

Smooth Projective Planes

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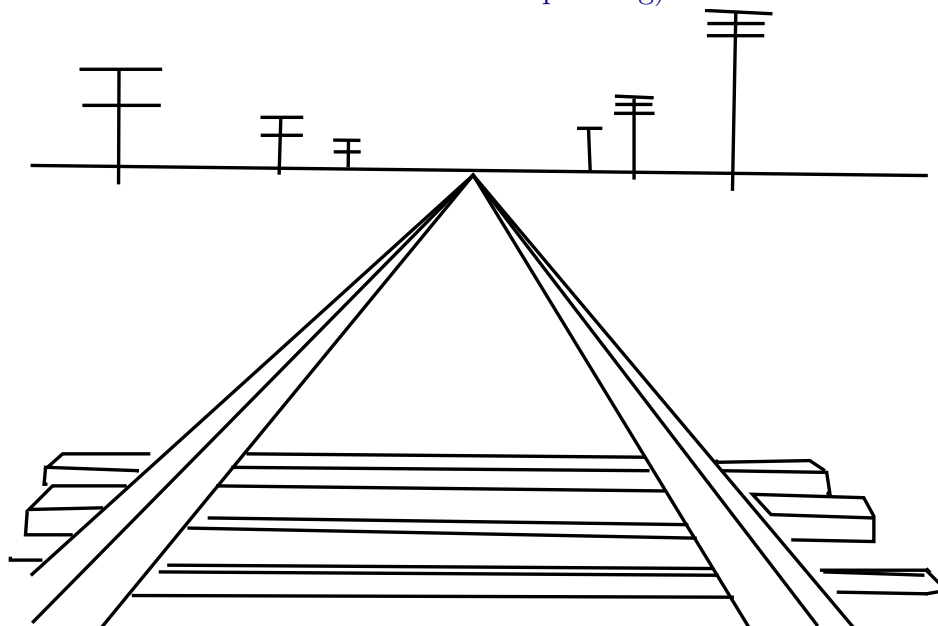
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1 History

~ 300 BCE	Euclid	Axioms of Euclidean geometry
~ 1413 CE	Brunelleschi & Alberti & della Francesca	Projective geometry (developed for painting)



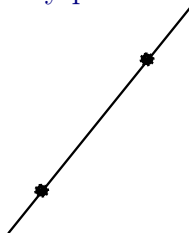
1899	D. Hilbert	Axioms
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2 Definitions

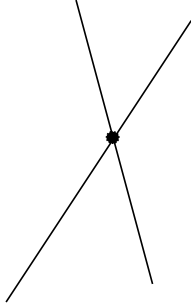
- A projective plane is a set P (the set of “points”) and a set L (the set of “lines”) and a subset $F \subset P \times L$ (the set of “pointed lines”).
- Say that a point p is on a line ℓ if $(p, \ell) \in F$.

3 Axioms

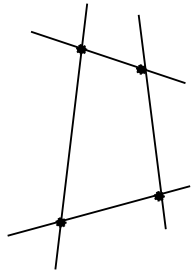
1. Any pair of distinct points p, q lie on a unique line pq .



2. Any pair of distinct lines ℓ, m contains a unique point ℓm .

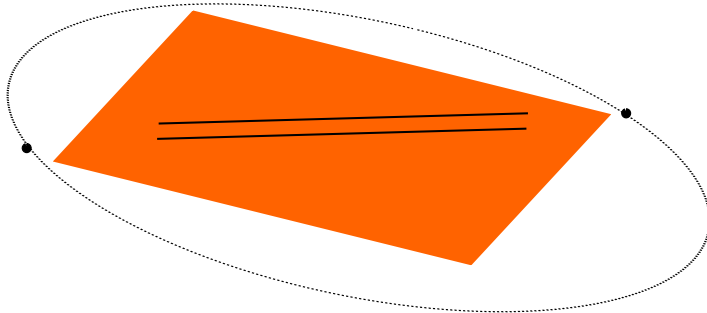


3. \exists 4 points, no 3 on a line (a *quadrilateral*)



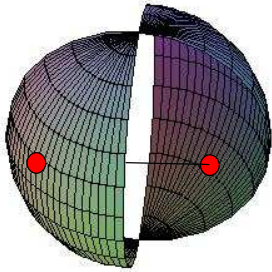
4 Standard model: \mathbb{RP}^2

\mathbb{RP}^2 is the Euclidean plane with points attached at ∞ ; 1 point in each direction, with opposite directions identified.

