THE BLASCHKE CONJECTURE

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Abstract.

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1. The conjecture

Conjecture 1. *Every Riemannian manifold whose injectivity radius equals its diameter is isometric to a compact rank one symmetric space.*

2. Motivation

The idea behind the Blaschke conjecture is to understand the Riemannian manifolds where light shining from a point, or sound emanating from a point, focuses all at the same moment, causing a "big flash" or "loud bang".

We picture the light rays travelling on geodesics. The first moment when a geodesic bangs into another is at the cut point: defined to be the last point where the geodesic is still minimizing. The distance to the first cut point is the injectivity radius—it is clear that before the injectivity radius, we still have no geodesic focusing, because the exponential map is a diffeomorphism. But beyond the injectivity radius, some geodesic no longer minimizes, so it must have already into a shorter one.

Clearly the diameter is never smaller than the injectivity radius, $D \geq I$, since the exponential map from a point preserves distance from that point up to the injectivity radius. But on the round sphere, the injectivity radius is the diameter, $D = I$. So we can have equality of injectivity radius and diameter. Conversely, suppose we have a complete Riemannian manifold on which every geodesic coming out of a given point p hits the cut locus of p at the same moment, say at distance L. Any point q can be connected to p by a shortest geodesic. But geodesics aren't shortest anymore after the cut locus. So the distance from p to q is never greater

Date: January 24, 2003.

than the distance to the cut locus, L. Therefore the diameter is at most L, and $D = L \leq I$. But the general inequality $D \geq I$ ensures $D = L = I$.

Putting this together: a manifold has cut locus at a constant distance precisely if that distance is the diameter and also the injectivity radius.

3. Warm up: surfaces

Consider a sphere made of a thin layer of glass. A pulse of light at the north pole will focus on the south pole. A burst of sound from the north pole will reach the south pole all at once, making a much louder noise there than at any point in between. The reason: geodesics from the north pole all concentrate at once at the south pole. Are there any other surfaces like this?

Consider a surface of revolution. Start at a fixed point of the revolution. Geodesics from that point reach out together, in circles, until they collide at another fixed point. But in general this will only work when you start from a fixed point of the revolution—the reason why is not at all obvious.

Reconsider the sphere. Take latitude and longitude coordinates (in radians) α and β. The usual metric is

$$ds^2 = d\alpha^2 + \cos^2 \alpha \, d\beta.$$

We can prove that longitudinal lines are precisely the shortest geodesics from south to north pole as follows: taking C any curve from south pole S to north pole N,

$$\begin{aligned} \pi &= \alpha(N) - \alpha(S) \\ &= \int_C d\alpha \\ &\leq \int_C \sqrt{d\alpha^2 + \cos^2 \alpha \, d\beta^2} \\ &= \int_C ds \\ &= \text{length}(C) \end{aligned}$$

with equality only if $d\beta = 0$, i.e. a longitudinal line. Call this phenomenon the calibration of longitude.

Define a *Blaschke point* on a surface to be a point S so that all geodesics from that point meet the cut locus at a given distance L, and at no smaller distance. In particular, all of these geodesics are minimal. Replace $\cos \alpha$ above by any positive function $f(\alpha, \beta)$. As long as f behaves like $\cos \alpha$ near the north and south poles, the metric completes to a smooth metric over the poles, and we can use the same proof and see that the north and south poles are Blaschke points. Since α is determined by arc length, and β by angle, the function $f(\alpha, \beta)$ is determined geometrically, up to changing the angle variable β by a constant. So there are lots of metrics with Blaschke points on the sphere, and they are very nearly parameterized by choice of the function $f(\alpha, \beta)$. In fact, Blaschke points appear to be a generic phemenon.

Conjecture 2. *Every metric on the two dimensional sphere has a pair of Blaschke points, which are cut points of one another along every geodesic through either one.*

Why believe the conjecture? Because we have constructed metrics with Blaschke points depending on one function of two variables. But the generic metric on the sphere depends, in local coordinates, on three functions of two variables

$$E\,dx^2 + F\,dx\,dy + G\,dy^2$$

and diffeomorphisms of the sphere, i.e. changes of variables, depend on two functions of two variables, leaving one function of two variables worth of metrics.

A similar construction works on a sphere of any dimension, so there are lots of metrics on spheres which have Blaschke points.

