Primitivity Conditions for Full Group C*-Algebras

G.J. Murphy

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Abstract

If Γ is a discrete group, conditions are given that ensure that the full group C*-algebra $C^*(\Gamma)$ is primitive, thereby partially answering a question posed by P. de la Harpe.

1 Introduction

A C*-algebra is *primitive* if it has a faithful irreducible representation. A primitive C*-algebra is necessarily prime; that is, any two non-zero closed ideals in it have a non-zero intersection. It is a well known result of J. Dixmier [8] that if a C*-algebra is separable, then the converse holds; that is, it is primitive if it is prime. For the non-separable case it was a long-standing open problem whether this remains true, but this has been answered negatively recently by N. Weaver [21].

Primitive C*-algebras, and more generally prime C*-algebras, are, to some extent, analogues in the class of C*-algebras of factors in the class of von Neumann algebras; indeed, it is well known (and not hard to see) that a von Neumann algebra is a factor if, and only if, it is prime. In a somewhat vague sense, one thinks of the prime/primitive C*-algebras (and especially the subclass of simple C*-algebras) as the basic "building blocks" of the theory.

If Γ is a discrete group, we consider in this paper the question of when the full group C*-algebra $C^*(\Gamma)$ of Γ is primitive. This question was posed by P. de la Harpe in [6, Problem 13] (the question of the primitivity of the group algebra $\mathbf{C}[\Gamma]$ was raised much earlier by I. Kaplansky; primitivity and primeness of $\mathbf{C}[\Gamma]$ are briefly discussed in Section 2). De la Harpe mentions only one positive result on primitivity/primeness of a full group C*-algebra and as far as I can ascertain this is the only one that has appeared in the published literature. It is due to H. Yoshizawa [22] but was rediscovered independently much later by M.D. Choi [4]. The result asserts that if $\Gamma = \mathbf{F}_n$, where $n \geq 2$, then $C^*(\mathbf{F}_n)$ is primitive. Here $\mathbf{F}_n$ is the free group on a finite or countably infinite number n of generators. (Thus, n is a positive integer or $n = \infty$.) In this paper we considerably extend the class of examples of groups Γ for which one can assert that $C^*(\Gamma)$ is primitive or prime. Indeed, we show there are uncountably many, pairwise non-isomorphic, group C*-algebras for which this result holds. From the preceding remarks it is clear we also consider the question of whether $C^*(\Gamma)$ is prime. Of course, if Γ is countable, then $C^*(\Gamma)$ is separable and therefore primeness and primitivity are equivalent conditions. In light of

Weaver's result mentioned above, it is not clear whether primeness of $C^*(\Gamma)$ still implies its primitivity when Γ is not countable.

It is, of course, easy to give examples where $C^*(\Gamma)$ is not primitive or prime. For instance, if Γ is abelian, then clearly $C^*(\Gamma)$ is prime only if Γ is trivial. More generally, if $C^*(\Gamma)$ is prime, then its centre is trivial, and therefore, the centre of Γ is trivial. However, even if Γ has trivial centre, $C^*(\Gamma)$ may not be prime, for the centre of $C^*(\Gamma)$ may be non-trivial. To see this, let $\Gamma = \mathbf{Z}_2 * \mathbf{Z}_2$, the free product of the two-element group $\mathbf{Z}_2$ with itself. In this case $C^*(\Gamma)$ is generated by a canonical pair of symmetries, that is, unitaries u and v such that $u^2 = v^2 = 1$ and it is easily seen that the element $a = uv + vu$ is a non-scalar central element. Thus, $C^*(\Gamma)$ has non-trivial centre. On the other hand, the free product of two non-trivial groups can never have a non-trivial central element, as is well known.

An easy result below (Corollary 2.2) gives a necessary condition on a group Γ for the algebra $C^*(\Gamma)$ to be primitive. The condition is that Γ be an ICC group (that is, all the conjugacy classes of Γ, except the one consisting of the unit, be infinite sets). A non-abelian free group is an ICC group, as is the free product of two non-trivial groups, if one of them has more than two elements [6, p. 123].

