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Integral Equations and Operator Theory

The C*-Algebra of a Function Algebra

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Abstract. We associate to each function algebra a C*-algebra and investigate its properties. We are particularly interested in those of its properties that are important for the Toeplitz operator theory on Hardy spaces of representing measures of the function algebra.

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1. Introduction

Let G be a compact, Hausdorff space and $C(G)$ the C*-algebra of all continuous, complex-valued functions on G. Let A be a function algebra on G; that is, A is a norm-closed subalgebra of $C(G)$ containing the constants and separating the points of G. A *Toeplitz representation* of the pair (A, G) in a unital C*-algebra B is a unital, norm-decreasing, linear map $T\colon \varphi \mapsto T_\varphi$ from $C(G)$ to B that preserves adjoints and satisfies the multiplicative condition $T_\varphi T_\psi = T_{\varphi\psi}$, for all $\varphi \in C(G)$ and $\psi \in A$.

This definition is, of course, motivated by the classical theory of Toeplitz operators on the Hardy space H^2 on the unit circle $\mathbf{T}$ in $\mathbf{C}$. For, if A is the *disc* algebra on $\mathbf{T}$; that is, the closed unital subalgebra of $C(\mathbf{T})$ generated by the inclusion function $z\colon \mathbf{T} \to \mathbf{C}$, then the map from $C(\mathbf{T})$ to $B(H^2)$ that assigns to a continuous function φ its corresponding Toeplitz operator T_φ is a Toeplitz representation of $(A, \mathbf{T})$.

This example can be considerably generalized. Suppose again that G is a compact, Hausdorff space and that A is a function algebra on G. Let m be a regular, Borel probability measure on G. We call the Hilbert subspace $H^2(A, m)$ of $L^2(G, m)$, obtained by taking the norm closure of A in $L^2(G, m)$, the *Hardy space* of the pair (A, m). If $\varphi \in L^\infty(G, m)$, define the corresponding operator T_φ in $B(H^2(A, m))$ to be the compression to $H^2(A, m)$ of the multiplication operator on $L^2(G, m)$ associated to φ. We call T_φ a *generalised Toeplitz operator* with *symbol* φ. One easily verifies that the map, $\varphi \mapsto T_\varphi$, is a Toeplitz representation of (A, G).

To get a Hardy space theory analogous to the classical theory on the circle it suffices to impose two conditions on the pair (A, m), namely that m is the unique representing measure for a character of A and that m is not a point mass. In this situation not only does the Hardy space theory extend in a satisfactory manner, but the theory for the corresponding Toeplitz operators also extends very nicely. For details the reader is referred to [2, 6, 10].

The concept of a representing measure is important for the considerations of this paper, so we recall its definition: If τ is a norm-bounded linear functional on A, where A is a function algebra on a compact, Hausdorff space G, a *representing measure* for τ is a regular, Borel complex measure m on G for which $\tau(\varphi) = \int \varphi\, dm$, for all $\varphi \in A$ and for which $\|\tau\| = \|m\|$. Such a measure always exists, by the Hahn–Banach theorem combined with the Riesz–Kakutani theorem. If τ is a character, that is, a unital multiplicative linear functional, all its representing measures are necessarily probability measures.

Given an ordered (abelian) group Γ, there is associated a corresponding function algebra $A(\Gamma)$ on its compact Pontryagin dual group $G = \hat{\Gamma}$ (details are given in Section 2). The normalized Haar measure m on G is a representing measure for $A(\Gamma)$ and the Hardy space associated to the pair $(A(\Gamma), m)$ admits a Toeplitz operator theory particularly amenable to analysis. A fruitful approach in this case is provided by the use of a certain C*-algebra $C^*(\Gamma^+)$ associated to the positive cone Γ^+ of the ordered group Γ, see [8]. This C*-algebra facilitates the study of the corresponding Toeplitz operators and is vital in the theory developed in [8] and [11].

Motivated by this example, and many others in the literature where C*-algebras have assisted the study of generalized Toeplitz operators, we associate to each proper function algebra A on G a certain C*-algebra $C^*(A, G)$. We expect this algebra to be useful in the analysis of generalised Toeplitz operators on Hardy spaces of representing measures of characters of A. In the case of an ordered group Γ we have $C^*(A(\Gamma), \hat{\Gamma}) = C^*(\Gamma^+)$. In the case of an arbitrary proper function algebra A on G, the algebra $C^*(A, G)$ has the key property that it admits a universal Toeplitz representation of (A, G) and that every Toeplitz representation of (A, G) factors through this universal one via a $*$-homomorphism on $C^*(A, G)$. Thus, this algebra encodes the Toeplitz representation theory. In certain cases we can use it to show that all the possible non-trivial Toeplitz operator theories on various Hardy spaces of representing measures on A are the same—at least in so far as the C*-algebra generated by the Toeplitz operators (with continuous symbols) is always canonically isomorphic to $C^*(A, G)$. This is the situation when $C^*(A, G)$ has a property we call the *injective homomorphisms property.* In this case, if a Toeplitz representation of (A, G) is not multiplicative, the induced $*$-homomorphism on $C^*(A, G)$ is injective.

Apart from its applicability in Toeplitz operator theory, $C^*(A, G)$ has much interest in its own right as well. For instance, its closed commutator ideal is simple in the case that $C^*(A, G)$ has the injective homomorphisms property. It is always desirable to get new examples of simple C*-algebras and it may be that these ideals provide such new examples.