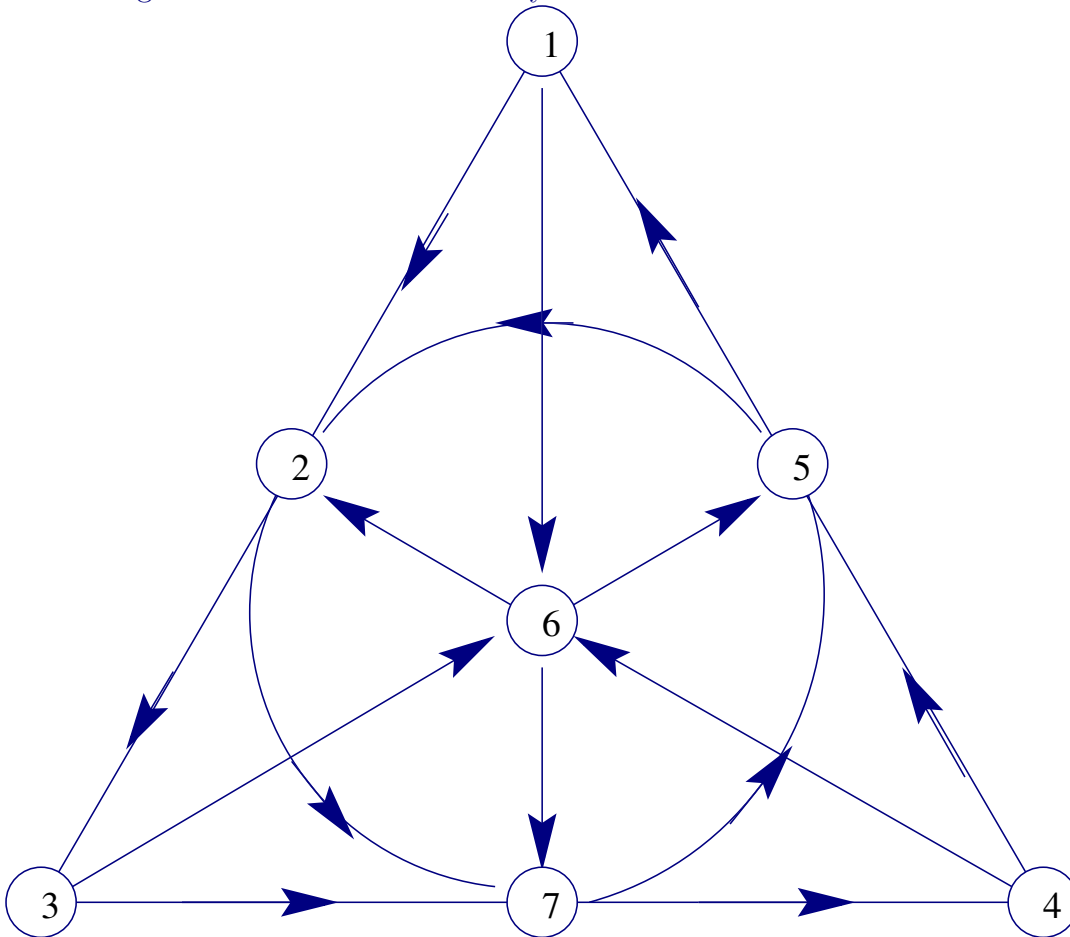


Parallel lines meet at these points.

The same space: identify opposite points of the sphere: $\mathbb{RP}^2 = S^2 / \pm 1$



Lines are great circles. Rotations are symmetries.