Recall that it is an old result of F.J. Murray and J. von Neumann [16] that the group von Neumann algebra $W^*(\Gamma)$ of Γ is a factor if, and only if, Γ is an ICC group. In view of the analogy relating factors and primitive/prime C*-algebras, this result suggests that $C^*(\Gamma)$ may be prime if Γ is an ICC group. Indeed, we show below that $C^*(\Gamma)$ *is* prime in the special case where Γ is amenable (and an ICC group). However, I personally do not think that all ICC groups have prime/primitive full group C*-algebras (this point is discussed further in Section 2). On the other hand, in the case of the reduced C*-algebra $C^*_r(\Gamma)$, its primeness is indeed equivalent to Γ being an ICC group—see Proposition 2.3.

We give a brief overview of the paper now. In Section 2 we show that if Γ is a (discrete) group, then it is an ICC group if $C^*(\Gamma)$ is prime and that primeness of $C^*_r(\Gamma)$ is equivalent to Γ being an ICC group, as mentioned above. We also show that if $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_1 and Γ_2 are groups for which $C^*(\Gamma_1)$ and $C^*(\Gamma_2)$ primitive, then $C^*(\Gamma)$ is also primitive, provided one, at least, of Γ_1 and Γ_2 is amenable. In Section 3 we show that certain semidirect products of a free group and an amenable group give rise to primitive full group C*-algebras, as does the free product of a non-trivial, countable, free group and a non-trivial, countable, amenable group. Another primitivity result is obtained for the full group C*-algebra of a free product of groups in Theorem 3.4, where a sort of "strong torsion freeness" condition is imposed on one of the factors of the free product. A consequence is that if $\Gamma = F * Z_1 * Z_2$, where F is a countable, non-abelian, free group and Z_1 and Z_2 are countable, torsion-free, amenable groups, then $C^*(\Gamma)$ is primitive.

2 Primitivity and ICC groups

In the sequel the term group *will always signify a discrete group, unless the contrary is indicated in a particular context.*

We begin by setting up some notation. Throughout, Γ will always denote a discrete group. We denote by $U\colon x \mapsto U_x$ the canonical unitary representation of Γ in $C^*(\Gamma)$. The characteristic property that $C^*(\Gamma)$ enjoys is that every other unitary representation of Γ factors through U. More precisely, if $V\colon\Gamma\to B$ is a unitary representation of Γ in a unital C*-algebra B, then there exists a unique $*$-homomorphism π from $C^*(\Gamma)$ to B for which $V=\pi\circ U$.

Let L be the left regular unitary representation of Γ on the Hilbert space $\ell^2(\Gamma)$. Thus, if $(\delta_x)_{x\in\Gamma}$ is the standard orthonormal basis of $\ell^2(\Gamma)$, then L_x is the unitary defined by $L_x(\delta_y)=\delta_{xy}$ $(y\in\Gamma)$. Recall that the C*-algebra generated by the family of operators $(L_x)_{x\in\Gamma}$ is denoted by $C^*_{\mathrm{r}}(\Gamma)$ and called the *reduced group C*-algebra* of Γ. Its weak closure $W^*(\Gamma)$ is the *group von Neumann algebra* of Γ.

It is clear the operators L_x $(x\in\Gamma)$ are linearly independent. Hence, since there exists a $*$-homomorphism π from $C^*(\Gamma)$ onto $C^*_{\mathrm{r}}(\Gamma)$ for which $\pi(U_x)=L_x$, for all $x\in\Gamma$, the unitaries U_x are also linearly independent. Moreover, as is well known, their linear span is dense in $C^*(\Gamma)$.

We shall give below (Theorem 2.4) a necessary and sufficient condition on a group Γ, in the case that Γ is amenable, to ensure that $C^*(\Gamma)$ is prime. However, our necessary condition applies more generally and its proof is standard in this context:

Proposition 2.1 *If Γ is a group for which $C^*(\Gamma)$ has trivial centre, then Γ is an ICC group.*

Proof. Suppose that C is a finite conjugacy class of Γ. To prove our result we have only to show that $C=\{e\}$, where e is the unit of Γ. Set $a=\sum_{x\in C}U_x$. Then $U_yaU_y^*=\sum_{x\in C}U_{yxy^{-1}}=\sum_{z\in yCy^{-1}}U_z=a$, since $yCy^{-1}=C$. Hence, a commutes with all the unitaries U_y, and therefore, a belongs to the centre of $C^*(\Gamma)$, since the unitaries U_y have dense linear span in $C^*(\Gamma)$. Consequently, $a=\lambda U_e$, for some scalar λ. Since the elements U_x are linearly independent, this implies that e belongs to C, so $C=\{e\}$, as required. □

Corollary 2.2 *If $C^*(\Gamma)$ is prime, then Γ is an ICC group.*

Proof. A prime C*-algebra has trivial centre. □

The following result is known, but I know of nowhere in the published literature where it is stated or proved. For this reason I have included a brief proof.