Suppose that we have any surface with a chosen point on it. We can take geodesic normal coordinates r, θ near that point, and then the Riemannian metric looks like $ds^2 = dr^2 + h^2\,d\theta^2$ where $h(r,\theta) = r - K\frac{r^3}{3} + \dots$ and K is the Gaußcurvature.

Suppose that S is a Blaschke point, and that the geodesics meet at a point N. Rescale the metric to arrange $L = \pi$. Now we let

$$\alpha = -\pi/2 + \text{distance from } S.$$

Then let β be the angle coordinate on the tangent plane at S. Define β on the surface to be constant along each geodesic, i.e. using the exponential map from the tangent plane at S. The metric near S must look like

$$ds^2 = dr^2 + f^2\,d\beta^2.$$

But by definition,

$$\alpha = -\pi/2 + r$$

so that

$$ds^2 = d\alpha^2 + f^2\,d\beta^2.$$

The 1-forms $d\alpha$ and $f\,d\beta$ are orthonormal and defined everywhere but at the poles. Looking now at the north pole in geodesic polar coordinates, we find

$$\alpha = \pi/2 - r$$

by definition, since the geodesics we are working with are minimal. So

$$d\alpha = -dr$$

and this determines that the orthogonal 1-form $f\,d\beta$ must be

$$f\,d\beta = -h\,d\theta = \left(r - K\frac{r^3}{3} + \dots\right)\,d\theta.$$

This shows that β is locally a function of θ. But they both increase by 2π under a complete rotation around the north pole, so $\beta = \beta(\theta)$ is globally a function of θ.

Take any function $\phi(\beta)$ which is 2π periodic and has vanishing mean value

$$\int_0^{2\pi} \phi(\beta)\,d\beta = 0$$

and satisfies $\phi(\beta) < 1$. Let $f(\alpha,\beta)$ be any nonnegative function which satisfies $f(\alpha,\beta) \sim \cos(\alpha)$ as $\alpha \to -\pi/2$ and

$$f(\alpha,\beta) \sim h\,(\pi/2 - \alpha)\,(1 - \phi(\beta))$$

as $\alpha \to \pi/2$, where $h(r)$ satisfies

$$h(r) \sim r - K\frac{r^3}{3!} + \dots$$

where the dots indicate higher order terms in r. The constant K sets the Gaußcurvature at the north pole.

In general, the geodesics of this metric will not be periodic, since they will leave each other from the south pole at an angle β and arrive at an angle $\theta = \beta - \int_0^\beta \phi$ at the north pole. This can be any reparameterization of angles.

A Riemannian manifold is called a *Blaschke manifold* if every point is a Blaschke point, i.e. the cut locus of every point is struck at the same time in each direction from that point.

Theorem 1 (Leon Green)**.** *A Blaschke surface is isometric to a round sphere or real projective plane.*

Conjecture 3. *Any surface with three distinct Blaschke points is isometric to a round sphere or round real projective plane.*

Notice that the same ideas work on the real projective plane: drawing the projective plane as a sphere, but treating each point of the projective plane as a pair of antipodal points of the sphere, we see that geodesics from the north/south pole collide in pairs at the equator. They all collide at the same time, but the loud bang is not as loud as on the sphere, because they don't all collide at the same point: only two at each point. Real projective spaces would appear to be the "softest" Blaschke manifolds, while spheres are the "hardest."

Using the above construction on the sphere, if the function $f(\alpha, \beta)$ satisfies

$$f(\alpha, \beta) = f(-\alpha, \beta + \pi)$$

then we can quotient the metric onto the real projective plane, so that the north/south pole has the equator as cut locus, and at the same distance in every direction. It is remarkable that the equator need not have constant geodesic curvature, and that the geodesics through the north/south pole might still not be periodic.

4. Examples: the compact rank one symmetric spaces

Consider a complex projective space $\mathbb{CP}^n$. It is a homogeneous space of $U(n+1)$. This group acts transitively on projective subspaces of all dimensions. The complex projective lines $\mathbb{CP}^1 \subset \mathbb{CP}^n$ are totally geodesic round spheres. (This is easy to see from the moving frame, using the Maurer–Cartan 1-form on the unitary group.) Given a pair of points $p, q \in \mathbb{CP}^n$, there is a unique projective line through them. If q is on the cut locus of p, then there is a unique projective line, i.e. totally geodesic round sphere, containing p and q; if p is the north pole, q is the south, and we know the precise distance: call it L, (L was π on the standard round sphere we discussed previously) in an appropriately scaled metric. The distance between any two points is never greater than the distance between mutually cut points. Moreover, the point q turns out to be on the hyperplane at infinity, in affine coordinates centered at p; i.e. if affine coordinates of p are $z = [1 : 0 : \cdots : 0]$ then the cut locus of p is at $z_0 = 0$. In fact, we only have to check this on a projective line, since every geodesic is contained in a totally geodesic projective line. But a projective line is a round 2-sphere, so it is obvious.