We describe now how the paper is organized. In Section 2 we construct the C*-algebra $C^*(A, G)$ of a function algebra A acting on a compact space G and investigate some of its principal properties. The most important result of this section is Theorem 2.10, which gives an internal characterization of when $C^*(A, G)$ has the injective homomorphisms property. If $C^*(A, G)$ has this property, then $C^*(A, G)$ has trivial centre. In Section 3 (Theorem 3.4) we show this in turn implies that A is antisymmetric. The converse is true—that is, $C^*(A, G)$ has trivial centre if A is antisymmetric—in the special case that A is a unimodular algebra (Theorem 3.8). In showing this we establish along the way a number of other useful properties of the algebra $C^*(A, G)$ and of the canonical generating elements V_φ, many of which (including some results of Section 2) show that the V_φ are a form of abstract Toeplitz operator. In Section 4 we show some results on Toeplitz theory on generalized Hardy spaces that are interesting in their own right and that allow us to obtain other nice results on the algebra $C^*(A, G)$. However, the most important result of this section is the last one: In Theorem 4.6 we show that if $C^*(A, G)$ has the injective homomorphisms property, then A is a maximal function algebra on G. We conclude by conjecturing that if A is both maximal and antisymmetric, then $C^*(A, G)$ has the injective homomorphisms property. This is a project for future research. If it is true, it will be an extremely nice result; however, I suspect it will be difficult to show.

2. The C*-algebra $C^*(A, G)$

To avoid continual repetition we shall make the following convention:

In the sequel G will always denote a compact, Hausdorff space and A will always denote a function algebra on G. Moreover, to avoid trivialities, we shall always assume that A is proper; that is, $A \neq C(G)$.

Our aim now is to associate to A and G a C*-algebra $C^*(A, G)$ that encodes much of the theory of the Toeplitz representations of (A, G). First, we need some useful observations.

Let T be a Toeplitz representation of (A, G) in a unital C*-algebra B. Then T is necessarily positive. For if φ is a positive element in $C(G)$ and $t = \|\varphi\|_\infty$, then $\|T_\varphi - t\| \leq \|\varphi - t\|_\infty \leq t$. Hence, since T_φ is self-adjoint, these inequalities imply that T_φ is positive, by Lemma 2.2.2 of [9]. Since $C(G)$ is, of course, a unital commutative C*-algebra, a well known theorem of Stinespring [3, p. 192] tells us that positivity of the map $\varphi \mapsto T_\varphi$ implies its complete positivity. Hence, if $\psi \in C(G)$ is such that $T_\psi^* T_\psi = T_{\bar{\psi}\psi}$, then $T_\varphi T_\psi = T_{\varphi\psi}$, for all $\varphi \in C(G)$ [14, p. 5].

Theorem 2.1. *There exists a unital C*-algebra $C^*(A, G)$ and a Toeplitz representation V of (A, G) in $C^*(A, G)$ whose range generates $C^*(A, G)$, with the property* *that if T is any other Toeplitz representation of (A, G) in a unital C*-algebra B, then there exists a unique $*$-homomorphism π from $C^*(A, G)$ to B for which*

$$\pi(V_\varphi) = T_\varphi \quad (\varphi \in C(G)).$$

Moreover, $C^*(A, G)$ is unique up to canonical $*$-isomorphism.

Proof. Uniqueness of $C^*(A, G)$ is obvious, so we confine ourselves to proving its existence. First we construct $C^*(A, G)$ and V and then we shall show that they have the requisite properties. Let Z be the unital C*-algebra free product of $C(G)$ and $\mathbf{C}^2$, that is, $Z = C(G) * \mathbf{C}^2$. If $p = (1, 0) \in \mathbf{C}^2$, let I be the closed ideal in Z generated by all elements of the form $p\varphi p\psi p - p\varphi\psi p$, where $\varphi \in C(G)$ and $\psi \in A$. Let ρ be the quotient map from Z onto Z/I. Set $C^*(A, G) = \rho(p)\rho(Z)\rho(p)$ and, for $\varphi \in C(G)$, set $V_\varphi = \rho(p)\rho(\varphi)\rho(p)$. Clearly, $C^*(A, G)$ is a unital C*-algebra and it is readily verified that V is a Toeplitz representation of (A, G) in $C^*(A, G)$ and that $C^*(A, G)$ is generated by the elements V_φ $(\varphi \in C(G))$.

Suppose now that T is another Toeplitz representation of (A, G) in a unital C*-algebra B. Since T is completely positive, as we saw in the remarks preceding this theorem, we may suppose that B is a unital C*-subalgebra of $B(H)$, for some Hilbert space H, and apply Stinespring's dilation theorem [3, p. 195] to deduce the existence of a Hilbert space K containing H as a closed linear subspace and the existence of a $*$-homomorphism α from $C(G)$ to $B(K)$ such that, for each function φ belonging to $C(G)$, the operator T_φ is the compression to H of the operator $\alpha(\varphi)$ on K. Now let β be the unital $*$-homomorphism from Z to $B(K)$ that extends α and that maps $\mathbf{C}^2$ to $B(K)$ by mapping p onto the projection P on K with range H. If φ and ψ are functions in $C(G)$ and A respectively, then the equation $T_\varphi T_\psi = T_{\varphi\psi}$ implies that

$$\beta(p\varphi p\psi p - p\varphi\psi p) = P\beta(\varphi)P\beta(\psi)P - P\beta(\varphi\psi)P = 0.$$

It follows that β vanishes on the ideal I and therefore that we have an induced $*$-homomorphism γ from $C^*(A, G)$ to $B(K)$. Since $\gamma(1) = P$, and so $\gamma(C^*(A, G)) \subseteq PB(K)P$, we get a $*$-homomorphism π from $C^*(A, G)$ by composing γ with the compression map from $B(K)$ onto $B(H)$. Since $\pi(V_\varphi)$ is the compression of $\beta(\varphi)$ to H, we have $\pi(V_\varphi) = T_\varphi$, for all $\varphi \in C(G)$, and, since the elements V_φ generate $C^*(A, G)$, we have $\pi(C^*(A, G)) \subseteq B$. Thus, $C^*(A, G)$ and V have the properties indicated in the statement of the theorem and π is the $*$-homomorphism induced by T that is stated to exist. □

We call $C^*(A, G)$ the *Toeplitz* C*-algebra associated to the pair (A, G). The elements V_φ $(\varphi \in C(G))$ have some properties similar to those enjoyed by Toeplitz operators and are representable as generalized Toeplitz operators, as we'll see below.