Proposition 2.3 *Let Γ be an arbitrary group. Then $C^*_{\mathrm{r}}(\Gamma)$ is prime if, and only if, Γ is an ICC group.*

Proof. Clearly, the proof of Proposition 2.1 can be adapted easily to show that if $C^*_{\mathrm{r}}(\Gamma)$ is prime, then Γ is an ICC group. Hence, we'll suppose that Γ is an ICC group and deduce that $C^*_{\mathrm{r}}(\Gamma)$ is prime:

Because Γ is an ICC group, $R=W^*(\Gamma)$ is a factor, by the result of Murray and von Neumann in [16] mentioned in the Introduction. Let I and J be closed ideals of $C^*_{\mathrm{r}}(\Gamma)$ for which $I\cap J=IJ=0$. We shall show that either I or J must be zero and this will prove that $C^*_{\mathrm{r}}(\Gamma)$ must be prime.

Let K be the closed vector subspace of $\ell^2(\Gamma)$ spanned by the vectors of the form Tf, where $T\in I$ and $f\in\ell^2(\Gamma)$. Let P be the projection of $\ell^2(\Gamma)$ onto K. Since K is clearly invariant under $C^*_{\mathrm{r}}(\Gamma)$, P commutes with $C^*_{\mathrm{r}}(\Gamma)$ and therefore with R. Also, if $(E_i)_i$ is any approximate unit for I, then $P=\lim_i E_i$ in the strong operator topology. Hence, $P \in R$, since all the elements $E_i \in R$. Thus, P is a central projection of R, implying that P is equal to 0 or 1, since R is a factor. Hence, either $K = 0$, in which case $I = 0$; or $K = \ell^2(\Gamma)$, in which case $J = 0$ (since $J = 0$ on K). □

Theorem 2.4 *Let Γ be an amenable group. Then $C^*(\Gamma)$ is prime if, and only if, Γ is an ICC group.*

Proof. Since Γ is amenable, the Hulanicki–Reiter theorem says that the canonical $*$-homomorphism from $C^*(\Gamma)$ onto $C^*_{\rm r}(\Gamma)$ is an isomorphism [17, Theorem 7.3.9]. Our result now follows immediately from the preceding proposition. □

There exists a vast number of amenable ICC groups. For instance, in [20] an uncountable family of amenable ICC groups is constructed for which the group C*-algebras are pairwise non-isomorphic. For a discussion of these points and other examples of amenable ICC groups, see [6, Section 2.4].

Let $\mathbf{C}[\Gamma]$ denote the group algebra of Γ. This can be identified in an obvious fashion with the linear span of the unitaries U_x in $C^*(\Gamma)$ or of the unitaries L_x in $C^*_{\rm r}(\Gamma)$. A brief discussion on how primitivity/primeness of the group C*-algebras of Γ relate to primitivity/primeness of $\mathbf{C}[\Gamma]$ is pertinent.

First we consider primeness of $\mathbf{C}[\Gamma]$. Recall that a ring R is *prime* if, whenever $a, b \in R$ and $aRb = 0$, then $a = 0$ or $b = 0$. For a C*-algebra, this ring-theoretic formulation of primeness is equivalent to primeness as we defined it in the Introduction. If Γ is an ICC group, then $\mathbf{C}[\Gamma]$ is a prime ring, since $C^*_{\rm r}(\Gamma)$ is prime. On the other hand, primeness of $\mathbf{C}[\Gamma]$ does not imply Γ is an ICC group. For example, an elementary argument shows that $\mathbf{C}[\mathbf{Z}]$ is prime. In fact, a special case of an old result of I.G. Connell [5] asserts that $\mathbf{C}[\Gamma]$ is prime if, and only if, Γ has no non-trivial finite normal subgroups.

