Rational Curves and Parabolic Geometries

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Outline

Outline

Proviso

All manifolds, bundles, Lie groups, Lie algebras are complex

Definition

A *rational homogeneous variety* G/P is a quotient of a semisimple Lie group by a parabolic subgroup.

What is a parabolic subgroup?

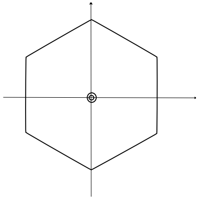
- A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a *Cartan* if
  - its elements act diagonalizably on $\mathfrak{g}$ (adjoint rep).
  - It is maximal among such.
- A Lie subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is *Borel* if it is maximal solvable containing $\mathfrak{h}$.
- A Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is *parabolic* if it contains $\mathfrak{b}$.
- A Lie subgroup $P \subset G$ is *parabolic* if its Lie algebra $\mathfrak{p} \subset \mathfrak{g}$ is parabolic.

Outline

Root lattices

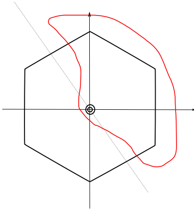
$\mathfrak{h} \subset \mathfrak{g}$ Cartan, so each $H \in \mathfrak{h}$ acts like $[H, A] = \alpha(H)A$, eigenvalue $\alpha(H)$, basis of eigenvectors A. $\alpha \in \mathfrak{h}^*$, root.

Example: $\mathfrak{sl}(3, \mathbb{C})$



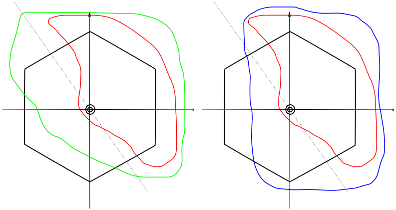
The space $\mathfrak{h}$ is the eigenspace of $\alpha = 0$.

Example: $\mathfrak{sl}(3, \mathbb{C})$



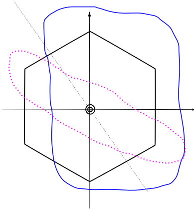
Pick a hyperplane containing only 0 root. One side of it is a Borel. Unique up to automorphism.

Example: $\mathfrak{sl}(3, \mathbb{C})$



Two parabolic subgroups: G/P is $\mathbb{P}^2$ and $\mathbb{P}^{2*}$. Matched up by automorphism.

Example: $\mathfrak{sl}(3,\mathbb{C})$



What is this line? Copy of $\mathfrak{sl}(2,\mathbb{C})$. Borel of $\mathfrak{sl}(2,\mathbb{C})$ is in P. $\mathrm{SL}(2,\mathbb{C})/B_{\mathrm{SL}(2,\mathbb{C})} = \mathbb{P}^1$. So this is a homogeneous $\mathbb{P}^1 \subset \mathbb{P}^2$, a line.

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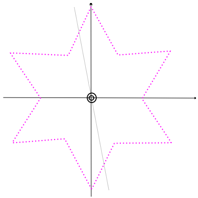
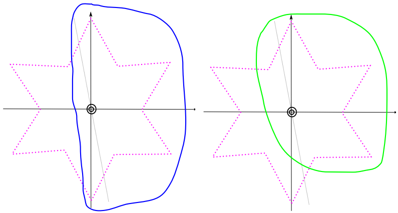
Example: G_2

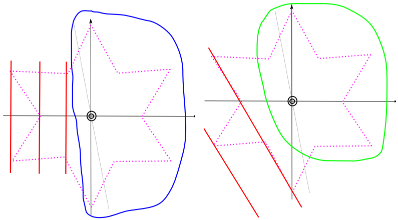


Two parabolic subgroups:



Example: G_2

The tangent bundle is $T(G/P) = G \times_P (\mathfrak{g}/\mathfrak{p})$. $\mathfrak{g}/\mathfrak{p}$ is quotient, so roots outside of $\mathfrak{p}$. We spot filtrations of the tangent bundle:

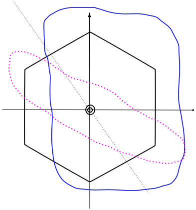


Brackets $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$. The second picture is a contact structure.

Outline

Phantoms and Shadows

Take $G_0 \subset G$ a subgroup. Call $G_0/(G_0 \cap P) \subset G/P$ a "phantom" (M-). If $G_0 \subset G$ parabolic subgroup, "shadow" (Tits). Example:



A $\mathbb{P}^1$-phantom (not a shadow).

