, for a unique integer n and some function $\psi \in C(\mathbf{T})$, where z is the inclusion function $\mathbf{T} \to \mathbf{C}$, we have, for any topological index ω on $C(\mathbf{T})$, $\omega(\varphi) = \omega(z^n) = n\omega(z)$. Hence, $\omega = c\,\mathrm{wn}$, where $c = \omega(z)$. Thus, there is only one non-zero topological index on $C(\mathbf{T})$, up to multiplication by a non-zero constant.

However, uniqueness of the index is not typical in the general setting. For example, there are topological indices ω_l and ω_r on $C(\mathbf{T}^2)$ defined by setting, for $\varphi \in \operatorname{Inv} C(\mathbf{T}^2)$,

$$\omega_l(\varphi) = \operatorname{wn}(\varphi(\cdot, 1)) \text{ and } \omega_r(\varphi) = \operatorname{wn}((\varphi(1, \cdot)).$$

A moment's reflection shows neither of these indices is a multiple of the other.

Of course, many C*-algebras admit no topological index other than the zero map. This is the case for the C*-algebra $M_N(\mathbf{C})$ of all complex matrices of order N and, more generally, for any von Neumann algebra. It is also the case for any unital AF-algebra. In each case the reason is that every invertible element is a product of exponential elements e^{φ}. Nevertheless, we shall see that many algebras, including many non-commutative ones, do admit useful topological index functions. Our objective now is to give a general procedure for constructing such functions. This will be based on the concept of an indicial triple $\Omega = (\mathcal{L}, F, \mathrm{tr})$ consisting of a unital C*-algebra $\mathcal{L}$, a self-adjoint unitary F and a trace function tr satisfying certain axioms (the precise definition is given below). We shall use Ω not only to define a topological index, but to set up a generalised theory of Toeplitz elements and to derive an index theory generalizing the Gohberg–Krein theorem. This will be done in the next section; here we will confine ourselves to the topological index. We begin by considering a more general situation where we get some partial or auxiliary results and then we specialize by making extra assumptions in order to obtain a definitive result.

As we said earlier, we shall make use of the quantized calculus of Connes. Recall that this, in its most fundamental form, is defined in terms of an ordered pair (H, F), where H is a Hilbert space and F is a self-adjoint unitary operator in $B(H)$ (the C*-algebra of norm-bounded linear operators on H). If C is an algebra of operators acting on H—to be precise, a C*-subalgebra of $B(H)$—we define the *quantized differential* of $\varphi \in C$ by setting $\mathrm{d}\varphi = [F, \varphi]$. To conform with Connes's theory, one should require that $\mathrm{d}\varphi$ be "infinitesimal" in the sense that it is a compact operator. In the even stronger case that $\mathrm{d}\varphi$ belongs to the trace-class ideal or the Dixmier ideal, and ψ is an arbitrary element of C, one can then define the "noncommutative integral" $\int \psi \, \mathrm{d}\varphi = \mathrm{tr}(\psi \mathrm{d}\varphi)$, where tr is the usual trace or the Dixmier trace.

It is clear that one can modify this in a natural way by replacing these traces by some other trace. We now generalize this set-up.

We suppose now that $\mathcal{L}$ is a unital C*-algebra containing a unital C*-subalgebra C and a self-adjoint unitary F. We set

$$\mathrm{d}\varphi = [F, \varphi], \quad \text{for all } \varphi \in C.$$

Clearly, the map

$$\mathrm{d} : C \to \mathcal{L}, \quad \varphi \mapsto \mathrm{d}\varphi,$$

is a derivation; that is,

$$\mathrm{d}(\varphi\psi) = (\mathrm{d}\varphi)\psi + \varphi \mathrm{d}\psi,$$

for all $\varphi, \psi \in C$.

We suppose also that $\mathcal{L}$ admits a trace $\mathrm{tr} : \mathcal{L}^+ \to [0, +\infty]$. We denote by $\mathcal{M}_{\mathrm{tr}}$ its definition ideal, the linear span of all elements $\varphi \in \mathcal{L}^+$ such that $\mathrm{tr}(\varphi) < +\infty$. Let C_0 be the unital $*$-subalgebra of C consisting of all elements φ in C for which $\mathrm{d}\varphi$ belongs $\mathcal{M}_{\mathrm{tr}}$. Then C_0 is *inverse-closed in* C in the sense that if φ is an element of C_0 invertible in C, then $\varphi^{-1} \in C_0$. This is immediate from the easily-verified formula $\mathrm{d}(\varphi^{-1}) = -\varphi^{-1}(\mathrm{d}\varphi)\varphi^{-1}$. Note also that

$$\mathrm{tr}(\mathrm{d}\varphi) = 0, \quad \text{for all } \varphi \in C_0. \tag{2}$$

This equation will play a fundamental role in our considerations. To see that it holds, first define $P = (F+1)/2$. Of course, P is a projection and $\mathrm{d}\varphi = 2[P, \varphi]$. Now we observe that $(P\varphi - \varphi P)P = P\varphi P - \varphi P$ and $P(P\varphi - \varphi P) = P\varphi - P\varphi P$ belong to $\mathcal{M}_{\mathrm{tr}}$ and $\mathrm{tr}((P\varphi - \varphi P)P) = \mathrm{tr}(P(P\varphi - \varphi P)) = \mathrm{tr}(P(P\varphi - \varphi P)P) = \mathrm{tr}(0) = 0$; hence, $\mathrm{tr}(P\varphi P - \varphi P) = \mathrm{tr}(P\varphi P - P\varphi) = 0$. Therefore $\mathrm{tr}(\mathrm{d}\varphi) = 2\mathrm{tr}(P\varphi - \varphi P) = 0$, as required.

We now define $\omega(\varphi)$, for φ in $\mathrm{Inv}\, C_0$, by setting

$$\omega(\varphi) = \mathrm{tr}(\varphi^{-1}\mathrm{d}\varphi)/2. \tag{3}$$

Our formula is obviously analogous to the usual integral formula

$$\mathrm{wn}(\varphi) = \frac{1}{2\pi i}\int \varphi(z)^{-1}\mathrm{d}\varphi(z),$$

for the winding number $\mathrm{wn}(\varphi)$ of a non-vanishing differentiable function φ on $\mathbf{T}$, with the integral replaced by the trace function. Of course, this is the case that motivates our theory. The factor of 1/2 in Equation 3 is used as a normalization constant to ensure our definition agrees with the definition of the winding number wn in the case of the circle and it also gives the correct normalization for the general index theorem, Theorem 3.1, in Section 3.

Lemma 2.1 *If* ψ *and* φ *are elements of* $\mathrm{Inv}\, C_0$*, then*

$$\omega(\varphi\psi) = \omega(\varphi) + \omega(\psi). \tag{4}$$

Proof. This follows from the following elementary calculation:

$$\begin{aligned} 2\omega(\varphi\psi) &= \mathrm{tr}((\varphi\psi)^{-1}\mathrm{d}(\varphi\psi)) = \mathrm{tr}(\psi^{-1}\varphi^{-1}(\mathrm{d}\varphi)\psi) + \mathrm{tr}(\psi^{-1}\varphi^{-1}\varphi\mathrm{d}\psi) \\ &= \mathrm{tr}(\varphi^{-1}(\mathrm{d}\varphi)\psi\psi^{-1}) + \mathrm{tr}(\psi^{-1}\mathrm{d}\psi) = 2\omega(\varphi) + 2\omega(\psi). \square \end{aligned}$$

The following result will be needed in the proof of Lemma 2.3 below.