The ring $\mathbf{C}[\Gamma]$ is *primitive* if it admits a maximal left ideal containing no non-zero two-sided ideal. If $\mathbf{C}[\Gamma]$ is primitive, it is necessarily prime [9]; the converse does not hold, since $\mathbf{C}[\mathbf{Z}]$ is clearly not primitive, although it is prime. The problem of characterizing primitivity of general group rings $F[\Gamma]$, for F a field, in terms of properties of Γ was raised in 1970 by I. Kaplansky in [12]. The question turned out to be a very hard one and is still open. Indeed, D. Passman has informed me in a private communication that there is not even a viable conjecture, although some partial results are known. For the considerations of this paper one of the most pertinent results is that $\mathbf{C}[\Gamma]$ may be primitive and yet the centre of Γ may be non-trivial [3] (and in this case the centre of $\mathbf{C}[\Gamma]$ will then be non-trivial). It is clear from the proof of Proposition 2.1 and from the statement of Proposition 2.3 that $\mathbf{C}[\Gamma]$ has trivial centre if, and only if, Γ is an ICC group. If follows therefore that primitivity of $\mathbf{C}[\Gamma]$ does not imply primitivity/primeness of $C^*(\Gamma)$ or of $C^*_{\rm r}(\Gamma)$. It seems that the primitivity questions for $C^*(\Gamma)$ and for $\mathbf{C}[\Gamma]$ may be totally unrelated, although I have not an example to show that primitivity of $C^*(\Gamma)$ does not imply primitivity of $\mathbf{C}[\Gamma]$. I suspect such an example exists, however.

Given that, in the case Γ is an amenable group, $C^*(\Gamma)$ is prime if, and only if, Γ is an ICC group, one might be tempted to conjecture that this still holds even if Γ is not amenable. In my view the preceding remarks suggest that this is unlikely to be the case, since the ICC property for Γ is equivalent to a somewhat weak condition for the group algebra $\mathbf{C}[\Gamma]$ (triviality of its centre) and this appears to be a long way from primeness of $C^*(\Gamma)$. However, I know of no example of an ICC group whose full group C*-algebra is not primitive/prime. On the other hand, I suspect that $C^*(\Gamma)$ is not primitive if $\Gamma = \mathbf{F}_2 \times \mathbf{F}_2$, the (ordinary) product of the free group on two generators with itself (Γ is clearly an ICC group, since $\mathbf{F}_2$ is one).

If Γ_1 and Γ_2 are any pair of groups, then the unique $*$-homomorphism from $C^*(\Gamma_1 \times \Gamma_2)$ to the maximal C*-tensor product $C^*(\Gamma_1) \otimes_{\max} C^*(\Gamma_2)$ that sends $U_{(x,y)}$ onto $U_x \otimes U_y$, for all $x \in \Gamma_1$ and $y \in \Gamma_2$, is a $*$-isomorphism, as is well known and easily proved (see Corollary 1 in Section 3 of [10]). We use this in the next result.

Theorem 2.5 *Let $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_1 and Γ_2 are groups for which $C^*(\Gamma_1)$ and $C^*(\Gamma_2)$ are primitive. Then $C^*(\Gamma)$ is also primitive, if (at least) one of Γ_1 and Γ_2 is amenable.*

Proof. It is a well known result of E.C. Lance [13] that a discrete group is amenable if, and only if, its full group C*-algebra is nuclear. The hypothesis of the theorem, combined with the observation preceding it, implies that $C^*(\Gamma) = C^*(\Gamma_1) \otimes_{\max} C^*(\Gamma_2) = C^*(\Gamma_1) \otimes C^*(\Gamma_2)$, where the second tensor product is the spatial one. Choose now faithful irreducible representations π_1 and π_2 of $C^*(\Gamma_1)$ and $C^*(\Gamma_2)$ on Hilbert spaces H_1 and H_2 respectively. Then their tensor product $\pi = \pi_1 \otimes \pi_2$ is a faithful representation of $C^*(\Gamma)$ on $H = H_1 \otimes H_2$. Faithfulness of π is a consequence of the fact that we are dealing with a spatial tensor product [14, Theorem 6.5.1]. Irreducibility of π is a well known result, see Section 2 of [10]. Thus, $C^*(\Gamma)$ is primitive, as required. □