The identity automorphism on $C(G)$ is a Toeplitz representation of (A, G) and therefore induces a $*$-homomorphism π from $C^*(A, G)$ onto $C(G)$. The existence of π implies that $\|V_\varphi\| = \|\varphi\|_\infty$, for all $\varphi \in C(G)$, since $\|\varphi\|_\infty = \|\pi(V_\varphi)\| \leq \|V_\varphi\| \leq \|\varphi\|_\infty$.

Theorem 2.2. *The kernel of the canonical map π from $C^*(A,G)$ onto $C(G)$ is the closed commutator ideal $K^*(A,G)$ of $C^*(A,G)$.*

Proof. We show first that $V_\varphi V_\psi - V_{\varphi\psi}$ belongs to $K^*(A,G)$, for all $\varphi, \psi \in C(G)$. To see this let B be the closed linear subspace of $C(G)$ consisting of all elements ψ for which $V_\varphi V_\psi - V_{\varphi\psi} \in K^*(A,G)$, for all $\varphi \in C(G)$. We shall show that B is a self-adjoint subalgebra of $C(G)$. First, B is self-adjoint: For if $\psi \in B$, then $V_\varphi V_{\bar\psi} - V_{\varphi\bar\psi} = (V_\psi V_{\bar\varphi} - V_{\bar\varphi} V_\psi)^* + (V_{\bar\varphi} V_\psi - V_{\bar\varphi\psi})^*$. The first bracketed term is a commutator and therefore belongs to $K^*(A,G)$ by definition and the second bracketed term belongs to $K^*(A,G)$ since $K^*(A,G)$ is self-adjoint. Hence, $\bar\psi \in B$, so B is self-adjoint. Second, B is a subalgebra: Let ψ_1 and ψ_2 be elements of B. Then $V_\varphi V_{\psi_1\psi_2} - V_{\varphi\psi_1\psi_2} = V_\varphi(V_{\psi_1\psi_2} - V_{\psi_1}V_{\psi_2}) + (V_\varphi V_{\psi_1} - V_{\varphi\psi_1})V_{\psi_2} + (V_{\varphi\psi_1}V_{\psi_2} - V_{\varphi\psi_1\psi_2})$. The first and third term belong to $K^*(A,G)$ because $\psi_2 \in B$, and the second because $\psi_1 \in B$. Hence, $\psi_1\psi_2 \in B$. Thus, B is a self-adjoint subalgebra of $C(G)$, as claimed. Since B clearly contains A, it separates the points of G, and therefore it follows from the Stone–Weierstrass theorem that $B = C(G)$. Hence, $V_\varphi V_\psi - V_{\varphi\psi}$ belongs to $K^*(A,G)$, for all $\varphi, \psi \in C(G)$.

It follows that the map from $C(G)$ to $C^*(A,G)/K^*(A,G)$ obtained by sending φ onto $V_\varphi + K^*(A,G)$ is a $*$-homomorphism. Moreover, because the elements V_φ generate $C^*(A,G)$, this map is surjective. Since $C(G)$ is abelian, $K^*(A,G) \subseteq \ker(\pi)$ and the reverse inclusion is a consequence of the surjectivity result that we have just proved. For, if $c \in \ker(\pi)$, then $c = V_\varphi + d$, for some elements $\varphi \in C(G)$ and $d \in K^*(A,G)$. Hence, $\varphi = \pi(c) = 0$, so $c \in K^*(A,G)$. □

Since $C^*(A,G)/K^*(A,G) = C(G)$, we see explicitly that $C^*(A,G)$ depends on G as well as A. For example, if $\mathbf{D}$ is the closed unit disc in $\mathbf{C}$ and B is the closed subalgebra of $C(\mathbf{D})$ generated by the inclusion function $z : \mathbf{D} \to \mathbf{C}$, and if A is the closed subalgebra of $C(\mathbf{T})$ obtained by restricting functions in B to $\mathbf{T}$, then the restriction map gives an isometric algebra isomorphism of B onto A. However, $C^*(B,\mathbf{D})$ is not isomorphic to $C^*(A,\mathbf{T})$, since otherwise these algebras would be isomorphic modulo their closed commutator ideals and therefore $\mathbf{D}$ and $\mathbf{T}$ would be homeomorphic, which is, of course, false.

It follows from the details of the proof of Theorem 2.2 that the map from $C(G)$ to $C^*(A,G)/K^*(A,G)$, defined by sending the function φ onto the coset $V_\varphi + K^*(A,G)$, is a $*$-isomorphism. The only point to be checked is that this map is injective, and this is immediate, since if $V_\varphi \in K^*(A,G)$, then $\varphi = \pi(V_\varphi) = 0$. Hence, we have the following.

Theorem 2.3. *If S is an element of $C^*(A,G)$, then $S = V_\varphi + K$, for a unique element $\varphi \in C(G)$ and a unique element K in $K^*(A,G)$.*

This decomposition is similar to a classical one that says every operator in the C*-algebra generated by the Toeplitz operators with continuous symbols on the Hardy space of $\mathbf{T}$ is the sum of a Toeplitz operator and a compact operator. The ideal $K^*(A,G)$ plays a role in our theory similar to that played by the ideal of compact operators in the theory of Toeplitz operators on the circle $\mathbf{T}$.

The following result shows how the elements V_φ have properties similar to those of the Toeplitz operators on the Hardy space of the circle. The proof of this result follows along the same lines as that given for Theorem 3.1 in [12] and is therefore omitted.