Lemma 2.2 *If* φ *and* ψ *are commuting elements of* C_0 *and* C*, respectively, then*

$$\mathrm{tr}(\psi\mathrm{d}\varphi^n) = n\mathrm{tr}(\psi\varphi^{n-1}\mathrm{d}\varphi), \tag{5}$$

for any positive integer n*. If* φ *is invertible in* C_0*, Equation 5 holds for* $n \leq 0$ *also.*

Proof. We show this for $n \geq 1$ by induction. The result is trivially true for $n = 1$. Suppose it is true (for all φ and ψ) for the value n. Then

$$\begin{aligned} \mathrm{tr}(\psi\mathrm{d}\varphi^{n+1}) &= \mathrm{tr}(\psi\varphi^n\mathrm{d}\varphi) + \mathrm{tr}(\psi(\mathrm{d}\varphi^n)\varphi) = \mathrm{tr}(\psi\varphi^n\mathrm{d}\varphi) + \mathrm{tr}(\varphi\psi(\mathrm{d}\varphi^n)) \\ &= \mathrm{tr}(\psi\varphi^n\mathrm{d}\varphi) + n\mathrm{tr}(\varphi\psi\varphi^{n-1}\mathrm{d}\varphi) \quad \text{(induction hypothesis)} \\ &= \mathrm{tr}(\psi\varphi^n\mathrm{d}\varphi) + n\mathrm{tr}(\psi\varphi^n\mathrm{d}\varphi) = (n+1)\mathrm{tr}(\psi\varphi^n\mathrm{d}\varphi). \end{aligned}$$

To see Equation 5 holds for $n < 0$ if φ is invertible in C_0 (it obviously holds for $n = 0$ in this case), we simply apply the case $-n > 0$ to $\theta = \varphi^{-1}$: Then $\mathrm{tr}(\psi \mathrm{d}\theta^{-n}) = -n\mathrm{tr}(\psi\theta^{-n-1}\mathrm{d}\theta)$. Since $\mathrm{d}\theta = -\theta(\mathrm{d}\varphi)\theta$, therefore $\mathrm{tr}(\psi\mathrm{d}\theta^{-n}) = -n\mathrm{tr}(\psi\theta^{-n-1}(-\theta(\mathrm{d}\varphi)\theta))$; that is, $\mathrm{tr}(\psi\mathrm{d}\varphi^n) = n\mathrm{tr}(\psi\varphi^{n-1}\mathrm{d}\varphi)$. □

Lemma 2.3 *If $\varphi \in C_0$, then*

$$\mathrm{tr}(\varphi^n \mathrm{d}\varphi) = 0,$$

for all all integers $n \geq 0$. If φ is invertible in C_0, this equation holds for all integers $n \neq -1$.

Proof. We know that $\mathrm{tr}(\mathrm{d}\varphi) = 0$ by Equation 2. If n is a non-negative integer, then $(n+1)\mathrm{tr}(\varphi^n\mathrm{d}\varphi) = \mathrm{tr}(\mathrm{d}\varphi^{n+1}) = 0$, by Lemma 2.2 and Equation 2 again. Hence, $\mathrm{tr}(\varphi^n\mathrm{d}\varphi) = 0$. To see this holds if φ is invertible in C_0 and $n \leq -2$, set $m = -n-2$ in this case and note that $\mathrm{tr}((\varphi^{-1})^m\mathrm{d}\varphi^{-1}) = 0$, by what we have just proved (applied to φ^{-1}). Hence, $\mathrm{tr}(\varphi^n\varphi^2(-\varphi^{-1}(\mathrm{d}\varphi)\varphi^{-1})) = 0$; that is, $-\mathrm{tr}(\varphi^{-1}\varphi^n\varphi\mathrm{d}\varphi) = 0$ and therefore $\mathrm{tr}(\varphi^n\mathrm{d}\varphi) = 0$. □

To proceed further we need to make an additional assumption on the trace. This assumption is essential for the proofs that follow, but it is not clear that it is essential for the results obtained, although I suspect that this is the case.

> *We assume now that the trace function* $\mathrm{tr} : \mathcal{L}^+ \to [0, +\infty]$ *is lower semicontinuous.*

This condition implies that for each element $\psi \in \mathcal{M}_{\mathrm{tr}}$, the linear map

$$\mathcal{L} \to \mathbf{C}, \quad \varphi \mapsto \mathrm{tr}(\varphi\psi),$$

is continuous (see [10, Lemma A5], for example). We shall use this in the following lemma.

Lemma 2.4 *If $\varphi \in C_0$ and $\|\varphi\| < 1$, then*

$$\omega(1-\varphi) = 0.$$

Proof. Clearly,

$$(1-\varphi)^{-1} = \sum_{n=0}^{\infty} \varphi^n.$$

Since $\mathrm{d}\varphi \in \mathcal{M}_{\mathrm{tr}}$, and tr is lower semicontinuous, we have continuity of $\mathrm{tr}(\cdot\,\mathrm{d}\varphi)$, and therefore

$$\omega(1-\varphi) = -\frac{1}{2}\mathrm{tr}((1-\varphi)^{-1}\mathrm{d}\varphi) = -\frac{1}{2}\sum_{n=0}^{\infty}\mathrm{tr}(\varphi^n\mathrm{d}\varphi).$$

It follows immediately from this equation and Lemma 2.3 that $\omega(1-\varphi) = 0$. □

Lemma 2.5 *The map $\varphi \mapsto \omega(\varphi)$ is locally constant on the set* $\mathrm{Inv}\, C_0$.

Proof. Let $\varphi \in \operatorname{Inv} C_0$. We show that $\omega(\psi) = \omega(\varphi)$, for all ψ in some neighbourhood U of φ in C_0. We take U to be the open set of all $\psi \in C_0$ such that $\|\psi - \varphi\| < \|\varphi^{-1}\|^{-1}$. Then for any element $\psi \in U$, we have $\|1 - \psi\varphi^{-1}\| < 1$, so that $\omega(\psi\varphi^{-1}) = \omega(1 - (1 - \psi\varphi^{-1})) = 0$, by Lemma 2.4. Hence, $\omega(\varphi) = \omega(\psi\varphi^{-1}) + \omega(\varphi) = \omega(\psi)$, as required. □

If C_0 is dense in C, the set of invertible elements of C_0 is dense in the set of invertible elements of C. In this case, since ω is continuous (for the discrete topology on $\mathbf{C}$), we extend it uniquely to a locally constant function $\omega : \operatorname{Inv} C \to \mathbf{C}$. Clearly, Equation 4 holds now for all elements φ and ψ in $\operatorname{Inv} C$.

The trace takes on arbitrary complex values on $\mathcal{M}_{\mathrm{tr}}$ (provided it is non-zero), so there is no *a priori* reason to suppose that $\omega(\varphi)$ is a real number. Nevertheless, this turns out to be the case, as we'll see now: If $\varphi \in \operatorname{Inv} C_0$, then, since $\mathrm{d}(\varphi^*) = -(\mathrm{d}\varphi)^*$, we have $2\omega(\varphi^*) = \mathrm{tr}((\varphi^*)^{-1}\mathrm{d}(\varphi^*)) = -\mathrm{tr}((\varphi^{-1})^*(\mathrm{d}\varphi)^*) = -\mathrm{tr}((\mathrm{d}\varphi)\varphi^{-1})^- = -2\omega(\varphi)^-$. It follows by continuity that $\omega(\varphi^*) = -\omega(\varphi)^-$, for all $\varphi \in \operatorname{Inv} C$. On the other hand, $\varphi^*\varphi = e^{\psi}$, for some element $\psi \in C$, and therefore $\omega(\varphi^*\varphi) = 0$. Hence, $\omega(\varphi^*) = -\omega(\varphi)$. Putting these two facts together shows that $\omega(\varphi)$ is a real number.

Thus, ω is a topological index on C. We summarise our construction in the following theorem. First we make a definition:

An *indicial triple* $\Omega = (\mathcal{L}, F, \mathrm{tr})$ for a unital C*-algebra C consists of a unital C*-algebra $\mathcal{L}$ containing C as a unital C*-subalgebra, a self-adjoint unitary F belonging to $\mathcal{L}$ and a lower semicontinuous trace $\mathrm{tr} : \mathcal{L}^+ \to [0, +\infty]$ for which the $*$-subalgebra C_Ω of all elements φ of C such that $[F, \varphi]$ belongs to the definition ideal $\mathcal{M}_{\mathrm{tr}}$ is dense in C. As before, we set $\mathrm{d}\varphi = [F, \varphi]$, if $\varphi \in C$.