Outline

Definition of a Parabolic Geometry

Model

$$\begin{array}{ccc} P & \longrightarrow & G \\ & & \downarrow \\ & & G/P \end{array}$$

principal right

Left inv vec field $\vec{A}$ for $A \in \mathfrak{g}$

$\vec{A}$ gens right action for $A \in \mathfrak{p}$

$\vec{A}$ transforms in adj rep

Parabolic Geometry

$$\begin{array}{ccc} P & \longrightarrow & E \\ & & \downarrow \\ & & M \end{array}$$

principal right

Some vec fields $\vec{A}$, on E, for $A \in \mathfrak{g}$

$\vec{A}$ gens right action for $A \in \mathfrak{p}$

$\vec{A}$ transforms in adj rep

Examples

Conformal geometry $\to$ hyperquadric model
Projective connection $\to$ $\mathbb{P}^n$ model

Completeness

Definition

Complete if $\vec{A}$ vector fields (“left invariant”) on E are complete, every $A \in \mathfrak{g}$.

Theorem (M-)

Complete iff iso to model.

Outline

Lifting and Dropping

If $P \to E \to M$ modelled on G/P, and $Q \subset P$ smaller parabolic, then $Q \to E \to E/Q$ modelled on $Q \to G \to G/Q$: "lift" to bigger model—functor.
But can't always drop from bigger model to smaller model.

Example

2nd order ODE system $\to \mathbb{P}T\mathbb{P}^n$ model. Drops to $\mathbb{P}^n$ model just when geodesic equations of a connection.

Curvature

$$\overrightarrow{\kappa(A,B)} = \overrightarrow{[A,B]} - \left[\vec{A},\vec{B}\right],\ \kappa : E \to \mathfrak{g} \otimes \Lambda^2 \left(\mathfrak{g}/\mathfrak{p}\right)^*,\ \textit{curvature}.$$

Outline

Apparent structure group

$P =$ structure group, $\mathfrak{p} =$ structure algebra

$$\mathfrak{p}^{\text{app}} = \left\{ A \in \mathfrak{g} \,\middle|\, \vec{A} \lrcorner \kappa = 0 \right\}, P^{\text{app}} = \exp \mathfrak{p}^{\text{app}}.$$

Theorem (Čap)

Locally, a parabolic geometry drops like

$$\begin{array}{c} G/P \\ \downarrow \\ G/P^{\text{app}}. \end{array}$$

Natural structure group

$$\mathfrak{p}^{\text{nat}} = \left\{ A \in \mathfrak{p}^{\text{app}} \,\middle|\, \vec{A} \text{ complete} \right\}$$

Theorem (M-)

A parabolic geometry drops like

$$\begin{array}{c} G/P \\ \downarrow \\ G/P^{nat} \end{array}$$

Outline

Phantoms and Shadows

Phantom $G_0/(G_0 \cap P) \subset G/P$. Take parabolic geom $P \to E \to M$ modelled on $P \to G \to G/P$.

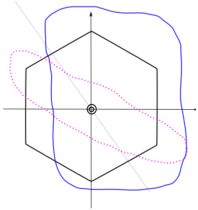
Definition

"Phantoms" are submanifolds $F \subset E$ of expected dimension tangent to $\vec{A}$ for $A \in \mathfrak{g}_0$.

If they exist, then they yield submanifolds $F/(G_0 \cap P) \subset M$.

Circles

If a phantom yields a curve in the model, it is called a *circle*.



Circles always exist in abundance.

Circles and Dropping

Theorem (M-)

The natural structure algebra of any parabolic geometry is

$$\mathfrak{p}^{nat} = \bigoplus_{\alpha} \mathfrak{g}^{\alpha},$$

sum over roots α whose circles are rational curves.

Example: $\mathfrak{sl}(n+1,\mathbb{C})$

Corollary

Geodesics of any connection are rational curves just when iso to model $\mathbb{P}^n$ (flat connection).

Proof.

Drops to bigger parabolic—only bigger is $\mathfrak{sl}(n+1,\mathbb{C})$. □

Summary

- Parabolic geometries are geometries like rational homogeneous varieties.
- They sometimes have phantoms, and always have circles.
- If circles for a given root are rational, then that root sits in the natural structure group, so we can drop, expanding the parabolic to contain that root.
- For example, if the lowest root has rational circles, then we can drop to G/G, without checking any other roots, so our geometry is isomorphic to the model.

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