Theorem 2.4. *Let φ be an element of $C(G)$.*

(1) $\|V_\varphi\| = r(V_\varphi) = \|\varphi\|_\infty$;

(2) V_φ is a positive operator if, and only if, φ is a positive element of $C(G)$;

(3) If V_φ is left or right invertible, then φ is invertible in $C(G)$;

(4) $\sigma(\varphi) \subseteq \sigma(V_\varphi) \subseteq \operatorname{co}\sigma(\varphi)$, where co *denotes the convex hull in* $\mathbf{C}$.

We shall show below (Corollary 2.9) that V is never multiplicative. Hence, $C^*(A, G)$ is always non-commutative (since there will always exist elements φ and ψ in $C(G)$ such that $V_\varphi V_\psi \neq V_{\varphi\psi}$ and therefore $K^*(A, G)$ contains the non-zero element $V_\varphi V_\psi - V_{\varphi\psi}$, as we saw in the proof of Theorem 2.2).

Lemma 2.5. *Let m be a regular probability measure on G and let T be the corresponding Toeplitz representation on the Hardy space $H^2(A, m)$. Let ψ be an element of $C(G)$ for which $T_{\bar{\psi}}T_\psi = T_{\bar{\psi}\psi}$. Then ψ belongs to $H^2(A, m)$.*

Proof. Let M_ψ be the multiplication operator on $L^2(G, m)$ corresponding to ψ and P the projection of $L^2(G, m)$ onto $H^2(A, m)$. Then $PM_{\bar{\psi}}PM_\psi P = PM_{\bar{\psi}\psi}P$ and therefore, $((1 - P)M_\psi P)^*(1 - P)M_\psi P = 0$. Hence, $(1 - P)M_\psi P = 0$. Consequently, $H^2(A, m)$ is an invariant space for M_ψ. Since $1 \in H^2(A, m)$, this implies that $\psi \in H^2(A, m)$. □

Lemma 2.6. *Let ψ be a function in $C(G)$ not belonging to A. Then there exists a regular probability measure m on G for which ψ does not belong to $H^2(A, m)$.*

Proof. The Hahn–Banach theorem implies there exists a regular complex measure μ on G for which $\int \varphi \, d\mu = 0$, for all $\varphi \in A$ and $\int \psi \, d\mu \neq 0$. Let $m = |\mu|$. Then m is a regular positive measure on G and, by multiplying μ by a suitable non-zero constant, if necessary, we can take m to be a probability measure. If $\psi \in H^2(A, m)$, there exists a sequence (φ_n) of functions belonging to A that converges to ψ in $L^2(G, m)$. Therefore, by the Cauchy–Schwarz inequality, (φ_n) converges to ψ in $L^1(G, m)$. However, $|\int \varphi_n - \psi \, d\mu| \leq \int |\varphi_n - \psi| \, dm$. Hence $\int \psi \, d\mu = \lim \int \varphi_n \, d\mu = 0$, a contradiction. Consequently, $\psi \notin H^2(A, m)$. □

Theorem 2.7. *There is a regular probability measure m on G for which the corresponding Toeplitz representation T from $C(G)$ to $B(H^2(A, m))$ is not multiplicative.*

Proof. Since $A \neq C(G)$, we may choose an element $\psi \in C(G) \setminus A$. By Lemma 2.6, there is a regular probability measure m on G for which ψ does not belong to $H^2(A, m)$. Let T be the canonical Toeplitz representation of (A, G) on $H^2(A, m)$. Then, by Lemma 2.5, $T_{\bar{\psi}}T_\psi \neq T_{\bar{\psi}\psi}$. Thus, T is not multiplicative. □

Theorem 2.8. *Let ψ be an element of $C(G)$ for which $V_{\bar{\psi}}V_\psi = V_{\bar{\psi}\psi}$. Then $\psi \in A$.*

Proof. Suppose $\psi \notin A$ and we'll get a contradiction. As we saw in the proof of Theorem 2.7, if $\psi \in C(G) \setminus A$, there is a regular probability measure m on G for which $T_{\bar{\psi}}T_\psi \neq T_{\bar{\psi}\psi}$, where T is the corresponding Toeplitz representation of (A, G) on $H^2(A, m)$. Let π be the $*$-homomorphism from $C^*(A, G)$ to $B(H^2(A, m))$ induced by T. Then $\pi(V_{\bar{\psi}}V_\psi) \neq \pi(V_{\bar{\psi}\psi})$. This implies that $V_{\bar{\psi}}V_\psi \neq V_{\bar{\psi}\psi}$, a contradiction. Hence, to avoid this, we must conclude that ψ belongs to A. □

Corollary 2.9. *The canonical map V from $C(G)$ to $C^*(A, G)$ is not multiplicative.*

Proof. This is an immediate consequence of the theorem, since $A \neq C(G)$. □

For the analysis of the generalised Toeplitz operator theory, it is important to know when the $*$-homomorphism induced by a Toeplitz representation is injective. Indeed, in the index theory developed in [11] injectivity of this homomorphism plays a vital role for a certain Toeplitz representation.

Theorem 2.10. *The following conditions are equivalent:*

(1) $C^*(A, G)$ is primitive and its commutator ideal $K^*(A, G)$ is simple;

(2) $K^*(A, G)$ is contained in every non-zero closed ideal of $C^*(A, G)$;

(3) For each non-multiplicative Toeplitz representation of (A, G) the induced homomorphism on $C^*(A, G)$ is injective.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. To show the converse, first note that $C^*(A, G)$ admits an irreducible representation $\pi : C^*(A, G) \to B(H)$ on some Hilbert space H of dimension greater than one, since $C^*(A, G)$ is not commutative. Since the algebra $\pi(C^*(A, G))$ is weakly dense in $B(H)$, it is not commutative, and therefore π does not vanish on $K^*(A, G)$. Hence, if (2) holds, $\ker(\pi)$ must be the zero ideal of $C^*(A, G)$ and therefore $C^*(A, G)$ is primitive. Simplicity of $K^*(A, G)$ is obvious, if (2) holds. Hence, (2) $\Rightarrow$ (1).