Theorem 2.6 *Let $\Omega = (\mathcal{L}, F, \mathrm{tr})$ be an indicial triple for a unital C*-algebra C. Then there is a unique locally constant function*

$$\omega : \operatorname{Inv} C \to \mathbf{R}, \quad \varphi \mapsto \omega(\varphi),$$

for which

$$\omega(\varphi) = \mathrm{tr}(\varphi^{-1}\mathrm{d}\varphi)/2,$$

for all invertible elements φ in C_Ω. Moreover, ω is a topological index on C.

We call ω the *topological index associated to*, or *derived from*, the triple Ω, and in cases of ambiguity we write ω_Ω in place of ω.

It is not at all clear that every topological index on a C*-algebra C is associated to an indicial triple for C in this manner. Indeed, I conjecture that this is not the case, but confess that this is not based on any supporting evidence.

Our aim in the next section will be to use topological index functions associated to indicial triples to derive index theorems for generalised Toeplitz operators. However, in this section we turn now to the question of presenting examples of indicial triples and computing the corresponding index functions. These examples will be revisited in Section 3 to derive analogues of the Gohberg–Krein index theorem.

Example 2.7 Suppose that H is the Hilbert space $L^2(\mathbf{T})$. We identify the C*-algebra $C(\mathbf{T})$ with the corresponding C*-algebra C of multiplication operators on H.

Let $(e_n)_n$ be the standard orthonormal basis of H, so $e_n = z^n$, where $z : \mathbf{T} \to \mathbf{C}$ is the inclusion map. The Hardy space H^2 on $\mathbf{T}$ is the closed linear span in H of all e_n, where $n \geq 0$. We denote the (Szego) projection of H onto H^2 by P and we set $F = 2P - 1$, so that F is a self-adjoint unitary.

We let $\mathcal{L}$ be the unital C*-subalgebra of $B(H)$ generated by C and F and we denote by $\mathrm{tr} : \mathcal{L}^+ \to [0, +\infty]$ the restriction of the canonical trace on $B(H)$. If $\mathrm{d}\varphi = [F, \varphi]$, for all $\varphi \in C$, a simple computation shows that if $m > 0$, then $\mathrm{d}(z^m)(e_n) = 2e_{m+n}$, for $n = -1, -2, \ldots, -m$, and $\mathrm{d}(z^m)(e_n) = 0$ otherwise. It follows that $\mathrm{d}(z^m)$ is of finite rank and therefore belongs to the ideal of definition $\mathcal{M}_{\mathrm{tr}}$ of tr. Hence, the $*$-algebra of all φ in C such that $\mathrm{d}\varphi \in \mathcal{M}_{\mathrm{tr}}$ contains all the trigonometric polynomials on $\mathbf{T}$ and is therefore dense in C. Therefore, $\Omega = (\mathcal{L}, F, \mathrm{tr})$ is an indicial triple for C. We denote the topological index on C associated to Ω by ω. Since $\mathrm{d}z = 2\Theta_{e_0, e_{-1}}$, where, for $f, g, h \in H$, the operator $\Theta_{f,g} \in B(H)$ is defined by setting $\Theta_{f,g}(h) = (\, h \mid g \,) f$, we have $\omega(z) = \mathrm{tr}(\bar{z}\mathrm{d}z)/2 = \mathrm{tr}(\Theta_{e_{-1}, e_{-1}}) = (\, e_{-1} \mid e_{-1} \,) = 1$. It now follows easily from the properties of ω and those of the classical winding number function that for any non-vanishing function φ in C the number $\omega(\varphi)$ is equal to the winding number of φ around the origin.

Example 2.8 In this example we set $H = L^2(\mathbf{U}(2))$, where $\mathbf{U}(2)$ is the compact group of unitary matrices of order 2. We identify $C(\mathbf{U}(2))$ with the corresponding unital C*-algebra C of multiplication operators on H. We denote by $P_0(\mathbf{U}(2))$ the unital subalgebra of C consisting of the polynomial functions on $\mathbf{U}(2)$ and by $R_0(\mathbf{U}(2))$ the algebra of all quotients of elements of $P_0(\mathbf{U}(2))$ (the rational functions on $\mathbf{U}(2)$). Obviously, $P_0(\mathbf{U}(2))$ is generated as an algebra by the coordinate functions

$$Z_{ij} : \mathbf{U}(2) \to \mathbf{C}, \quad g \mapsto g_{ij}.$$

In this setting the Hardy space H^2 on $\mathbf{U}(2)$ is defined to be the L^2-closure of $P_0(\mathbf{U}(2))$ in H. We denote by P the projection of H onto H^2. We then let $\mathcal{L}$ be the unital C*-algebra generated by C and the self-adjoint unitary $F = 2P - 1$. As usual, we set $\mathrm{d}\varphi = [F, \varphi]$, for all $\varphi \in C$.

Let $\mathbf{SU}(2)$ denote the special unitary (compact) group of all elements of $\mathbf{U}(2)$ of determinant one. It is shown in [1] that we can identify H as a Hilbert space tensor product, $H = L^2(\mathbf{T}) \otimes L^2(\mathbf{SU}(2))$, in such a way that the closed commutator ideal $\mathcal{K}$ of $\mathcal{L}$ is identified with the C*-tensor product $B_0(L^2(\mathbf{T})) \otimes B$, where we use B_0 to denote the C*-algebra of compact operators, and where B is a certain unital C*-subalgebra of $B(L^2(\mathbf{SU}(2))$. In this identification Z_{ij} is identified with an element of the form $1 \otimes a_{ij} + z \otimes b_{ij}$, where z is multiplication on $L^2(\mathbf{T})$ by the independent variable and a_{ij} and b_{ij} are certain elements of B. Also, $P = Q \otimes 1$, where Q is the projection of $L^2(\mathbf{T})$ onto the Hardy space of $\mathbf{T}$. Hence, $\mathrm{d}Z_{ij} = 2[Q, z] \otimes b_{ij}$. It follows easily from this and our calculations for $[Q, z^m]$ in the preceding example that for all $\varphi \in P_0(\mathbf{U}(2))$, $\mathrm{d}\varphi$ belongs to the subset $\mathcal{F}$ in $\mathcal{K}$ consisting of sums of elementary tensors $T \otimes b$, where T is a finite-rank norm-bounded linear operator on $L^2(\mathbf{T})$ and b is an element of B. Using the ideas of Section 3 of [9], one can easily see that $\mathcal{L}$ admits a lower semicontinuous trace $\mathrm{tr} : \mathcal{L}^+ \to [0, +\infty]$ with definition ideal containing $\mathcal{F}$ such that $\mathrm{tr}(T \otimes b) = \mathrm{Tr}(T)\tau(b)$, where $\mathrm{Tr}(T)$ is the usual operator trace of T in $B(L^2(\mathbf{T}))$ and τ is a certain tracial state on B. Moreover, one can choose to have $\mathcal{K}$ be equal to the closure of $\mathcal{M}_{\mathrm{tr}}$ (this will be important for Theorem 3.4). We'll refer to any such trace as a *tensor trace* in Theorem 3.4 below. We did not show lower semicontinuity of tr in [9], but this is easily verified. The $*$-algebra of all elements φ in C such that $\mathrm{d}\varphi \in \mathcal{M}_{\mathrm{tr}}$ is dense in C, since it contains $P_0(\mathbf{U}(2))$ (by what we have stated above), and therefore $R_0(\mathbf{U}(2))$, and since $R_0(\mathbf{U}(2))$ is dense in C. Hence, $\Omega = (\mathcal{L}, F, \mathrm{tr})$ is an indicial triple for C. As usual, we denote by ω its associated topological index.