We have a picture of the cut locus as the projective hyperplane, the shortest geodesics from p to a point q of the cut locus of p form a projective line, and the projective line strikes the cut locus (i.e. the hyperplane) orthogonally. The tangent space at q is decomposed orthogonally into the tangent space of the cut locus (the hyperplane) and that of the projective line.

Looking back at p, we can consider the exponential map

$$\exp : T_p\mathbb{CP}^n \to \mathbb{CP}^n.$$

Inside the ball $B \subset T_p\mathbb{CP}^n$ of radius L, we are within the injectivity radius, so the exponential map is a diffeomorphism. On sphere $S^{2n-1} \subset T_p\mathbb{CP}^n$ of radius L, the exponential map maps to the cut locus, i.e. the hyperplane at infinity. Take a projective line passing through p. It will strike the cut locus at a point q. Those of its tangent vectors at p of length L will all exponentiate to q, and nothing else will.

Because the projective line is a smooth sphere, its tangent vectors at p of length L form a circle. Because every tangent vector at p of length L must exponentiate to the cut locus, every tangent vector at p belongs to a unique one of these circles. Hence the sphere S^{2n-1} is fibered into circles. Each of these circles, being the intersection of S^{2n-1} with the tangent plane of a projective line, is a great circle, i.e. a geodesic circle on S^{2n-1}. In fact we can identify these circles: each projective line has as tangent plane a complex line $T_p\mathbb{CP}^1 \subset T_p\mathbb{CP}^n$. And every complex line in the tangent space at p is the tangent plane of a unique projective line. So the sphere S^{2n-1} is fibered into great circles which are the intersections $S^{2n-1} \cap \Lambda$ with complex lines Λ. This is precisely the Hopf fibration:

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n-1} \\ & & \downarrow \scriptstyle\text{Hopf} \\ & & \mathbb{CP}^{n-1}. \end{array}$$

Each of these great circles exponentiates to a single point of the cut locus, so $\mathbb{CP}^{n-1}$ = cut locus of p. Hence we see

$$\mathbb{CP}^n = \bar{B}^{2n} \cup \mathbb{CP}^{n-1}/\text{Hopf}$$

gluing together the boundary of the ball to the complex projective space, explicitly carried out by the exponential map.

Now we can repeat this story with $\mathbb{RP}^n, \mathbb{HP}^n$ and $\mathbb{OP}^n$. In each case, a the cut locus of a point is the projective hyperplane at infinity, and each point of it is reaching by a unique totally geodesic projective line. Recall

$$\mathbb{RP}^1 = S^1$$
$$\mathbb{CP}^1 = S^2$$
$$\mathbb{HP}^1 = S^4$$
$$\mathbb{OP}^1 = S^8$$

See table 1; the short form CROSS means *compact rank one symmetric space.* Analogy leads us to think of any sphere S^n as a projective line, with a hyperplane being just a point.

5. Ideas of Omori, Nakagawa & Shiohama

Recall that a *Blaschke manifold* is a Riemannian manifold with constant distance to the cut locus from each point in each direction.

Theorem 2 (Omori, Nakagawa & Shiohama)**.** *Let M^n be a Blaschke manifold, with diameter equal to injectivity radius of any point, say equal to L. Then L is also the distance between cut points along every geodesic. For any pair of points*