To see (2) $\Rightarrow$ (3), assume (2) holds and let T be a non-multiplicative Toeplitz representation of (A, G) in a unital C*-algebra B. Then there exists elements φ and ψ of $C(G)$ such that $T_\varphi T_\psi \neq T_{\varphi\psi}$. If π is the $*$-homomorphism induced by T, then $\pi(V_\varphi V_\psi - V_{\varphi\psi}) \neq 0$ and therefore, π does not vanish on $K^*(A, G)$, since $V_\varphi V_\psi - V_{\varphi\psi}$ belongs to $K^*(A, G)$, by the proof of Theorem 2.2. Hence, $\ker(\pi) = 0$, by Condition (2). Thus, Condition (3) holds, if (2) holds.

Finally, to show (3) $\Rightarrow$ (2), suppose that (3) holds and that I is a closed ideal of $C^*(A, G)$ that does not contain $K^*(A, G)$. To show (2) holds and complete the proof, we have only to show we must have $I = 0$. Since the elements $V_\varphi V_\psi - V_{\varphi\psi}$ generate $K^*(A, G)$ as a closed ideal in $C^*(A, G)$, by the proof of Theorem 2.2, one of them does not belong to I, and therefore the Toeplitz representation T of (A, G) in $C^*(A, G)/I$, defined by setting $T_\varphi = V_\varphi + I$, is non-multiplicative. However, the $*$-homomorphism associated to T is clearly the quotient map from $C^*(A, G)$ onto $C^*(A, G)/I$. By Condition (3), this is injective; hence, $I = 0$, as required. □

We shall say that the algebra $C^*(A,G)$ has the *injective homomorphisms* property if Condition (3) in the preceding theorem holds; that is, if for each non-multiplicative Toeplitz representation of (A,G) the induced homomorphism on $C^*(A,G)$ is injective.

We shall give now a class of examples of function algebras whose corresponding C*-algebras have the injective homomorphisms property. First, we need to introduce some concepts from the theory of ordered groups.

An *ordered group* is a pair $(\Gamma,\leq)$ consisting of a discrete abelian group Γ and a translation-invariant total order relation $\leq$ on it.

Ordered groups exist in great abundance, since every torsion-free abelian group admits a total order relation making it an ordered group [7]. A large and important class of examples is provided by the additive subgroups of $\mathbf{R}$, endowed with the order relation induced from $\mathbf{R}$.

Let Γ be any ordered group and $G=\hat{\Gamma}$ its the Pontryagin dual. Of course, G is a compact, Hausdorff group and, by Pontryagin duality, we may regard Γ as the set of continuous characters on G, so that $\Gamma \subseteq C(G)$. With this identification, it is clearly appropriate to use multiplicative notation for the operation in Γ and, of course, the group unit is the constant function 1.

We denote by Γ^+ the set of elements of Γ that are greater than or equal to the unit of Γ. (The group order relation will not be the same as the natural pointwise-defined partial order relation on $C(G)$, of course.) Let $A(\Gamma)$ denote the closed linear span in $C(G)$ of Γ^+. Then $A(\Gamma)$ is a function algebra on G and is proper if Γ is non-trivial. Note also that $A(\Gamma)$ is antisymmetric (that is, if $\bar{\psi},\psi \in A(\Gamma)$, then $\psi=\lambda 1$, for some $\lambda \in \mathbf{C}$). We shall use this well known and easily-proved fact in the proof of the following theorem.

An ordered group Γ is *archimedean* if it is order and group isomorphic to an ordered subgroup of the additive group $\mathbf{R}$.

Theorem 2.11. *Let Γ be a non-trivial ordered group. Then $C^*(A(\Gamma),\hat{\Gamma})$ is primitive. Moreover, $C^*(A(\Gamma),\hat{\Gamma})$ has the injective homomorphisms property if, and only if, Γ is archimedean.*

Proof. Set $A=A(\Gamma)$. First, observe that the map $\gamma \mapsto V_\gamma$, from Γ^+ to $C^*(A,G)$, is a semigroup of isometries in the sense of [8]. Hence, it induces a canonical surjective $*$-homomorphism π from the C*-algebra $C^*(\Gamma^+)$ of Γ^+ to $C^*(A,G)$. Moreover, it follows from Theorem 2.8 and the fact that $A(\Gamma)$ is antisymmetric that each element V_γ is non-unitary when $\gamma \neq 1$. Hence, by [8, Theorem 2.9], π is injective and so $C^*(\Gamma^+)$ and $C^*(A,G)$ are isomorphic. This implies $C^*(A,G)$ is primitive, since $C^*(\Gamma)$ is primitive [8, Theorem 3.14].

Suppose now that Γ is archimedean. Then the closed commutator ideal of $C^*(\Gamma)$ is simple [4] and therefore so is $K^*(A,G)$. Hence, by Theorem 2.10, $C^*(A,G)$ has the injective homomorphisms property.

On the other hand, if $C^*(A,G)$ has the injective homomorphisms property, then $K^*(A,G)$ is simple and therefore so is the closed commutator ideal of $C^*(\Gamma^+)$. It follows from [8, Theorem 4.3] that Γ is archimedean. □

We conclude this section by considering a natural question that, at first sight, one might be tempted to think had a positive answer, but that is easily seen not to have, after a short reflection. First observe that if $T : \varphi \mapsto T_\varphi$ is a Toeplitz representation of (A, G) on a Hilbert space H, then the map $A \to B(H)$, $\psi \mapsto T_\psi$, is a norm-decreasing unital homomorphism. It is natural to enquire whether every such homomorphism admits an extension to $C(G)$ that is a Toeplitz representation of (A, G). The answer is no, as we have already indicated. To see this, let A be the disc algebra on $\mathbf{T}$. Recall this is generated, as a closed algebra of $C(\mathbf{T})$, by the inclusion function $z : \mathbf{T} \to \mathbf{C}$. It is well known and easily seen that $1/z = \bar{z} \notin A$ and therefore there is a character τ on A for which $\tau(z) = 0$. Suppose τ were to admit an extension $\sigma : C(\mathbf{T}) \to \mathbf{C}$ that is a Toeplitz representation of (A, G) (we are identifying $\mathbf{C}$ with $B(\mathbf{C})$). Then we would get $0 = |\tau(z)|^2 = \sigma(z)^{-}\sigma(z) = \sigma(\bar{z}z) = \sigma(1) = 1$, a contradiction. Thus, τ admits no extension to a Toeplitz representation of $(A, \mathbf{T})$.