Now if φ is an invertible element of C, we may write $\varphi = \Delta^n e^{\psi}$, where Δ is the determinant function on $\mathbf{U}(2)$, n is some integer and ψ is some element of C (see [1, Lemma 17]). Hence, since $\Delta = z \otimes 1$, we have $\omega(\Delta) = \mathrm{tr}(\Delta^{-1}[P, \Delta]) = \mathrm{tr}((z^{-1} \otimes 1)[Q \otimes 1, z \otimes 1]) = \mathrm{Tr}(z^{-1}[Q, z])\tau(1) = \mathrm{wn}(z) = 1$ and therefore,

$$\omega(\varphi) = n, \quad \text{if } \varphi = \Delta^n e^{\psi}.$$

The construction of the trace function tr is not obvious, nor is it clear that there is any uniqueness involved. It would be preferable to obtain a more "natural" construction. Nevertheless, despite this "artificiality" in the construction of tr, this example shows that our topological index provides a significant and natural invariant for the non-vanishing continuous functions on $\mathbf{U}(2)$.

In the preceding two examples the "symbol algebra" C was commutative. However, there are important cases in the applications of this theory where this algebra is non-commutative; in particular, one will want to consider "symbol algebras" that are matrix algebras over commutative algebras. This situation is addressed in the following theorem. An application involving Toeplitz operators is given in the next section.

Commutativity of the algebra C in the second half of the following theorem is required so that the determinant of a matrix in $M_N(C)$ as an element of C makes sense. It is also required at some points in the proof.

Theorem 2.9 *Let $\Omega = (\mathcal{L}, F, \mathrm{tr})$ be an indicial triple over a unital C^*-algebra C. For each positive integer N, let $C_N = C \otimes M_N(\mathbf{C})$ and $\mathcal{L}_N = \mathcal{L} \otimes M_N(\mathbf{C})$. Denote by F_N the self-adjoint unitary $F \otimes 1_N$ in $\mathcal{L}_N$, where 1_N is the unit of $M_N(\mathbf{C})$. Let $\mathrm{tr}_N : \mathcal{L}_N^+ \to [0, +\infty]$ be the trace on $\mathcal{L}_N$ defined by setting*

$$\mathrm{tr}_N(\varphi) = \sum_{i=1}^{N} \mathrm{tr}(\varphi_{ii}),$$

where φ is the matrix (φ_{ij}) in $\mathcal{L}_N^+$. Then $\Omega_N = (\mathcal{L}_N, F_N, \mathrm{tr}_N)$ is an indicial triple for C_N. Suppose now C is commutative and denote by ω and ω_N the topological index functions corresponding to Ω and Ω_N. Then if φ is an element of $\mathrm{Inv}\, C_N$ with determinant $\det \varphi$ in $\mathrm{Inv}\, C$, the topological indices of these elements are related by the equation

$$\omega_N(\varphi) = \omega(\det \varphi). \tag{6}$$

Proof. All the statements in the theorem are easily verified apart from the formula in Equation 6, in the case C is commutative. To see this, first set $\mathrm{d}\psi = [F, \psi]$ and $\mathrm{d}_N(\varphi) = [F_N, \varphi]$, for all $\psi \in C$ and $\varphi \in C_N$. Now suppose

φ is invertible in C_{Ω_N} and set $\Delta = \det\varphi$. Then Δ is invertible in C_Ω. Also, $2\omega(\Delta) = \mathrm{tr}(\Delta^{-1}\mathrm{d}\Delta) = \mathrm{tr}(\Delta^{-1}\sum_\sigma \varepsilon_\sigma \mathrm{d}(\varphi_{1,\sigma 1}\cdots\varphi_{N\sigma N}))$, where the summation is over all permutations σ of the integers $1,\ldots,N$ and ε_σ is the signature of σ. Hence,

$$\begin{aligned} 2\omega(\Delta) &= \mathrm{tr}(\Delta^{-1}\sum_\sigma \varepsilon_\sigma \sum_{i=1}^N \varphi_{1\sigma 1}\cdots \mathrm{d}\varphi_{i\sigma i}\cdots\varphi_{N\sigma N}) \\ &= \mathrm{tr}(\Delta^{-1}\sum_{i,j=1}^N \sum_{\sigma,\sigma i=j} \varepsilon_\sigma\, \varphi_{1\sigma 1}\cdots\widehat{\varphi_{i\sigma i}}\cdots\varphi_{N\sigma N}\mathrm{d}\varphi_{ij}), \end{aligned}$$

where the hat $\hat{}$ indicates the symbol is to be omitted. Note that the commutativity of C is needed at this point (as well as to define $\det\varphi$ in the first place). The sum

$$\sum_{\sigma,\sigma i=j} \varepsilon_\sigma\, \varphi_{1\sigma 1}\cdots\hat{\varphi_{i\sigma i}}\cdots\varphi_{N\sigma N}$$

is clearly the ji-entry ψ_{ji} of the adjugate matrix $\psi = (\psi_{ij})_{ij}$ of φ, so $2\omega(\Delta) = \mathrm{tr}(\Delta^{-1}\sum_{i,j=1}^N \psi_{ji}\mathrm{d}\varphi_{ij}) = \mathrm{tr}_N(\Delta^{-1}\psi\mathrm{d}_N\varphi) = \mathrm{tr}_N(\varphi^{-1}\mathrm{d}_N\varphi) = 2\omega_N(\varphi)$. □

Example 2.10 In this example the topological index function will be seen to take on arbitrary real values. Here we take as our "symbol algebra" the C*-algebra $\mathrm{AP}(\mathbf{R})$ of almost-periodic functions on $\mathbf{R}$; that is, the sup-norm closure of the trigonometric polynomials on $\mathbf{R}$, where, of course, a trigonometric polynomial is a linear combination of the functions

$$e_x : \mathbf{R} \to \mathbf{C}, \quad y \mapsto e^{ixy},$$

for $x \in \mathbf{R}$. For $\varphi \in L^\infty(\mathbf{R})$, let M_φ be the operator on $L^2(\mathbf{R})$ given by multiplication by φ and for $x \in \mathbf{R}$, denote by U_x the unitary operator on $L^2(\mathbf{R})$ given by translation by x, so that

$$U_x(f)(y) = f(y-x) \quad (f \in L^2(\mathbf{R}),\ y \in \mathbf{R}).$$

Then if $\mathcal{F}$ is the unitary Fourier transform on $L^2(\mathbf{R})$, we have

$$\mathcal{F}M_{e_x}\mathcal{F}^* = U_x \ \text{ and } \ \mathcal{F}P\mathcal{F}^* = Q, \tag{7}$$

where P is the projection of $L^2(\mathbf{R})$ onto the Hardy space

$$H^2(\mathbf{R}) = \{f \in L^2(\mathbf{R}) \mid (\mathcal{F}f)(x) = 0 \ (x<0)\}$$

and $Q = M_{\chi_{\mathbf{R}^+}}$, where $\chi_{\mathbf{R}^+}$ is the characteristic function of $\mathbf{R}^+$ in $\mathbf{R}$.

We identify $\mathrm{AP}(\mathbf{R})$ with the corresponding C*-algebra C of multiplication operators on $L^2(\mathbf{R})$ and denote by $\mathcal{L}$ the unital C*-subalgebra of $B(L^2(\mathbf{R}))$ generated by C and F, where $F = 2P - 1$. It follows from [5] that the C*-algebra $\mathcal{B}$ on $L^2(\mathbf{R})$ generated by all multiplication operators M_φ, where $\varphi \in L^\infty(\mathbf{R})$, and by the operators U_x, where $x \in \mathbf{R}$, admits a lower semicontinuous trace Tr such that

$$\mathrm{Tr}(\sum_{j=0}^N M_{\varphi_j}U_{x_j}) = \sum_{x_j=0}\int_{-\infty}^{+\infty}\varphi_j(y)\,\mathrm{d}y, \tag{8}$$

where $\varphi_j \in L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$ and $x_j \in \mathbf{R}$. Using the inclusion of C*-algebras $\mathcal{F}\mathcal{L}\mathcal{F}^* \subseteq \mathcal{B}$ given by Equation 7, we get an induced lower semicontinuous trace $\mathrm{tr} : \mathcal{L}^+ \to [0, +\infty]$. If $x \in \mathbf{R}$, set $P_x = M_{e_x} P M_{e_x}^*$. Then it is clear that $P_x - P_y$ belongs to $\mathcal{M}_{\mathrm{tr}}$, for all $x, y \in \mathbf{R}$, and $\mathrm{tr}(P_x - P_y) = y - x$. It follows from the equation $[P, M_{e_x}]M_{e_x}^* = P_0 - P_x$ that $\mathrm{d}(M_{e_x}) = 2[P, M_{e_x}]$ belongs to $\mathcal{M}_{\mathrm{tr}}$. From this it follows that $\Omega = (\mathcal{L}, F, \mathrm{tr})$ is an indicial triple on $\mathrm{AP}(\mathbf{R})$. Let $\omega = \omega_\Omega$.