3 C*-algebras of semidirect products and free products of groups

Let (A, Γ, α) be a C*-dynamical system; thus, A is a C*-algebra, Γ—as always—is a (discrete) group and α is a homomorphism from Γ into the group $\mathrm{Aut}\, A$ of all automorphisms of A; α is called an *action* of Γ on A. Let ϕ be a non-zero representation of A on a Hilbert space H. We recall the well-known result that the induced representation $\bar{\phi}$ of the covariance algebra $C^*(A, \Gamma, \alpha)$ on the Hilbert space $l^2(\Gamma, H)$ is irreducible if, and only if, the following two conditions hold:

(1) ϕ is irreducible;

(2) if $x \in \Gamma$ and there exists a unitary operator U on H such that $\phi(\alpha_x(a)) = U\phi(a)U^*$, for all $a \in A$, then $x = e$, where e is the unit of Γ.

Moreover, if Γ is amenable, then $\bar{\phi}$ is faithful if ϕ is faithful.

This material is standard (see [15] and [17], for instance).

We use these remarks in the proof of the following lemma. Before stating the lemma, recall that the action α on A is said to be *effective* if $\alpha_x = \mathrm{id}$ (the identity map on A) only for $x = e$. A similar definition applies for actions of Γ on groups and we shall use the term in this context also.

Lemma 3.1 *Let (A, Γ, α) be a C*-dynamical system for which Γ is an amenable (discrete) group and the action α is effective. Suppose that A is generated as a C*-algebra by a subset S such that $\alpha_x(S) = S$, for all $x \in \Gamma$. Suppose also that*

A admits a non-zero, faithful, irreducible representation ϕ on a Hilbert space H and that for all $a, a' \in S$, the operators $\phi(a)$ and $\phi(a')$ are unitarily equivalent on H only if $a = a'$. Then the covariance algebra $C^*(A, \Gamma, \alpha)$ is primitive.

Proof. Since Γ is amenable, the induced representation $\bar{\phi}$ on $C^*(A, \Gamma, \alpha)$ is faithful, by the remarks preceding this lemma. Suppose that $x \in \Gamma$ and that there is a unitary U on H for which $\phi(\alpha_x(a)) = U\phi(a)U^*$, for all $a \in A$. Then, if a belongs to S, so does $\alpha_x(a)$, and the operators $\phi(\alpha_x(a))$ and $\phi(a)$ are unitarily equivalent. By hypothesis this can only happen if $\alpha_x(a) = a$. Thus, $\alpha_x = \mathrm{id}$ on S and therefore, $\alpha_x = \mathrm{id}$ on A. Hence, $x = e$, since α is effective. Again invoking the remarks preceding this lemma, we conclude that $\bar{\phi}$ is an irreducible representation. It follows that $C^*(A, \Gamma, \alpha)$ is primitive.□

The proof of the following theorem makes extensive use of results from [4]. The countability assumption on the free group appearing in the theorem is needed to use these results.

Theorem 3.2 *Suppose that $\Gamma = F \times_\beta Z$, the semidirect product of a countable, non-abelian, free group F by an amenable group Z, via an effective action β. Suppose also that F admits a free basis X for which $\beta_z(X) = X$, for all $z \in Z$. Then $C^*(\Gamma)$ is primitive.*

Proof. The action β induces an obvious action α of Z on $C = C^*(F)$ in such a way that $\alpha_z(S) = S$, for all $z \in Z$, where $S = \{U_j\}_j$ is the set of canonical unitary generators of $C^*(F)$. The index j runs over a finite or infinite sequence of successive integers, starting at 1. Moreover, α is clearly effective and, by Proposition 6.6 of [18], we have $C^*(\Gamma) = C \times_\alpha Z$. Hence, by the preceding lemma, to prove the result we need only show that C admits a faithful, irreducible representation ϕ for which the operators $\phi(U_j)$ are pairwise non-equivalent.

We may suppose that C is a C*-subalgebra of $B(H)$, for some separable Hilbert space H (we need F countable to ensure H can be taken to be separable). Moreover, by a slight adaptation of the Weyl–von Neumann–Berg theorem for normal operators (see [4, Lemma 4]), we may write every unitary operator on H as a compact perturbation of a unitary operator which is diagonal relative to some orthonormal basis of H and has all diagonal entries distinct. Let us apply this to the unitaries U_j, for each index $j > 1$, and denote by V_j the corresponding unitary diagonal operator, with $U_j - V_j$ a compact operator (we are *not* supposing all the V_j are diagonal relative to the same basis). By Lemma 3 combined with Lemma 5 of [4], we can find a compact perturbation V_1 of U_1 that is a unitary operator having no common non-trivial invariant subspace with V_2. It follows that the C*-algebra B generated by all of the operators V_j $(j \geq 1)$ acts irreducibly on H.

