Smooth Projective Planes and Symplectic Topology

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Outline

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1. Projective Planes
    - Definitions
    - Smoothness
    - The Problem
2. Affine Charts
3. Rings from projective planes
4. Cohomology
5. Radon Transform
6. Plane Curves
7. Singularities

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Definition

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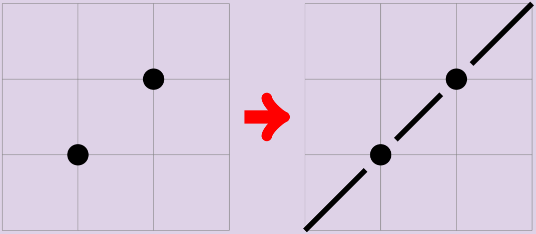
An *incidence geometry* is a set P (the set of *points*), a set L (the set of *lines*) and a set $F \subset P \times L$ (the set of *pointed lines*).

We say a point p *lies on* a line λ, (or λ *lies on* p), if $(p, \lambda) \in F$. Swapping the roles of points and lines gives the *dual* incidence geometry.

Definition (Hilbert)

An incidence geometry is a *projective plane* if

1. any two points p_1, p_2 lie on a unique line $p_1 p_2$ and



2. any two lines λ_1, λ_2 lie on a unique point $\lambda_1 \lambda_2$ and
3. there is a quadrilateral: 4 points, no three on the same line.

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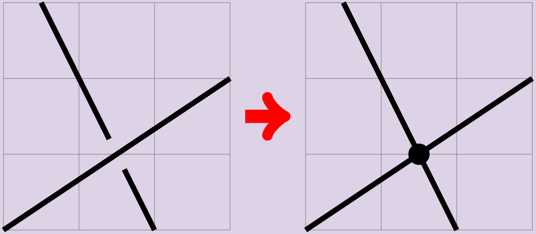

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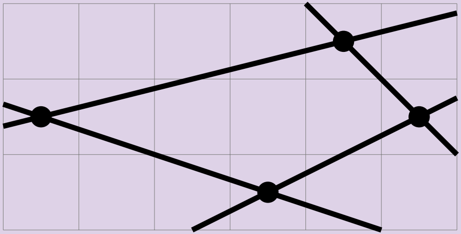

Duality

The dual of a projective plane is a projective plane.

Outline

Definition

A projective plane is *smooth* if P, L and F are compact smooth manifolds and the maps

$$p_1, p_2 \mapsto p_1 p_2$$

and

$$\lambda_1, \lambda_2 \mapsto \lambda_1 \lambda_2$$

are smooth maps.

Example

$$\mathbb{RP}^2, \mathbb{CP}^2, \mathbb{HP}^2, \mathbb{OP}^2$$

$\mathbb{H} =$ quaternions, $\mathbb{O} =$ octave numbers (octonions)

-

Theorem (Freudenthal)

$$\dim P = 0, 2, 4, 8 \textit{ or } 16.$$

- Connected iff $\dim P \neq 0$.
- Henceforth assume connected.

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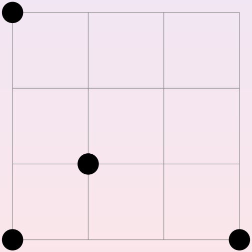
The Problem

If dim $P = 4$, then P diffeo to $\mathbb{CP}^2$.

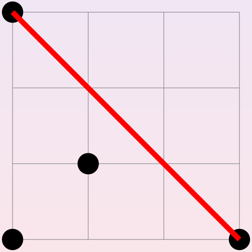
Affine Charts

Take a quadrilateral:



Affine Charts

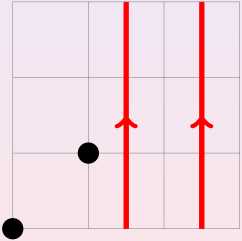
Call this:



the *line at infinity*. Henceforth, draw it far away.

Affine Charts

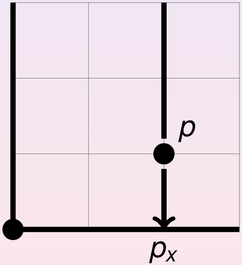
Two lines:



are *parallel* if they meet at the line at infinity.

Affine Charts

Map



Affine Charts

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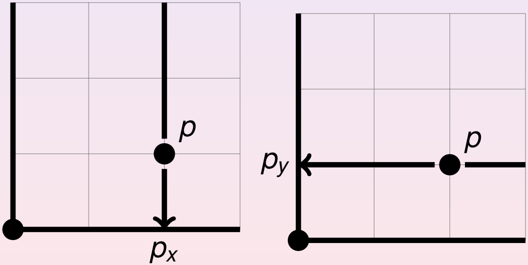


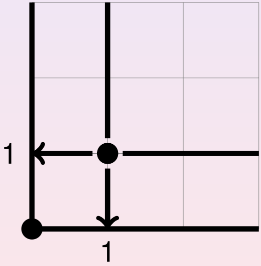
Affine Charts

$$p \mapsto \left(p_x, p_y\right)$$
$$\left(p_x, p_y\right) \mapsto p = \left(p_x \infty_x\right)\left(p_y \infty_y\right),$$

diffeos.

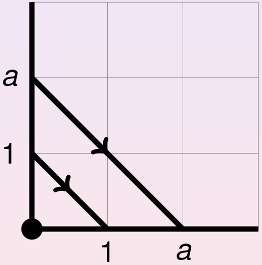
Identity

From a quadrilateral, draw:



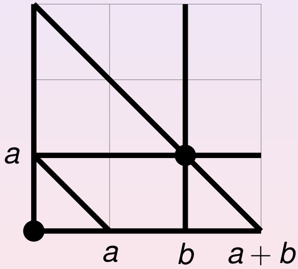
Call these points both 1.