We now calculate $\omega(\varphi)$, where φ is an invertible element of $\mathrm{AP}(\mathbf{R})$; that is, $\varphi : \mathbf{R} \to \mathbf{C}$ is an almost-periodic function bounded away from zero. A classical result of H. Bohr [2] says that $\varphi = e_x e^\psi$, for some number $x \in \mathbf{R}$ and some function $\psi \in \mathrm{AP}(\mathbf{R})$. Then $\omega(\varphi) = x$ in this case. For, $\omega(\varphi) = \omega(e_x) = \mathrm{tr}(M_{e_x}^*[P, M_{e_x}]) = \mathrm{tr}(P_{-x} - P_0) = x$.

If $\varphi \in \mathrm{AP}(\mathbf{R})$, then it is well known, and easy to check, that the limit

$$L(\varphi) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(t)\, \mathrm{d}t$$

exists and defines a state on $\mathrm{AP}(\mathbf{R})$. Clearly, $L(e_x) = \delta_0^x$, for all x in $\mathbf{R}$. On the other hand, $\mathrm{tr}(\cdot\, \mathrm{d}e_x^*)$ is a bounded linear functional such that, for all $y \in \mathbf{R}$, we have $\mathrm{tr}(e_y \mathrm{d}e_x^*)/2 = \mathrm{tr}(e_y P e_x^* - e_{y-x} P) = \mathrm{Tr}(U_y M_{\chi_{\mathbf{R}^+}} U_{-x} - U_{x-y} M_{\chi_{\mathbf{R}^+}}) = \mathrm{Tr}(M_{\chi_{[y,\infty)}} U_{y-x} - M_{\chi_{[x-y,\infty)}} U_{x-y}) = \delta_y^x \mathrm{Tr}(M_{\chi_{[x,\infty)}} - M_{\chi_{[0,\infty)}}) = -x\delta_y^x$, by Equation 8. Thus, $\mathrm{tr}(e_y \mathrm{d}e_x) = 2L(e_y(xe_x))$, for all $x, y \in \mathbf{R}$. Hence, if ψ and φ are trigonometric polynomials on $\mathbf{R}$, then $\mathrm{tr}(\psi \mathrm{d}\varphi) = -2iL(\psi\varphi')$, where φ' is the derivative of φ. It follows by continuity of $\mathrm{tr}(\cdot\, \mathrm{d}\varphi)$ and $L(\cdot\, \varphi')$ and density of the trigonometric polynomials in $\mathrm{AP}(\mathbf{R})$ that $\mathrm{tr}(\psi \mathrm{d}\varphi) = -2iL(\psi\varphi')$, for all functions ψ in $\mathrm{AP}(\mathbf{R})$. Hence, if φ is bounded away from zero,

$$\omega(\varphi) = \lim_{T\to\infty} \frac{1}{2iT} \int_{-T}^{T} \frac{\varphi'(t)}{\varphi(t)}\, \mathrm{d}t.$$

This classical formula justifies the usual terminology in which $\omega(\varphi)$ is referred to as the *average winding number* of φ in this case.

Our next example shows that the range of a topological index can be an arbitrary subgroup of $\mathbf{R}$.

Example 2.11 Let Γ be a subgroup of $\mathbf{R}$ endowed with the discrete topology. It is well known and easy to see that the group C*-algebra of Γ can be canonically identified with the C*-subalgebra $\mathrm{AP}(\Gamma)$ of $\mathrm{AP}(\mathbf{R})$ generated by all the characters e_x on $\mathbf{R}$ for which x belongs to Γ, where $e_x(t) = e^{ixt}$, for all $t \in \mathbf{R}$, as in the preceding example. We now identify $\mathrm{AP}(\Gamma)$ with the corresponding C*-algebra C of multiplication operators on $L^2(\mathbf{R})$. Let P be the projection of $L^2(\mathbf{R})$ onto $H^2(\mathbf{R})$ and let tr be the restriction of the trace, defined in the preceding example, to the C*-algebra $\mathcal{L}$ on $L^2(\mathbf{R})$ generated by C and P. Then $\Omega = (\mathcal{L}, F, \mathrm{tr})$ is an indicial triple on $\mathrm{AP}(\Gamma)$, where $F = 2P - 1$. As in the preceding example, if $P_x = M_{e_x} P M_{e_x}^*$, then for all $x, y \in \Gamma$, $P_x - P_y$ belongs to $\mathcal{M}_{\mathrm{tr}}$ and $\mathrm{tr}(P_x - P_y) = y - x$.

We are now in a position to compute the topological index ω associated to Ω. If φ is an invertible element of $\mathrm{AP}(\Gamma)$, then an extension, due to van Kampen [11], of the theorem of Bohr mentioned in the preceding example asserts that $\varphi = e_x e^\psi$, for an element $x \in \Gamma$ and some function ψ belonging to $\mathrm{AP}(\Gamma)$. Hence,

$\omega(\varphi) = \omega(e_x) = \mathrm{tr}(M_{e_x}^*[P, M_{e_x}]) = \mathrm{tr}(P_{-x} - P_0) = x$. Thus, the range of ω in this example is equal to Γ.

Thus, in all of these examples the topological index we defined has been seen to represent a significant and natural invariant for the function or C*-algebra element concerned. We turn now to developing appropriate extensions of the Gohberg–Krein index theorem that involve our generalized notion of winding number, the topological index.

3 The generalized Fredholm index and the index theorem

We begin this section with a discussion of generalized Fredholm index theory by recalling some of our earlier work in this direction [7]. Suppose that $\mathcal{A}$ is a unital C*-algebra and let $\mathrm{tr} : \mathcal{A}^+ \to [0, +\infty]$ denote a trace on $\mathcal{A}$. In this setting, one can define a generalized Fredholm index of an element of $\mathcal{A}$ by analogy with the case of the C*-algebra $B(H)$, where H is an infinite-dimensional Hilbert space, with the role of tr played by the usual trace function. The new theory one obtains reduces to the usual one in this case.

We fix then a unital C*-algebra $\mathcal{A}$ and a trace tr on $\mathcal{A}$. As usual, we let $\mathcal{M}_{\mathrm{tr}}$ be the definition ideal of tr. We suppose that tr is not finite; that is, $\mathcal{M}_{\mathrm{tr}} \neq \mathcal{A}$.

An element a of $\mathcal{A}$ is *Fredholm* relative to tr, or tr-*Fredholm*, if there exists an element $b \in \mathcal{A}$ such that $1 - ab$ and $1 - ba$ belong to $\mathcal{M}_{\mathrm{tr}}$. The element b is then a *partial inverse* of a. The *Fredholm index* of a relative to tr, or tr-*index* of a, is defined by setting $\mathrm{index}\,(a) = \mathrm{tr}(ab - ba)$. This is easily seen to be well-defined (even though b is not unique). Observe that although the trace can take on arbitrary complex values on $\mathcal{M}_{\mathrm{tr}}$ (unless it is trivial), this is not the case for the index, which is real-valued.