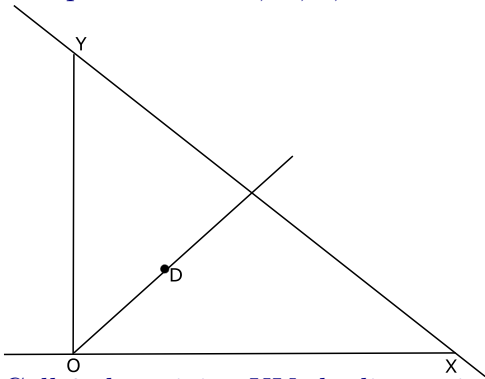


5 Today's problem

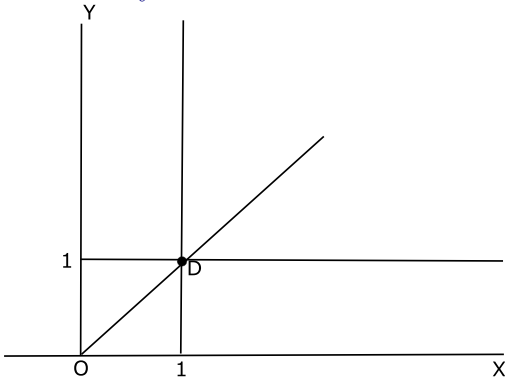
Find all projective planes for which P, L, F closed manifolds, and $pq(lm)$ depends smoothly on $p, q(l, m)$: *smooth projective planes*.

6 Drawing in projective planes

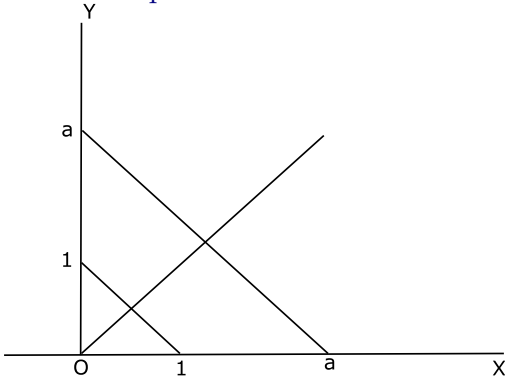
Take a quadrilateral O, X, Y, D .



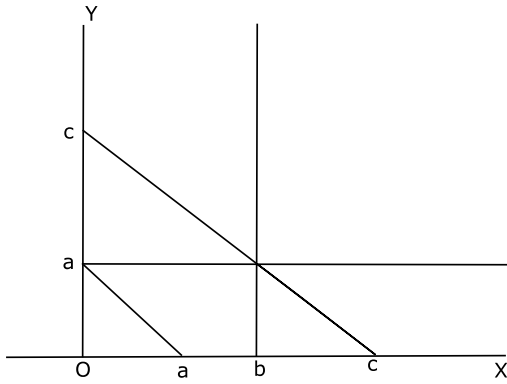
Call O the origin, XY the line at infinity, OD the diagonal, OX the x -axis, etc. Call lines parallel if they meet on the line at infinity.



Call these points 1.



Identify points on the axes lying on parallels to 11.



$a + b = c$, defines $+$ on the axes of the quadrilateral
 Can define \times similarly.

7 Rings and projective planes

- For \mathbb{RP}^2 , $+$, \times make axes into \mathbb{R} as a ring.
- For any proj plane, every quadrilateral gives ring structure to its lines.
- Conversely, given a ring, can build many proj planes with that ring coming up.
-

Proj Planes with Quadrilateral \rightarrow Rings

functor.

- Many-to-1, and depends on quadrilateral.

(1)	\mathbb{RP}^2	$= \{\mathbb{R}\text{-lines through } 0 \text{ in } \mathbb{R}^3\}$	\rightarrow	\mathbb{R}
(2)	\mathbb{CP}^2	$= \{\mathbb{C}\text{-lines through } 0 \text{ in } \mathbb{C}^3\}$	\rightarrow	\mathbb{C}
(3)	\mathbb{HP}^2	$= \{\mathbb{H}\text{-lines through } 0 \text{ in } \mathbb{H}^3\}$	\rightarrow	\mathbb{H}
(4)	Moufang (1933): $\exists! \mathbb{OP}^2 \rightarrow \mathbb{O}$			

\mathbb{O} has multiplication table $e_i e_j + e_j e_i = -2\delta_{ij}$ and $e_i e_j = e_k$ when i, j, k in a line on Segré plane: $e_3 e_6 = e_5$, etc.

For smooth projective planes,

- expand xy into Taylor series
- infinitesimal algebra
- for generic quadrilateral, has no zero divisors
- because lines are generically transverse (Sard's lemma)

Theorem 1. Any \mathbb{R} -alg without 0-divisors has dimension 1, 2, 4 or 8 ($= \dim \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$).

Proof involves: let S^n be unit sphere in the alg, and map $S^n \times S^n \rightarrow S^n$ (multiply and rescale). Algebraic topology of maps of spheres.