Since V_2 has non-empty point spectrum $\sigma_p(V_2)$, there exists $\lambda \in \mathbf{T}$ (the unit circle of $\mathbf{C}$) such that λV_2 and V_1 are not unitarily equivalent. For otherwise, we would have $\lambda\sigma_p(V_2) = \sigma_p(V_1)$, for all $\lambda \in \mathbf{T}$, and therefore, $\mathbf{T} \subseteq \sigma_p(V_1)$. However, this is clearly impossible, since the point spectrum of a unitary operator on a separable Hilbert space must be countable. Set $V_1' = V_1$ and $V_2' = \lambda V_2$, where $\lambda \in \mathbf{T}$ is chosen so that V_1' and V_2' are inequivalent. We may repeat this kind of argument, to find a suitable scalar λ of modulus one such that $V_3' = \lambda V_3$ is inequivalent to both V_1' and V_2'. Proceeding in this fashion we obtain for each

V_j a unitary V_j', which is a multiple of V_j by a unit-modulus scalar, such that the set of unitaries V_j' are all pairwise inequivalent.

By the universal property of C, there exists a $*$-homomorphism ψ from C onto B mapping each canonical generator U_j onto V_j. Let π be the quotient homomorphism from $B(H)$ onto the Calkin algebra $B(H)/K(H)$ and let π_C and π_B denote the restrictions of π to C and B, respectively. Then $\pi_C(U_j) = \pi_B(V_j) = \pi_B\psi(U_j)$, since V_j is just a compact perturbation of U_j. Hence, $\pi_C = \pi_B\psi$. Moreover, $\ker(\pi_C) = K(H) \cap C$ and this intersection is equal to zero by [4, Corollary 2]. Hence, π_C is injective and therefore, so is ψ. Now, applying the universal property to B and the unitaries V_j in place of C and U_j, there is clearly a $*$-isomorphism ψ' of B sending each operator V_j onto V_j'. Finally, the representation $\phi = \psi'\psi$ of C is faithful and irreducible, and the operators $\phi(U_j) = V_j'$ are pairwise inequivalent, as required.□

If F and Z are groups, we denote their free product by $F * Z$, as is customary.

Theorem 3.3 *Let F be a non-trivial, countable, free group and Z a non-trivial, countable, amenable group. Then $C^*(F * Z)$ is primitive.*

Proof. Let the non-empty set S be a free basis for F. Let $F_{Z\times S}$ be the free group on $Z \times S$, where Z is considered merely as a set, so that $Z \times S$ is a free basis of $F_{Z\times S}$. Define an action β of Z on $F_{Z\times S}$ by setting $\beta_z(t, s) = (zt, s)$, for all $z, t \in Z$ and $s \in S$. Clearly β is effective. Let $\Gamma = F_{Z\times S} \times_\beta Z$. We shall show that Γ is isomorphic to $F * Z$. An application of Theorem 3.2 will then suffice to show the present theorem, since Z is amenable and $F_{Z\times S}$ has countable rank greater than one (because Z and S are countable and Z is non-trivial).

We show now that the required isomorphism exists: Let $V: z \mapsto V_z$ be the canonical homomorphism from Z into Γ. Thus, if $z, t \in Z$ and $s \in S$, then $\beta_z(t, s) = V_z(t, s)V_z^{-1}$. By the universal property of the free product, there exists a homomorphism α from $F*Z$ to Γ such that $\alpha(s) = (e, s)$ and $\alpha(z) = V_z$, for all $s \in S$ and $z \in Z$. Moreover, since $(z, s) = \beta_z(e, s) = V_z(e, s)V_z^{-1}$, it follows that α is surjective. To see that α is injective, we construct a left inverse α' for it. First, let γ be the homomorphism from $F_{Z\times S}$ to $F * Z$ for which $\gamma(z, s) = zsz^{-1}$, for all $z \in Z$ and $s \in S$. Since $\gamma\beta_z(t, s) = zts(zt)^{-1} = z\gamma(t, s)z^{-1}$, for all $z, t \in Z$ and $s \in S$, it follows from the universal property of the semidirect product $F_{Z\times S} \times_\beta Z$ that there exists a homomorphism α' from Γ to $F * Z$ such that α' extends γ and $\alpha'(V_z) = z$, for all $z \in Z$. From this one easily sees that $\alpha'\alpha = \mathrm{id}$, as required. Hence α is an isomorphism.□