Let $\mathcal{K}_{\mathrm{tr}}$ denote the closure of $\mathcal{M}_{\mathrm{tr}}$ in $\mathcal{A}$ (we shall always use $\mathcal{K}_{\mathrm{tr}}$ to denote the closure of the definition ideal $\mathcal{M}_{\mathrm{tr}}$); of course, $\mathcal{K}_{\mathrm{tr}}$ is a proper ideal in $\mathcal{A}$. If $a \in \mathcal{A}$, then it is clear that a is tr-Fredholm if, and only if, it is invertible modulo $\mathcal{K}_{\mathrm{tr}}$. Hence, if π is the quotient map from $\mathcal{A}$ to $\mathcal{A}/\mathcal{K}_{\mathrm{tr}}$, the set Φ of tr-Fredholm elements of $\mathcal{A}$ is equal to $\pi^{-1}\mathrm{Inv}\,(\mathcal{A}/\mathcal{K}_{\mathrm{tr}})$. It follows immediately that Φ is open in the norm topology of $\mathcal{A}$ and that it is closed under multiplication.

It is shown in [7] that the map

$$\Phi \to \mathbf{R}, \quad a \mapsto \mathrm{index}\,(a),$$

is locally constant and, for all $a, b \in \Phi$ and $x \in \mathcal{K}_{\mathrm{tr}}$,

$$\mathrm{index}\,(ab) = \mathrm{index}\,(a) + \mathrm{index}\,(b),$$

and

$$\mathrm{index}\,(a + x) = \mathrm{index}\,(a).$$

Thus, the generalized Fredholm index defined here has the right kind of properties one requires of such a function.

Suppose now that $\Omega = (\mathcal{L}, F, \mathrm{tr})$ is an indicial triple associated to a unital C*-algebra C. Let P be the projection $(F + 1)/2$ and, for φ in C, set $T_\varphi = P\varphi P$.

Let $\mathcal{A}$ be the C*-subalgebra of $\mathcal{L}$ generated by the elements T_φ. Clearly, $\mathcal{A}$ is unital, with unit P. We shall regard the T_φ as analogues of Toeplitz operators and shall call them the *Toeplitz elements* associated to Ω. We call $\mathcal{A}$ the *Toeplitz algebra* associated to Ω. The trace tr on $\mathcal{L}$ restricts to a trace $\mathrm{Tr} : \mathcal{A}^+ \to [0, +\infty]$ on $\mathcal{A}$. We shall denote by index_Ω the generalized Fredholm index associated to Tr and call it the index associated to Ω. We refer to Tr-Fredholm elements of $\mathcal{A}$ as Ω-Fredholm elements.

We now derive our index theorem for these Toeplitz elements and shall then apply it to various classes of Toeplitz operators.

Theorem 3.1 *Let $\Omega = (\mathcal{L}, F, \mathrm{tr})$ be an indicial triple for a unital C*-subalgebra C and let $\mathcal{A}$ be the Toeplitz algebra associated to Ω. Then if φ is an invertible element of C, the Toeplitz element T_φ of $\mathcal{A}$ is an Ω-Fredholm element. Moreover, in this case*

$$\mathrm{index}_\Omega(T_\varphi) = -\omega_\Omega(\varphi). \tag{9}$$

Proof. We retain the notation of the preceding paragraphs. Let $\varphi \in \mathrm{Inv}\, C$. First note that $P - T_\varphi T_{\varphi^{-1}} = P(P - \varphi P \varphi^{-1})P = P[P, \varphi]\varphi^{-1}P$. Hence, $P - T_\varphi T_{\varphi^{-1}}$ belongs to the closure $\mathcal{K}_{\mathrm{Tr}}$ of the definition ideal $\mathcal{M}_{\mathrm{Tr}}$ of Tr, since $[P, \psi]$ belongs to $\mathcal{M}_{\mathrm{tr}}$ if $\psi \in \mathrm{Inv}\, C_\Omega$. Replacing φ by φ^{-1}, we see that $P - T_{\varphi^{-1}}T_\varphi$ also belongs to $\mathcal{K}_{\mathrm{Tr}}$, for all $\varphi \in \mathrm{Inv}\, C$. Hence, T_φ is invertible in $\mathcal{A}$ modulo $\mathcal{K}_{\mathrm{Tr}}$ and therefore T_φ is Tr-Fredholm.

Using continuity of the generalized Fredholm index function and of the topological index function, and density of $\mathrm{Inv}\, C_\Omega$ in $\mathrm{Inv}\, C$, to see Equation 9 holds it suffices to show it in the case φ belongs to $\mathrm{Inv}\, C_\Omega$, and we now assume this is the case. Since $[P, \varphi] \in \mathcal{M}_{\mathrm{tr}}$ and $[P, \varphi^{-1}] \in \mathcal{M}_{\mathrm{tr}}$ is this case, the calculations of the preceding paragraph show that $P - T_\varphi T_{\varphi^{-1}}$ and $P - T_{\varphi^{-1}}T_\varphi$ belong to $\mathcal{M}_{\mathrm{Tr}}$. Hence, $T_{\varphi^{-1}}$ is a partial inverse for T_φ and $\mathrm{index}\,(T_\varphi) = \mathrm{Tr}(T_\varphi T_{\varphi^{-1}} - T_{\varphi^{-1}}T_\varphi) = \mathrm{tr}(P\varphi P\varphi^{-1}P - P\varphi^{-1}P\varphi P)$. Now $\varphi P\varphi^{-1}P - \varphi^{-1}P\varphi P = (\varphi P\varphi^{-1} - P)P - (\varphi^{-1}P\varphi - P)P$ belongs to $\mathcal{M}_{\mathrm{tr}}$, since $\varphi P\varphi^{-1} - P = [\varphi, P]\varphi^{-1}$ and $\varphi^{-1}P\varphi - P = [\varphi^{-1}, P]\varphi$ belong to $\mathcal{M}_{\mathrm{tr}}$. Hence, $\mathrm{index}\,(T_\varphi) = \mathrm{tr}(P(\varphi P\varphi^{-1}P - \varphi^{-1}P\varphi P)) = \mathrm{tr}((\varphi P\varphi^{-1}P - \varphi^{-1}P\varphi P)P) = \mathrm{tr}(\varphi P\varphi^{-1}P - \varphi^{-1}P\varphi P) = \mathrm{tr}(\varphi^{-1}[\varphi, P]P) + (\mathrm{tr}([\varphi, P]\varphi^{-1}P) = \mathrm{tr}([\varphi, P]P\varphi^{-1}) + (\mathrm{tr}([\varphi, P]\varphi^{-1}P) = -\mathrm{tr}(P - \varphi P\varphi^{-1}) + \mathrm{tr}(\varphi P\varphi^{-1}P - P\varphi P\varphi^{-1})$. Now $\omega(\varphi) = \mathrm{tr}([P, \varphi]\varphi^{-1}) = \mathrm{tr}(P - \varphi P\varphi^{-1})$ and $\mathrm{tr}(\varphi P\varphi^{-1}P - P\varphi P\varphi^{-1}) = \mathrm{tr}([\varphi P\varphi^{-1} - P, P]) = 0$, since $\varphi P\varphi^{-1} - P$ belongs to $\mathcal{M}_{\mathrm{tr}}$. Hence, $\mathrm{index}\,(T_\varphi) = -\omega(\varphi)$, as required. □

It is natural to consider the question of whether invertibility of the symbol φ is not only sufficient, but also necessary, for T_φ to be Ω-Fredholm. The following result addresses this issue.

Theorem 3.2 *Let $\Omega = (\mathcal{L}, F, \mathrm{tr})$ be an indicial triple for a unital C*-subalgebra C and let $\mathcal{A}$ be the Toeplitz algebra associated to Ω. Let $\mathcal{K}_{\mathrm{Tr}}$ be the norm closure of $\mathcal{M}_{\mathrm{Tr}}$, where* Tr *is the restriction of* tr *to $\mathcal{A}$. The following are equivalent conditions:*

(1) For all $\varphi \in C$, the Toeplitz element T_φ of $\mathcal{A}$ is an Ω-Fredholm element if, and only if, φ is invertible in C.