Using same alg top machine:

Theorem 2 (Bott). *If P a smooth proj plane, then cohom of P is cohom of just one of $\mathbb{R}P^2, \mathbb{C}P^2, \mathbb{H}P^2, \mathbb{O}P^2$ (the model of P).*

Theorem 3 (Salzmann). *Lines are homeo to spheres.*

8 Main theorem

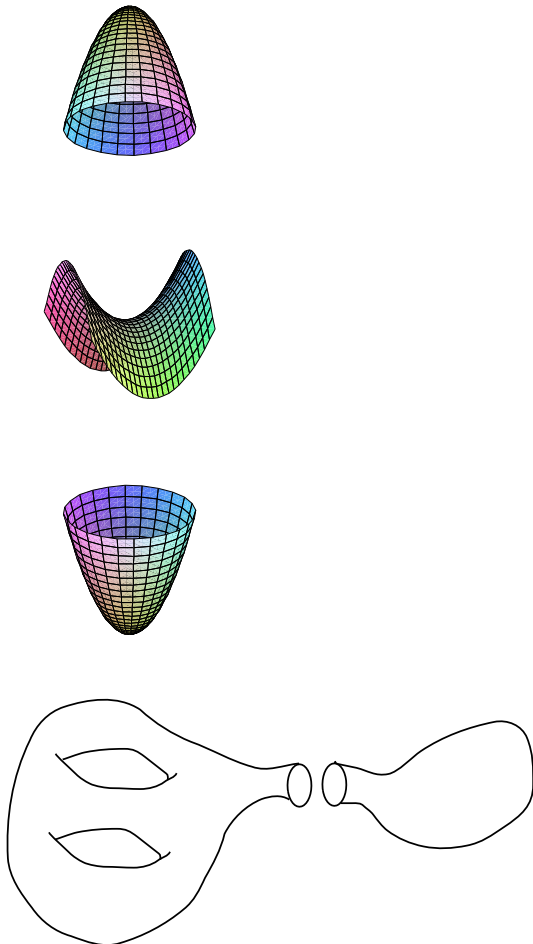
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Theorem 4 (McKay–Kramer–Stolz). *Proj planes are diffeo to their models.*

- Conjectured by Kolmogorov and Freudenthal.
- Remainder of talk is outline of proof.
- $\mathbb{R}P^2$ as model: graduate topology exercise

Theorem 5 (Eels & Kuiper). *$H^*(\mathbb{H}P^2) = \mathbb{Z}^{(0)} \oplus \mathbb{Z}^{(4)} \oplus \mathbb{Z}^{(8)}$ and if P has this cohom, then P has a function with 3 critical points (max, min, and saddle).*

Glue together whole of P out of very simple pieces:



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- **Corollary 1.** P is uniquely determined up to connected sum with homotopy sphere
- Homotopy spheres form a group Θ_n under connected sum.
- Θ_n acts on n -dimensional manifolds by connected sum.
- Kramer & Stolz (2005): Every cohomology- $\mathbb{H}\mathbb{P}^2$ is a fixed point of Θ_8 , diffeo to $\mathbb{H}\mathbb{P}^2$.
- Similar for $\mathbb{O}\mathbb{P}^2$.

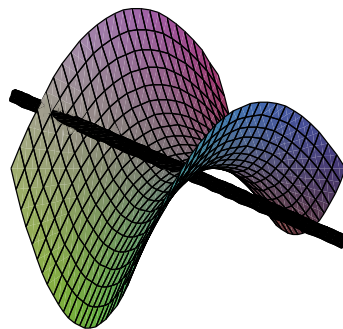
9 Projective planes modelled on the complex projective plane

- Trouble: low dimensional topology is hard (Poincaré conjecture).
- If $P \cong \mathbb{C}\mathbb{P}^2$,



10 Radon transform

- Pick any measure μ on L .
- Pick any surface $S \subset P$



- Define measure $\hat{\mu}$ on S by

$$\int_S f(p) d\hat{\mu}(p) = \int_L \sum_{p \in \ell \cap S} f(p) d\mu(\ell)$$

- $\hat{\mu}$ is Radon transform of μ .
 - μ a smooth measure gives $\hat{\mu}$ a closed 2-form on P , positive on lines, and $\hat{\mu}^2$ a smooth measure
 -
- Theorem 6 (Taubes).** *A closed 4-manifold with closed 2-form ω , positive on a 2-sphere, with ω^2 positive, is diffeo to $\mathbb{C}P^2$.*
- Proof uses Seiberg–Witten string/gauge theory

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