Rational Curves and Parabolic Geometries

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4 December 2007

Outline

Contents

1 Introduction

What is geometry?

- Klein: geometry=G/H
- Riemannian manifolds don't fit.
- Cartan: examples of "generalized spaces", "bent" G/H
- Chern: G-structures
- Ehresmann: Cartan geometries

2 Local Picture of G/H

Local picture of G/H

- Bundle $\begin{array}{c} G \longleftarrow H \\ \downarrow \\ G/H \end{array}$
- Open sets in G/H differ by G-action iff their pullbacks of this bundle differ by diffeo matching $g^{-1}\,dg$.
- Therefore the bundle with $g^{-1}\,dg$ encodes local picture of G/H.

3 Bending G/H

Bending G/H

Definition 3.1. A *Cartan geometry* on a manifold M, modelled on G/H, is a bundle $\begin{array}{c} E \longleftarrow H \\ \downarrow \\ M \end{array}$ and a 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$ so that

1. ω can be identified with $g^{-1}\,dg$ by linear identification of tangent spaces of E and G, matching up infinitesimal H-action
2. ω transforms in the adjoint rep

Example: Riemannian geometry

Model

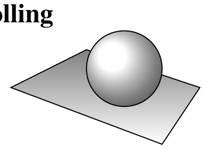
- $G/H = \mathbb{R}^n$
- $G =$ rigid motions of $\mathbb{R}^n$
- $H = O(n)$.
- $$g^{-1}\,dg = \begin{pmatrix} h^{-1}\,dh & h^{-1}\,dx \\ 0 & 0 \end{pmatrix}$$

Riemannian geometry

- M any Riem mfld
- $E =$ o.n. frm bndl of M
- $H = O(n)$
- $$\omega = \begin{pmatrix} \text{Levi-Civita} & \text{soldering} \\ 0 & 0 \end{pmatrix}$$

4 Rolling

Rolling



- Draw a curve in the plane.
- Roll a ball along that curve.
- The point of contact moves on a curve on the sphere.
- The ball and plane stay tangent: tangent planes identified.
- So orthonormal frames are identified.

Rolling

- $$\begin{array}{ccccccc} E_{\mathbb{R}^2} & \longleftarrow & \phi_1^* E & \xrightarrow{\sim} & \phi_2^* E & \longrightarrow & E_{S^2} \\ \downarrow & & \searrow & & \swarrow & & \downarrow \\ \mathbb{R}^2 & \longleftarrow & & C & & \longrightarrow & S^2 \end{array}$$
- $\widetilde{\longrightarrow}$ also matches up ω (rolling without slipping or twisting).
- Obvious analogous definition of rolling for any Cartan geometries.
- Classically called *development*.

5 Lifting

Lifting

- Suppose $H_- \subset H_+ \subset G$ are subgps.
- $G/H_- \to G/H_+$ is a G-invariant bundle map.
- "Bend it": If $E \to M$ is modelled on G/H_+, then $E \to E/H_-$ is modelled on G/H_-. (Same ω).
- New Cartan geometry in higher dimension.

Example: Riemannian geometry

Model

- $G =$ rigid motions of $\mathbb{R}^n$
- $H_+ = O(n)$
- $H_- = O(n-1)$
- $G/H_+ = \mathbb{R}^n$
- $G/H_- =$ unit $T\mathbb{R}^n$

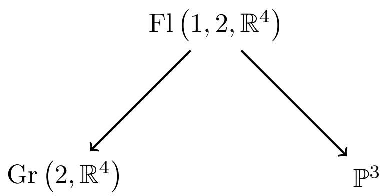
Riemannian manifold M

- $E =$ o.n. frm bndl
- $H_+ = O(n)$
- $H_- = O(n-1)$
- $E/H_+ = M$
- $G/H_- =$ unit TM

6 Twistor Theory

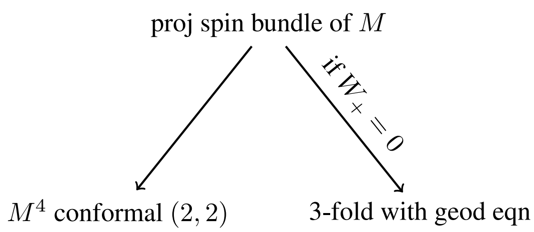
Example: conformal $(2,2)$ geometries

- A *twistor correspondence* is a lifting followed by a dropping.
- A model example:



Example: conformal $(2,2)$ geometries

Geometries with these models:



Dropping

The hard question: when does a geometry drop? (i.e. when is it a lift?)

7 Parabolic Geometries

Definition

- A *complex parabolic geometry* is a holomorphic Cartan geometry modelled on a rational homogeneous variety G/P.
- Example: holomorphic conformal geometry, modelled on hyperquadric.

Rational curves

Theorem 7.1 (M-)**.** *A parabolic geometry on a compact Kähler manifold drops iff the manifold contains a rational curve.*

Proof.

- Roll to the model.
- Bend-and-break (Mori).
- Roll back to M.
- Part of curvature bundle is negative.
- Curvature vanishes in direction of curve.

□

Example: conformal geometry

Theorem 7.2 (M-)**.** *A compact Kähler manifold bearing a conformal geometry contains a rational curve iff it is the hyperquadric with its usual conformal geometry.*

Example: rigidity of model

Theorem 7.3 (M-). *The only rationally connected manifold M with a G/P-geometry is G/P. The only G/P-geometry on G/P is the model geometry.*

8 2-Plane Fields

Definition 8.1. A *2-plane field* on a manifold M is a rank 2 subbundle of TM. A 2-plane field on a 5-fold M is *nondegenerate* if near each point it has local sections X and Y so that $X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]]$ are linearly independent.

Global Theory of 2-Plane Fields

Theorem 8.2 (É. Cartan). *Every nondegenerate 2-plane field on a 5-fold imposes a parabolic geometry modelled on G_2/P.*

Theorem 8.3 (Boucksom, Demailly, Paun, Peternell). *A smooth projective variety with plane field not closed under bracket contains a rational curve.*

Corollary 8.4 (M-). *The only smooth projective 5-fold with a nondegenerate 2-plane field is G_2/P, with its G_2-invariant 2-plane field.*

Proof. Has rational curve, so parabolic geometry drops. Parabolic subgroup is maximal, so isomorphic to model. □

9 Characteristic Classes

Theorem 9.1 (M-). *Any polynomial relation among characteristic classes on G/P holds true on any closed Kähler manifold M with G/P-geometry.*

Example: if the model is $\mathbb{P}T\mathbb{P}^2$, then

$$c_2 = 2\left(\frac{c_1}{2}\right)^2$$
$$c_3 = \left(\frac{c_1}{2}\right)^3.$$

10 Open Problems

- Conjecture: Parabolic geometries on compact complex manifolds are locally homogeneous.
- Classification on smooth projective varieties?
- Extend results proven for smooth projective varieties to compact complex manifolds.