(2) There is a unique $*$-homomorphism $\pi : \mathcal{A} \to C$ such that $\pi(T_\varphi) = \varphi$, for all $\varphi \in C$. Moreover, $\ker(\pi) = \mathcal{K}_{\mathrm{Tr}}$.

(3) If $\varphi \in C$ and $T_\varphi \in \mathcal{K}_{\mathrm{Tr}}$, then $\varphi = 0$.

(4) If $\varphi \in C$ and $P\varphi \in \mathcal{K}_{\mathrm{tr}}$, then $\varphi = 0$.

Proof. That Condition 3 implies 4 is obvious from the equality $\mathcal{K}_{\mathrm{Tr}} = \mathcal{K}_{\mathrm{tr}} \cap \mathcal{A}$. Conversely, to see that Condition 4 implies 3 first note that $P\varphi - \varphi P$ belongs to the closure $\mathcal{K}_{\mathrm{tr}}$ of $\mathcal{M}_{\mathrm{tr}}$, for all $\varphi \in C$, since $P\psi - \psi P$ belongs to $\mathcal{M}_{\mathrm{tr}}$, for all $\psi \in C_\Omega$ and C_Ω is dense in C. Hence, $P\varphi - P\varphi P = P(P\varphi - \varphi P)$ belongs to $\mathcal{K}_{\mathrm{tr}}$. Therefore, if $T_\varphi \in \mathcal{K}_{\mathrm{Tr}}$, we have $P\varphi P \in \mathcal{K}_{\mathrm{tr}}$ and so $P\varphi \in \mathcal{K}_{\mathrm{tr}}$. Hence, by Condition 4, $\varphi = 0$. Thus, Condition 3 holds, if 4 does.

Now suppose φ and ψ are elements of C_Ω. Since $(1 - P)\psi P = -[P, \psi]P \in \mathcal{M}_{\mathrm{tr}}$, $T_{\varphi\psi} - T_\varphi T_\psi = P\varphi(1 - P)\psi P$ belongs to $\mathcal{M}_{\mathrm{Tr}}$. Hence, by density of C_Ω in C, $T_{\varphi\psi} - T_\varphi T_\psi$ belongs to $\mathcal{K}_{\mathrm{Tr}}$ for all φ and ψ in C. This shows that the $*$-linear map

$$\rho : C \to \mathcal{A}/\mathcal{K}_{\mathrm{Tr}}, \quad \varphi \mapsto T_\varphi + \mathcal{K}_{\mathrm{Tr}},$$

is multiplicative and therefore a $*$-homomorphism. It is clearly surjective, since the elements T_φ generate $\mathcal{A}$. If now Condition 3 holds, then ρ is a $*$-isomorphism. We therefore get a $*$-homomorphism $\pi : \mathcal{A} \to C$ by composing the quotient map from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{K}_{\mathrm{Tr}}$ with ρ^{-1}. Obviously π is the unique $*$-homomorphism for which $\pi(T_\varphi) = \varphi$, for all $\varphi \in C$. Moreover, $\ker(\pi) = \mathcal{K}_{\mathrm{Tr}}$. Thus, we've shown Condition 3 implies 2. That Condition 2 implies 1 is obvious.

Finally, we show Condition 1 implies 3. For, if Condition 1 holds, then $\sigma(\varphi) \subseteq \sigma(T_\varphi + \mathcal{K}_{\mathrm{Tr}})$, for all $\varphi \in C$, where $\sigma(\cdot)$ denotes the spectrum. Therefore, for all φ in C for which $T_\varphi \in \mathcal{K}_{\mathrm{Tr}}$, we have $\sigma(\varphi) = 0$. However, the set of all φ in C for which $T_\varphi \in \mathcal{K}_{\mathrm{Tr}}$ is the closed ideal $\ker(\rho)$ and since, in a C*-algebra, the only closed ideal consisting of quasinilpotent elements is the zero ideal we must have $\ker(\rho) = 0$. Therefore Condition 3 holds. □

We illustrate the theory we have derived by applying it to the case of Toeplitz operators on generalized Hardy spaces. We begin by recalling basic definitions.

Let G be a compact Hausdorff space and A a function algebra on G; that is, A is a norm-closed unital subalgebra of $C(G)$ separating the points of G. If τ is a character on A, then an application of the Hahn–Banach theorem and the Riesz–Kakutani theorem shows that there exists a regular Borel probability measure m on G such that

$$\tau(\varphi) = \int \varphi \, \mathrm{d}m \quad (f \in A).$$

We call m a *representing measure* for τ and denote by $H^2(A, m)$ the closure in $L^2(m)$ of A. This is a *generalized Hardy space.* In the case that m is a unique representing measure for τ and is not a point mass many of the properties of the Hardy space on the circle extend to $H^2(A, m)$. Indeed, the classical case is a special case of this general theory, where $G = \mathbf{T}$, A is the subalgebra of $C(\mathbf{T})$ consisting of the continuous functions on $\mathbf{T}$ that admit an analytic extension to the open unit disc, and m is the Haar measure on $\mathbf{T}$.

However, the uniqueness hypothesis on m is too strong for many of the applications. A very useful theory still exists in the non-unique case provided that A is a *unimodular (function) algebra* on G in the sense that every continuous function on G can be uniformly approximated by functions of the form $\varphi\bar{\theta}$, where φ

and θ belong to A and $|\theta(x)| = 1$, for all $x \in G$ (that is, θ is *unimodular*). For details, see [8].

Returning to the general case, suppose again that m is the representing measure for a character on a function algebra A on G. If $\varphi \in C(G)$, we denote by T_φ the compression to $H^2(A, m)$ of the multiplication operator on $L^2(m)$ associated to φ and we call T_φ the *Toeplitz operator* on $H^2(A, m)$ with *symbol* φ. The C*-algebra $\mathcal{A}$ generated by all the T_φ ($\varphi \in C(G)$) is called the *Toeplitz algebra* associated to the pair (A, m).

Now denote by P the projection of $L^2(m)$ on the space $H^2(A, m)$, and set $F = 2P - 1$. Let $\mathcal{L}$ be the unital C*-algebra on $L^2(m)$ generated by F and $C(G)$, identified with the corresponding C*-algebra of multiplication operators on $L^2(m)$. The map that sends an operator T on $L^2(m)$ to its compression to $H^2(A, m)$ is easily seen to be a $*$-isomorphism from $P\mathcal{L}P$ onto $\mathcal{A}$ and is the unique one sending $P\varphi P$ onto T_φ, for all $\varphi \in C(G)$. Thus, we can, and we will, identify $\mathcal{A}$ with $P\mathcal{L}P$ and the Toeplitz operator T_φ with the corresponding Toeplitz element $P\varphi P$ in $\mathcal{L}$.

Of course, one can also define Toeplitz operators with matrix symbols. Write H^2 for $H^2(A, m)$. If $\varphi = (\varphi_{ij}) \in C(G) \otimes M_N(\mathbf{C})$, we define the Toeplitz operator T_φ on $H^2 \otimes \mathbf{C}^N$ by setting $T_\varphi = \sum_{i,j=1}^N T_{\varphi_{ij}} \otimes e_{ij}$, where the elements e_{ij} constitute the standard set of matrix units of $B(\mathbf{C}^N) = M_N(\mathbf{C})$.

We now apply our index theorem and Theorem 3.2 in a number of special cases to derive some known theorems. As we shall see, in each case we still need to provide some proofs to show that our theorems apply, but we do not have to go back to first principles and can give proofs that are short compared to the original proofs of these results. One should note that our proofs are, in essence, totally different from the original proofs of these results.