3. The centre of $C^*(A, G)$

We have seen that, for any proper function algebra A, the C*-algebra $C^*(A, G)$ is not commutative. It is pertinent to try to characterize the centre and to determine under what conditions it is trivial. In the case of a large class of functions algebras—the unimodular ones—we obtain complete answers to these questions; for more general functions algebras we obtain a necessary condition for triviality of the centre that I suspect is also sufficient (it is sufficient in the case of unimodular algebras, as we show below).

First we present a result in general C*-algebra theory that will show the relevance of the condition of triviality of its centre to the question of whether $C^*(A, G)$ has the injective homomorphisms property. This result is presumably known, but I have never seen it stated or proved anywhere.

Theorem 3.1. *Let B be a C*-algebra with closed commutator ideal K. If $b \in B$ and $bK = 0$ or $Kb = 0$, then b belongs to the centre of B.*

Proof. Let I denote the set of all elements $b \in B$ such that $bK = 0$. Clearly, I is a closed ideal of B and therefore, in particular, it is self-adjoint. It follows that $bK = 0$ if, and only if, $Kb = 0$. We shall show now that I is contained in the centre $Z(B)$ of B and to do this it suffices to show $I^+ = I \cap A^+ \subseteq Z(B)$. Suppose then b is a positive element of I. We may write $b = c^2$, where $c = b^{1/2}$ belongs to I. Since $cK = Kc = 0$, we have $c[b_1, b_2] = [b_1, b_2]c = 0$, for all elements $b_1, b_2 \in B$. Hence, taking $b_2 = c$, we get $c^2 b_1 = cb_1c = b_1c^2$. Therefore, $b \in Z(B)$, as required. □

Corollary 3.2. *If B has trivial centre and $K \neq 0$, then K is an essential ideal in B.*

Theorem 3.3. *If $C^*(A, G)$ has trivial centre, then $C^*(A, G)$ has the injective homomorphisms property if, and only if, $K^*(A, G)$ is simple (as a C*-algebra).*

Proof. This is immediate from Theorem 2.10 and Corollary 3.2, since $K^*(A,G)$ has non-zero intersection with every non-zero closed ideal of $C^*(A,G)$. □

In the following result, we obtain a necessary condition on A for the centre of $C^*(A,G)$ to be trivial.

Theorem 3.4. *If $C^*(A,G)$ has trivial centre, then A is antisymmetric.*

Proof. Let ψ and $\bar{\psi}$ be elements of A. If $\varphi \in C(G)$, then $V_\varphi V_\psi = V_{\varphi\psi}$ and $V_{\bar{\varphi}} V_{\bar{\psi}} = V_{\bar{\varphi}\bar{\psi}}$. Hence, taking adjoints, we get $V_\psi V_\varphi = V_{\psi\varphi}$. Therefore, $V_\varphi V_\psi = V_\psi V_\varphi$. Since the elements V_φ $(\varphi \in C(G))$ generate $C^*(A,G)$, V_ψ commutes with all elements of $C^*(A,G)$ and therefore V_ψ lies in the centre of $C^*(A,G)$. Hence, by hypothesis, $V_\psi = \lambda 1$, for some element $\lambda \in \mathbf{C}$. Consequently, $\psi = \lambda 1$. □

It is natural to ask whether the converse of the preceding theorem is true; that is, whether $C^*(A,G)$ has trivial centre if A is antisymmetric. I do not know the answer for *all* function algebras, but for an extremely large class of function algebras I shall show that the answer is positive.

First it is necessary to introduce this class of algebras and prove some results concerning their C*-algebras that are interesting in their own right and that will be needed to solve the question posed in the preceding paragraph.

A function algebra A on G is *unimodular* if every element of $C(G)$ can be uniformly approximated by elements of the form $\bar{\theta} f$, where $f, \theta \in A$ and θ is unimodular (that is, $|\theta| = 1$).

The class of unimodular algebras is vast, see [12] for a wide variety of examples. In particular, it contains the class of algebras $A(\Gamma)$, where Γ is any ordered group.

For the purposes of illustration, we present another class of examples of unimodular algebras. Denote by $P(\mathbf{U}_n)$ the function algebra generated by the polynomial functions on the group $\mathbf{U}_n$ of unitary matrices of size n. Then $P(\mathbf{U}_n)$ is a unimodular algebra [12, Proposition 2.1]. Note in passing that the Haar measure m of $\mathbf{U}_n$ is a representing measure for a character of $P(\mathbf{U}_n)$ [12, Corollary 2.3].

If θ is a unimodular function in an arbitrary (proper) function algebra A, set $P_\theta = 1 - V_\theta V_{\bar{\theta}}$. Since V_θ is an isometry, P_θ is a projection belonging to $K^*(A,G)$. It is easily checked that (P_θ) is an increasing net, where we define $\theta_1 \leq \theta_2$ to mean that $\theta_2 \bar{\theta}_1 \in A$.