In the preceding theorem we are allowed to suppose that F is of rank one, that is, we can take $F = \mathbf{Z}$, but we need to have Z amenable (for the proof to work). We now show, in the following theorem, that if we suppose that the rank of F is greater than one, then we may drop the amenability requirement on Z. However, we have to replace it by another assumption, namely, non-existence of non-trivial projections in $C^*(Z)$.

The question under what conditions on a group Z the C*-algebra $A = C^*(Z)$, or $A = C^*_{\mathrm{r}}(Z)$, has no non-trivial projections is an important open one in the theory of operator algebras. If Z admits an element of finite order, not equal to the unit, then spectral theory applied to the corresponding unitary in A implies that A admits a non-trivial projection.

If Z is amenable and torsion free, then $C^*(Z)$ has no non-trivial projections. This is also the case the case if Z is a free product of two torsion-free amenable groups. See [7] and [11] for these results.

The famous Kadison–Kaplansky conjecture is that $C^*_{\mathrm{r}}(Z)$ has no non-trivial projections if Z is an arbitrary torsion-free group. If Z is a torsion-free group for which the even-more-famous Baum–Connes conjecture holds, then the Kadison–Kaplansky conjecture also holds for Z. We refer to [1] and [19] for surveys on these questions.

Many of the ideas of the following proof occur in Choi's paper [4] and, indeed, in the proof above of Theorem 3.2. However, we give full details for the sake of completeness.

Theorem 3.4 *Let $\Gamma = F * Z$, where F is a countable, non-abelian free group and Z is a countable group for which $C^*(Z)$ admits no non-trivial projection. Then $C^*(\Gamma)$ is primitive.*

Proof. First, observe that $C^*(\Gamma) = C^*(F) * C^*(Z)$ (the unital free product of C*-algebras [2, pp. 105–6]). We shall show the more general result that for any separable unital C*-algebra A not admitting non-trivial projections, the free product $C = C^*(F) * A$ is primitive.

We first show that C admits no non-trivial projections: Represent C faithfully as a C*-subalgebra of $B(H)$, for some separable Hilbert space H. Define B to be the C*-subalgebra of $C([0,1], B(H))$ consisting of elements f for which $f(0) \in A$. If p is a projection in B, then $p(0)$ is a projection in A and therefore $p(0) = 0$ or $p(0) = 1$. Hence, to show that $p = 0$ or $p = 1$, we may suppose that $p(0) = 0$. Since the map $t \mapsto \|p(t)\|$ is continuous and integer-valued, and its domain $[0,1]$ is connected, it is constant. Therefore, $\|p(t)\| = \|p(0)\| = 0$ for all t; that is, $p = 0$. Thus, B has no non-trivial projections. Now let the unitaries U_j be the canonical generators of $C^*(F)$ and write $U_j = \exp(iT_j)$, where T_j are self-adjoint operators on H. The functions f_j defined by $f_j(t) = \exp(itT_j)$ clearly belong to B and are unitaries, so there exists a $*$-homomorphism ρ_1 from $C^*(F)$ into B such that $\rho_1(U_j) = f_j$, for all j. Also, if $a \in A$, we get an element g_a in B by setting $g_a(t) = a$, for all t. The map ρ_2 from A to B that sends a to g_a is clearly a $*$-homomorphism. By the universal property of the free product, there exists a $*$-homomorphism ρ from C to B extending both ρ_1 and ρ_2. The $*$-homomorphism from B to $B(H)$ obtained by sending f onto $f(1)$ is clearly a left inverse for ρ, so ρ is injective. Hence, since B has no non-trivial projections, neither has C. It follows by spectral theory that $C \cap K(H) = 0$.