Theorem 3.3 *(Gohberg–Krein) Let $\varphi \in M_N(C(\mathbf{T}))$ and let T_φ be the corresponding Toeplitz operator on $H^2 \otimes \mathbf{C}^N$, where H^2 is the Hardy space of the circle. Then T_φ is a Fredholm operator if, and only if, $\det\varphi$ never vanishes on $\mathbf{T}$ and in this case*

$$\operatorname{index}(T_\varphi) = -\operatorname{wn}(\det\varphi),$$

where wn *denotes the winding number around the origin.*

Proof. Let $\Omega = (\mathcal{L}, F, \mathrm{tr})$ be the indicial triple for $C = C(\mathbf{T})$ defined in Example 2.7 and $\Omega_N = (\mathcal{L}_N, F_N, \mathrm{tr}_N)$ be the corresponding indicial triple for $M_N(C)$ constructed in Theorem 2.9. As usual we identify C with the corresponding algebra of multiplication operators on $L^2(\mathbf{T})$. The only operators on $L^2(\mathbf{T})$ commuting with all the elements of C are the multiplication operators (with symbols in $L^\infty(\mathbf{T})$) and since the only multiplication operators commuting with F are the scalars, it follows that $\mathcal{L}$ acts irreducibly on $L^2(\mathbf{T})$. Since $\mathcal{L}$ contains finite-rank operators (for example, $\mathrm{d}z = [F, z]$), it contains the ideal $B_0(L^2(\mathbf{T}))$ of compact operators on $L^2(\mathbf{T})$. Hence, $\mathcal{M}_{\mathrm{tr}}$ is the set of trace-class operators, and its closure is $B_0(L^2(\mathbf{T}))$. If Tr denotes the restriction of tr to the Toeplitz algebra $\mathcal{A} = P\mathcal{L}P$, where P is the Szego projection, then it follows that $\mathcal{M}_{\mathrm{Tr}}$ is norm-dense in $B_0(H^2)$. If an operator belongs to $B_0(H^2)$, then its diagonals converge to zero; hence, if $T_\varphi \in B_0(H^2)$, it must be equal to zero, since its diagonals are clearly constant. It follows from Theorem 3.2

there is a unique $*$-homomorphism $\pi : \mathcal{A} \to C$ such that $\pi(T_\varphi) = \varphi$, for all $\varphi \in C$, and $\ker(\pi) = B_0(H^2)$. Now for $\varphi \in M_N(C)$, T_φ is Fredholm if, and only if, it is invertible modulo $M_N(B_0(H^2))$. Using this, and the inflation homomorphism $\pi_N = \pi \otimes \mathrm{id} : \mathcal{A} \otimes M_N(\mathbf{C}) \to C \otimes M_N(\mathbf{C})$, we see that T_φ is Fredholm if, and only if, φ is invertible in $M_N(C)$. By Theorem 3.1, $\mathrm{index}_{\,\Omega_N}(T_\varphi) = -\omega_{\Omega_N}(\varphi)$. But $\mathrm{index}_{\,\Omega_N}$ is just the classical Fredholm index and $\omega_{\Omega_N}(\varphi) = \omega(\det\varphi) = \mathrm{wn}(\det\varphi)$, by Theorem 2.9 (and our observations in Example 2.7). This proves the theorem. □

The following theorem was proved by the author in [9] in the case of scalar symbols. The extension to matrix symbols here is new. For the terminology "tensor trace" used here, see Example 2.8.

Theorem 3.4 *Let H^2 be the Hardy space of $\mathbf{U}(2)$ and let Tr be a tensor trace on the Toeplitz algebra $\mathcal{A}$ of all Toeplitz operators on H^2. Let Δ be the determinant function on $\mathbf{U}(2)$. If $\varphi \in M_N(C(\mathbf{U}(2)))$ and T_φ is the corresponding Toeplitz operator on $H^2 \otimes \mathbf{C}^N$, then T_φ is a Tr-Fredholm operator if, and only if, $\det\varphi$ never vanishes on $\mathbf{U}(2)$ and in this case*

$$\mathrm{index}\,(T_\varphi) = -n, \text{ where } \det\varphi = \Delta^n e^{\psi},$$

for some integer n and some continuous function ψ on $\mathbf{U}(2)$.

Proof. We retain the notation used in Example 2.8, so that $\Omega = (\mathcal{L}, F, \mathrm{tr})$ is the indicial triple for $C = C(\mathbf{U}(2))$ defined there. We let $\Omega_N = (\mathcal{L}_N, F_N, \mathrm{tr}_N)$ be the corresponding indicial triple for $M_N(C)$ constructed in Theorem 2.9. Our tensor trace Tr is the restriction of tr to $\mathcal{A}$. Since $\mathcal{K}_{\mathrm{tr}} = B_0(L^2(\mathbf{T})) \otimes B$, $\mathcal{K}_{\mathrm{Tr}} = B_0(H^2) \otimes B$. In [1, Theorem 15] it is shown that there is a $*$-homomorphism $\pi : \mathcal{A} \to C$ such that $\pi(T_\varphi) = \varphi$, for all $\varphi \in C$, and $\ker(\pi) = B_0(H^2) \otimes B$. Using this, and the inflation homomorphism $\pi_N = \pi \otimes \mathrm{id} : \mathcal{A} \otimes M_N(\mathbf{C}) \to C \otimes M_N(\mathbf{C})$, we see that T_φ is Tr-Fredholm (more precisely, T_φ is Fredholm relative to the trace on $M_N(\mathcal{A})$ induced by Tr_N, the restriction of tr_N) if, and only if, φ is invertible in $M_N(C)$. By Theorem 3.1, $\mathrm{index}_{\,\Omega_N}(T_\varphi) = -\omega_{\Omega_N}(\varphi)$ in this case. But $\omega_{\Omega_N}(\varphi) = \omega(\det\varphi)$, by Theorem 2.9. On the other hand, we showed in Example 2.8 that if ψ is an invertible element of C, then $\omega(\psi) = n$, where $\psi = \Delta^n e^{\theta}$, for some n in $\mathbf{Z}$ and some function θ in C. This proves the theorem. □

We ascribe the following result to [5], but the form of the result given here is not quite the same as in [5], where the Breuer Fredholm index is used. Similar reasoning to that employed in the proofs of the preceding two theorems enables one to deduce the result from Theorem 3.1.

Theorem 3.5 *(Coburn–Douglas–Schaeffer–Singer) Let $\varphi \in M_N(\mathrm{AP}(\mathbf{R}))$ and let T_φ be the corresponding Toeplitz operator on $H^2(\mathbf{R}) \otimes \mathbf{C}^N$. Then T_φ is a Fredholm operator relative to the trace tr in Example 2.10 if $\det\varphi$ is bounded away from zero and in this case*

$$\mathrm{index}_{\,\mathrm{tr}}(T_\varphi) = -\omega(\det\varphi),$$

where ω denotes the average winding number of $\det\varphi$.

References

[1] C.A. Berger and L.A. Coburn, Wiener–Hopf operators on U_2, *J. Integr. Equ. Oper. Theory* **2** (1979), 139–173.

[2] H. Bohr, Über die Argumentvariation einer fastperiodischen Funktion, *Danske vidensk Selskab.* **10** (1930), 10.

[3] M. Breuer, Fredholm theories in von Neumann algebras I, *Math. Ann.* **178** (1968), 243–254.

[4] M. Breuer, Fredholm theories in von Neumann algebras II, *Math. Ann.* **180** (1969), 313–325.

[5] L.A. Coburn, R.G. Douglas, D. Schaeffer and I.M. Singer, C*-algebras of operators on a half-space II. Index theory, *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 69–79.

[6] A. Connes, Noncommutative Geometry, *Academic Press*, San Diego (1994).

[7] G.J. Murphy, Fredholm index theory and the trace, *Proc. Royal Irish Acad.* **94** (1994), 161–166.

[8] G.J. Murphy, Toeplitz operators associated to unimodular algebras, *J. Integr. Equ. Oper. Theory* **46** (2003), 363–375.

[9] G.J. Murphy, The index theory associated to a non-finite trace on a C*-algebra, *Canadian Math. Bull.* (to appear).

[10] J. Phillips and I. Raeburn, An index theorem for Toeplitz operators with noncommutative symbol space, *J. Funct. Anal.* **120** (1994), 239–263.

[11] E. Van Kampen, On almost periodic functions of constant absolute value, *J. London Math. Soc.* **12** (1937), 3–6.

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