Theorem 3.5. *Let A be a unimodular algebra. Then the centre of $K^*(A,G)$ is trivial and the net of projections (P_θ) is an increasing approximate unit for $K^*(A,G)$.*

Proof. The proof that the net (P_θ) is an approximate unit for $K^*(A,G)$ is the same as that of the proof of Theorem 3.2 of [12] and is therefore omitted. We shall show only that a central element of $K^*(A,G)$ is necessarily equal to zero. Suppose then K is a central element of $K^*(A,G)$. Since $\lim_\theta KP_\theta = K$, we have $\lim_\theta KV_\theta = 0$. Hence, $\lim_\theta V_\theta K = 0$, since K is central, and therefore $K = \lim_\theta V_{\bar{\theta}} V_\theta K = 0$, as required. □

Theorem 3.6. *Let A be a unimodular algebra and S an element of $C^*(A,G)$. Then $SV_\psi = V_\psi S$, for all $\psi \in A$, if, and only if, $S = V_\varphi$, for some element $\varphi \in A$.*

Proof. The backward implication is obvious, so we shall only prove the forward one. Suppose then $SV_\psi = V_\psi S$, for all $\psi \in A$. We may write $S = V_\varphi + K$, for some element $\varphi \in C(G)$ and some $K \in K^*(A,G)$. Since (P_θ) is an approximate unit for $K^*(A,G)$, we have $\lim_\theta KV_\theta = 0$ and therefore $\lim_\theta V_{\bar\theta} KV_\theta = 0$. Hence, $S = \lim_\theta V_{\bar\theta} SV_\theta = \lim_\theta (V_\varphi + V_{\bar\theta} KV_\theta) = V_\varphi$. Therefore, if $\psi \in A$, we have $V_\psi V_\varphi = V_\varphi V_\psi = V_{\varphi\psi} = V_{\psi\varphi}$. It follows from this immediately that $V_\psi V_\varphi = V_{\psi\varphi}$, for all elements of the form $\psi = \bar\psi_1 \psi_2$, where $\psi_1, \psi_2 \in A$, and therefore, since such elements have dense linear span in $C(G)$, by the Stone–Weierstrass theorem, we have $V_\psi V_\varphi = V_{\psi\varphi}$, for all elements $\psi \in C(G)$. This implies, by Theorem 2.8, that $\varphi \in A$. □

Corollary 3.7. *S is central if, and only if, $S = V_\psi$, for some element $\psi \in A \cap \bar A$.*

Proof. If S is central, then $S = V_\varphi$ and $S^* = V_\psi$, for some elements φ and ψ in A. Hence, $\psi = \bar\varphi$ belongs to $A \cap \bar A$. □

Theorem 3.8. *Let A be an antisymmetric unimodular algebra. Then the centre of $C^*(A,G)$ is trivial.*

Proof. This is immediate from Corollary 3.7 and the fact that $A \cap \bar A = \mathbf{C}1$. □

4. The injective homomorphisms property

One of our principal aims in this section is to see what properties are implied for a function algebra A if $C^*(A,G)$ has the injective homomorphisms property. Representing measures for characters of A, especially those that are not point masses, play an important part in our considerations.

We begin with the following useful result. Its proof is the same as that of Theorem 3.5 of [12]. (The assumption made in [12] that the function algebra is unimodular and the measure has support equal to G is not needed to get the result here.)

Theorem 4.1. *Let m be a representing measure for a character of A. Then the only closed linear subspaces of $H^2(A,m)$ that are invariant for all the Toeplitz operators T_φ, with $\varphi \in C(G)$, are the trivial spaces 0 and $H^2(A,m)$.*

The following lemma is also important for our considerations. First, it is relevant to point out that it is *not* true that all proper function algebras admit representing measures that are not point masses. The counterexample of B.J. Cole to the peak point conjecture, given in [2, Appendix A], provides an example of a proper function algebra all of whose representing measures are point masses. (I would like to thank Prof. A. Browder for directing me to this example.)

Lemma 4.2. *Let m be a representing measure for A that is not a point mass. Then the canonical Toeplitz representation T of (A,G) on $H^2(A,m)$ is not multiplicative.*

Proof. If T is multiplicative, then for every element $\psi \in C(G)$ we have $T_{\bar{\psi}}T_\psi = T_{\bar{\psi}\psi}$. Hence, by Lemma 2.5, $\psi \in H^2(A,m)$. Since $C(G)$ is norm dense in $L^2(G,m)$ this implies that $H^2(A,m) = L^2(A,m)$. Hence, the functional

$$C(G) \to \mathbf{C}, \quad \psi \mapsto \int \psi \, dm,$$

is multiplicative. Therefore, m is a point mass, contradicting the hypothesis. We conclude that T cannot be multiplicative. □

If Γ is a non-trivial ordered group, then the function algebra $A(\Gamma)$ admits a representing measure that is not a point mass, namely its normalized Haar measure m [10, Example 3.1] (of course, the support of m is G itself).

Theorem 4.3. *Suppose that $C^*(A,G)$ has the injective homomorphisms property. If m is a representing measure for A that is not a point mass, then the support of m is equal to G.*

Proof. If M is the support of m and $M \neq G$, then there is a non-zero function $\varphi \in C(G)$ that vanishes on M, by elementary topology. Clearly, in this case, $T_\varphi = 0$, where T denotes the canonical Toeplitz representation of (A,G) on $H^2(A,m)$. By Lemma 4.2, T is not multiplicative. Hence, the induced $*$-homomorphism $\pi : C^*(A,G) \to B(H^2(A,m))$ is injective, by hypothesis. Consequently, since $\pi(V_\varphi) = T_\varphi = 0$, we have $V_\varphi = 0$ and therefore $\varphi = 0$. This is a contradiction, since $\varphi \neq 0$, by construction. To avoid the contradiction we must conclude that $M = G$. □

Theorem 4.4. *Suppose that A admits a unique representing measure m for one of its characters and that the support of m is equal to G. If the commutator ideal $K^*(A,G)$ of $C^*(A,G)$ is simple, then $C^*(A,G)$ has the injective homomorphisms property.*

Proof. Since $A \neq C(G)$, G is not reduced to a point, so that m is not a point mass. By Theorem 2.10, we need only show that $C^*(A,G)$ is primitive. In fact, we shall show that the canonical map π from $C^*(A,G)$ to $B(H^2(A,m))$ is injective. This will suffice to show primitivity, since $\pi(C^*(A,G))$ acts irreducibly on $H^2(A,m)$, by Theorem 4.1.