Using the same ideas we used in the proof of Theorem 3.2, we can find unitaries V_j on H such that $U_j - V_j \in K(H)$, for all j and only the trivial subspaces of H are common invariant closed vector subspaces for the V_j. Now, again using the universal property of the free product, we see that there exists a $*$-homomorphism ψ from C to $B(H)$ such that $\psi(U_j) = V_j$, for all j and such that $\psi(a) = a$, for all $a \in A$. Let π be the quotient map from $B(H)$ onto $B(H)/K(H)$ and π_C its restriction to C. Then $\pi_C = \pi\psi$, since $\pi(U_j) = \pi(V_j)$, for all j. Also, $\ker(\pi_C) = C \cap K(H) = 0$, so π_C is injective. Hence, ψ is injective. Since $\psi(C)$ acts irreducibly on H (as the V_j have no non-trivial common invariant subspace), the representation ψ of C is a faithful, irreducible one and therefore C is primitive, as required. □

Corollary 3.5 *Let $\Gamma = F * Z_1 * Z_2$, where F is a countable, non-abelian, free group and Z_1 and Z_2 are countable, torsion-free, amenable groups. Then $C^*(\Gamma)$ is primitive.*

Proof. This follows immediately from the theorem and the remarks that precede it. □

References

[1] P. Baum and A. Connes, *K-Theory for discrete groups*, Operator Algebras and Application, Vol 1, London Math. Soc. Lecture Note Series **135**, Cambridge Univ. Press (1988), 1–20.

[2] B. Blackadar, *K-Theory for Operator Algebras*, Springer, New York, 1986.

[3] K.A. Brown, *Primitive group rings of soluble groups*, Arch. Math. (Basel) **36** (1981), 404–413.

[4] M-D. Choi, *The full C*-algebra of the free group on two generators*, Pacific J. Math. **87** (1980), 41–48.

[5] I.G. Connell, *On the group ring*, Canadian J. Math **15** (1963), 650–685.

[6] P. de la Harpe, *Operator algebras, free groups and other groups*, Astérisque **232** (1995), 121–153.

[7] P. de la Harpe and K.J. Dykema, *Some groups whose reduced C*-algebras have stable rank one*, J. Math. Pures Appl. **78** (1999), 591–608.

[8] J. Dixmier, *Sur les C*-algèbras*, Bull. Soc. Math. France **88** (1960), 95–112.

[9] E. Formanek and R.L. Snider, *Primitive group rings*, Proc. Amer. Math. Soc. **36** (1972), 357–360.

[10] A. Guichardet, *Tensor Products of C*-algebras, Part I*, Aarhus Universitet Lect. Note Ser. **1**2 (1969).

[11] R. Ji and S. Pedersen, *Certain full group C*-algebras without proper projections*, J. Oper. Theory **24** (1990), 239–252.

[12] I. Kaplansky, *"Problems in the theory of rings" revisited*, Amer. Math. Monthly **77** (1970), 445–454.

[13] E.C. Lance, *On nuclear C*-algebras*, J. Funct. Anal. **12** (1973), 157–176.

[14] G.J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, San Diego–London, 1990.

[15] G.J. Murphy, *Crossed products of C*-algebras by semigroups of automorphisms*, Proc. London Math. Soc. **68** (1994), 423–448.

[16] F.J. Murray and J. von Neumann, *On rings of operators IV*, Ann. of Math. **44** (1943), 716–808.

[17] G.K. Pedersen, *C*-Algebras and their Automorphism Groups*, Academic Press, London, 1979.

[18] J. Tomiyama, *The Interplay between Topological Dynamics and Theory of C*-Algebras*, University Press, National University Seoul, 1992.

[19] A. Valette, *The conjecture of idempotents: a survey of the C*-algebraic approach*, Bull. Soc. Math. Belg. **41** (1989), 485–521.

[20] Y. Watatani, *The character groups of amenable group C*-algebras*, Math. Japonica **24** (1979), 141–144.

[21] N. Weaver, *A prime C*-algebra that is not primitive*, Preprint, Washington University, St Louis, USA, 2001 (arXiv:math.OA/0106252 v2).

[22] H. Yoshizawa, *Some remarks on unitary representations of the free group*, Osaka Math. J. **3** (1951), 55–63.

Department of Mathematics
National University of Ireland, Cork
Western Road
Cork, Ireland
Email: gjm@ucc.ie