Axis Matching



Parallel lines. Identify these points called *a*.

Addition



Picture defines addition on each axis. Multiplication similar.

Cohomology

Theorem (Breitsprecher)

1. *Lines are diffeo to spheres*
2. $H^* \left(P, \mathbb{Z}\right) = \mathbb{Z}[pt] \oplus \mathbb{Z}[line] \oplus \mathbb{Z}[P] = H^* \left(\mathbb{KP}^2, \mathbb{Z}\right)$

Proof uses

1. ring on line minus point → diffeo to $\mathbb{R}^n$
2. P has “Schubert cell” decomposition

Intersection theory

- Lines are transverse (axes in affine chart).
- So can orient to get unique $+$ive intersection of any two lines.

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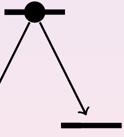
Forgetful maps

$$\begin{tikzcd}
& F \arrow[dl] \arrow[dr] & \\
P & & L
\end{tikzcd}$$



Radon transform

1. Take a volume form dL on L.
2. Pull back to F.
3. $\int$ over fibers to P.
4. Call the result $\widehat{dL}$, the *Radon transform*.

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Crofton formula

Theorem

Take $\Sigma \subset P$ a compact submanifold (with boundary, corners). Then

$$\int_{\Sigma} \widehat{dL} = \int_{L} \# \left(\Sigma \cap \lambda\right) \, dL(\lambda),$$

average $\#$ of intersections with a line.

Symplectic form

If dim $P = 4$,

$$\begin{tikzcd} & F^6 \arrow[dl] \arrow[dr] & \\ P^4 & & L^4 \end{tikzcd}$$

- $\widehat{dL}$ a 2-form
- $\widehat{dL} > 0$ on each line. (Positivity of intersection.)
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Lalonde–McDuff theorem

Theorem (Lalonde–McDuff)

Suppose that P^4 is a closed symplectic 4-manifold, and

$$H^* (P, \mathbb{Z}) = H^* \left(\mathbb{CP}^2, \mathbb{Z}\right),$$

and P contains a sphere on which the symplectic form is > 0. Then P is symplecto to $\mathbb{CP}^2$, up to constant rescaling of symplectic form.

Corollary

Smooth 4-dimensional projective planes are diffeomorphic to $\mathbb{CP}^2$.

Definition

Definition

A submanifold $C \subset P$ is a *smooth plane curve* if C is tangent to a line at each point.

Problem

How can we define *singular* plane curves?

Lifting

Definition

The *lift* of a smooth plane curve $C \subset P$ is the set of its pointed tangent lines $(p, \lambda) : T_p\lambda = T_pC$.

$$\begin{tikzcd}
 & F \arrow[dd] \\
C \arrow[ur] \arrow[dr] & \\
 & P
\end{tikzcd}$$

The Polycontact Plane Field

$$\begin{tikzcd}
 & F \arrow[dl, "\pi_P"'] \arrow[dr, "\pi_L"] & \\
P & & L
\end{tikzcd}$$

Let $\Theta \subset TF$,

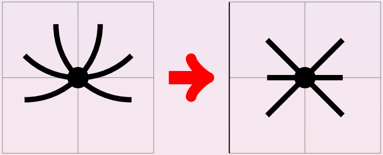
$$\Theta_{(p,\lambda)} = \ker \pi'_P \oplus \ker \pi'_L .$$

Lemma

The lift of any smooth plane curve is Θ-tangent.

Gauss Map

Let $\bar{p}$ be the set of lines through point p.



$$\lambda \in \bar{p} \mapsto T_p\lambda \in \mathrm{Gr}\left(2, T_pP\right)$$

Regularity

Definition

A smooth projective plane is *regular* if the Gauss map is an immersion.

Duality

Theorem

Regularity is invariant under duality.

Almost complex geometry

Theorem

If P^4 *is regular, then there is a unique*

$$\begin{tikzcd}\Theta \arrow[rr, "J"] \arrow[dr] & & \Theta \arrow[dl] \\ & F & \end{tikzcd}$$

with $J^2 = -1$ *for which lifts of smooth plane curves are* J*-holomorphic.*

Allowing plane curves to have singularities

Definition

A *plane curve* is a Θ-tangent J-holomorphic curve.

Theorem

Plane curves are duality invariant.

Singularities

Think of plane curves as living in P (identify with their projections $C \to F \to P$).

Theorem (Micallef and White)

Near each point of a plane curve in a regular 4-dim projective plane, there are complex coordinates, Lipschitz continuous, taking the curve to a complex curve.

Corollary

The singularities are the same as for complex plane curves.

Degree

Definition

The *degree* of a plane curve is

$$\deg C = \frac{[C]}{[\lambda]},$$

ratio of homology classes.

Quadrics

Theorem

In a regular 4-dim projective plane, through any 5 pts (no 3 in a line), there is a unique smooth quadric (degree 2). Any smooth quadric is diffeo to S^2.

Cubics

Theorem (Sikorav)

In a regular 4-dim projective plane, any cubic curve is

1. *diffeo to T^2 or*
2. *diffeo to S^2 and has*
   1. *a node or*
   2. *a cusp or*
3. *is line $\cup$ quadric or*
4. *is triple line.*

“Node”, “cusp” mean singularities Lipschitz homeo to usual node, cusp.