Let T denote the canonical Toeplitz representation of (A,G) on $H^2(A,m)$.

First note that the restriction of π to $K^*(A,G)$ is non-zero. For otherwise, for all $\varphi, \psi \in C(G)$, we have $\pi(V_\varphi V_\psi - V_{\varphi\psi}) = 0$, and therefore $T_\varphi T_\psi - T_{\varphi\psi} = 0$; that is, T is multiplicative. This contradicts Lemma 4.2. Therefore, as claimed, π is not the zero map on $K^*(A,G)$. Hence, π is injective on $K^*(A,G)$, since $K^*(A,G)$ is simple, by hypothesis.

We now use [10, Theorem 9.5], which tells us there is a $*$-homomorphism ρ, with kernel the commutator ideal, from the C*-algebra $\pi(C^*(A,G))$ onto $C(G)$ such that $\pi(T_\varphi) = \varphi$, for all $\varphi \in C(G)$.

Now let $S \in C^*(A,G)$ and suppose that $\pi(S) = 0$. Write $S = V_\varphi + K$, for some element φ of $C(G)$ and K of $K^*(A,G)$—this is possible by Theorem 2.3. Then $0 = \pi(S) = T_\varphi + \pi(K)$. Hence, since $\pi(K)$ is in the commutator ideal of $\pi(C^*(A,G))$, we have $\rho\pi(K) = 0$ and therefore $0 = \rho\pi(S) = \rho(T_\varphi) = \varphi$. Consequently, $S = K$ and therefore $\pi(K) = 0$, which implies $K = 0$, since π is injective on $K^*(A,G)$. Thus, $S = 0$ and π is injective on $C^*(A,G)$, as required. □

Recall that a function algebra A on G is a *Dirichlet algebra* if $A + \bar{A}$ is dense in $C(G)$, where $\bar{A} = \{\bar{\varphi} \mid \varphi \in A\}$. These algebras are probably the most important class in function algebra theory and include the group function algebras $A(\Gamma)$ of ordered groups Γ.

Theorem 4.5. *Suppose that $C^*(A,G)$ has the injective homomorphisms property and that A is a Dirichlet algebra. Then G is connected.*

Proof. By Theorem 2.10, $C^*(A,G)$ is primitive and therefore has trivial centre. Hence, by Theorem 3.4, A is antisymmetric. Now let E be a clopen subset of G and let φ be its characteristic function. Since A is a Dirichlet algebra, it contains φ, by [6, p. 181]. Hence, since A is antisymmetric and $\bar{\varphi} = \varphi$, φ is a constant and therefore, $E = \emptyset$ or $E = G$. Hence G is connected, as required. □

Recall that a function algebra on G is said to be *maximal* if it is proper and not properly contained in any other proper function algebra on G.

If Γ is a non-trivial ordered group, then the function algebra $A(\Gamma)$ on $G = \hat{\Gamma}$ is maximal if, and only if, Γ is archimedean. Note that, in general, $A(\Gamma)$ may not even be contained in a maximal function algebra on G. For more on these facts, see [5, pp. 166–168].

Theorem 4.6. *Suppose that $C^*(A,G)$ has the injective homomorphisms property. Then A is maximal.*

Proof. Suppose that B is a proper function algebra on G containing A. Then the canonical map W from $C(G)$ to $C^*(B,G)$ is not multiplicative, by Corollary 2.9. Since W is a Toeplitz representation of (A,G), the corresponding $*$-homomorphism π from $C^*(A,G)$ to $C^*(B,G)$ that maps V_φ onto W_φ is injective, by hypothesis. Now let $\psi \in B$. Then $W_{\bar{\psi}} W_\psi = W_{\bar{\psi}\psi}$; that is, $\pi(V_{\bar{\psi}} V_\psi) = \pi(V_{\bar{\psi}\psi})$. Therefore, $V_{\bar{\psi}} V_\psi = V_{\bar{\psi}\psi}$. Hence, by Theorem 2.8, $\psi \in A$. Thus, $B = A$ and A is maximal, as required. □

It follows from this result that the C*-algebra $C^*(A, \mathbf{U}_n)$ does *not* have the injective homomorphisms property, where $A = P(\mathbf{U}_n)$, the function algebra generated by the polynomial functions on the group $\mathbf{U}_n$ of unitary matrices of size n. For otherwise, A is maximal and, since it is antisymmetric, it is *pervasive* [13, p. 127], in the sense that $A_F = \{\psi_F \mid \psi \in A\}$ is dense in $C(F)$, for every non-empty, proper, closed subset F of $\mathbf{U}_n$. Of course, ψ_F denotes, as usual, the restriction of

ψ to F. It is clear, however, that A is not pervasive, since if F is the set of all scalar matrices $\lambda 1_n$ in $\mathbf{U}_n$, then A_F is clearly not dense in $C(F)$.

Combining Theorem 4.6 with Theorem 3.4, a natural question presents itself at this point: If A is antisymmetric and maximal, does $C^*(A, G)$ necessarily have the injective homomorphisms property? My hunch is that it does; however, I have been unable to prove it.

A particular class of examples where this might be tested is provided by the polynomial algebras. Let K be a simply-connected compact subset of $\mathbf{C}$ with boundary $G = \partial K$ and suppose that the interior of K is non-empty. Let A be the function algebra obtained as the closure of the polynomial functions on G. It is a theorem of E. Bishop [1] that A is a maximal function algebra on G. It is not true that A is necessarily antisymmetric, but in the case where it is (a case that occurs frequently), my hunch would imply that $C^*(A, G)$ has the injective homomorphisms property. However, I have not been able to show this.

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