CROSS	**Hopf fibration**	**Construction**
$\mathbb{RP}^{n+1}$	$S^0 \longrightarrow S^n \xrightarrow{\pi} \mathbb{RP}^n$	$\mathbb{RP}^{n+1} = \bar{B}^{n+1} \cap \mathbb{RP}^n/\pi$
$\mathbb{CP}^{n+1}$	$S^1 \longrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n$	$\mathbb{CP}^{n+1} = \bar{B}^{2n+2} \cap \mathbb{CP}^n/\pi$
$\mathbb{HP}^{n+1}$	$S^3 \longrightarrow S^{4n+3} \xrightarrow{\pi} \mathbb{HP}^n$	$\mathbb{HP}^{n+1} = \bar{B}^{4n+4} \cap \mathbb{HP}^n/\pi$
$\mathbb{OP}^2$	$S^7 \longrightarrow S^{15} \xrightarrow{\pi} \mathbb{OP}^1 = S^8$	$\mathbb{OP}^2 = \bar{B}^{16} \cap \mathbb{OP}^1/\pi$
S^n	$S^n \longrightarrow S^n \xrightarrow{\pi} \bullet$	$S^n = \bar{B}^n \cap \bullet/\pi$

TABLE 1. Hopf fibrations

$p, q \in M$, either there is a unique shortest geodesic between them, of length less than L, or q belongs to the cut locus of p, and the shortest geodesics between p and q form a sphere, called a projective line. *Moreover, this projective line is orthogonal to the cut locus of p. All projective lines have the same dimension and are smoothly embedded. The unit tangent spheres of the projective lines at p for various choices of q form a smooth great sphere fibration*

$$\begin{array}{ccc} S^{k-1} & \longrightarrow & S^{n-1} \subset T_pM \\ & & \downarrow \\ & & Cut(p) \end{array}$$

A fortiori, the cut locus is a smooth submanifold.

The proof is long, but very elegant (particularly as found in Besse's book [1]). It consists in repeated applications of the first and second variation equations.

We will call these spheres of geodesics the *projective lines* of the Blaschke manifold, and call the cut loci the *hyperplanes* of the Blaschke manifold.

Model	**Adjective**	**Mathematician**
$\mathbb{RP}^n$	isometric	Berger
S^n	isometric	Berger, Kazdan, Weinstein, Yang
$\mathbb{CP}^n$	diffeomorphic	McKay
$\mathbb{HP}^2$	homeomorphic	Gluck, Warner, Yang
$\mathbb{OP}^2$	homeomorphic	Gluck, Warner, Yang

Table 2. Blaschke manifolds with model **Model** are **Adjective** to their model, by a theorem of **Mathematician**.

Theorem 3 (Browder)**.** *Every sphere fibration of a sphere is one of*

$$\begin{array}{ccc} S^0 & \longrightarrow & S^n \\ & & \downarrow \\ & & \mathbb{RP}^n \end{array}$$

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & X^{2n} \end{array}$$

$$\begin{array}{ccc} S^3 & \longrightarrow & S^{4n+3} \\ & & \downarrow \\ & & X^{4n} \end{array}$$

$$\begin{array}{ccc} S^7 & \longrightarrow & S^{15} \\ & & \downarrow \\ & & X^{8} \end{array}$$

$$\begin{array}{ccc} S^n & \longrightarrow & S^n \\ & & \downarrow \\ & & \bullet \end{array}$$

The possibilities for the spaces labelled X are as yet unclassified.

A Blaschke manifold M is said to be *modelled* on a compact rank one symmetric space M_0 if they have the same dimensions and their projective lines have the same dimensions (or equivalently, their hyperplanes have the same dimensions). By Browder's theorem, every Blaschke manifold has a unique model.

6. State of the art

For the sphere, try the embolic inequality of Berger & Kazdan:

$$\operatorname{Vol}(M^n) \geq \operatorname{Vol}(S^n) \left(\frac{\operatorname{Inj}(M)}{\pi}\right)^n$$

with equality only for the standard sphere. Then estimate the volume of a Blaschke manifold modelled on the sphere. None of this is easy.

For $\mathbb{CP}^n$, I took a very different approach. I followed Cartan's method of the moving frame, treating a great circle fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \pi \\ & & X^{2n} \end{array}$$

as a geometric structure on S^{2n+1}. I calculated its local differential invariants, and found that one of them is a map

$$X \to \mathbb{CP}^{2n+1}$$

which is an embedding. Pick a generic $\mathbb{CP}^n = \mathbb{CP}^n_0 \subset \mathbb{CP}^{2n+1}$, which I call a *hinge*. Then look at all $\mathbb{CP}^{n+1}$ which contain that hinge. Each of these $\mathbb{CP}^{n+1}$ turn out to strike X at a unique point transversely. So X is diffeomorphic to the set of $\mathbb{CP}^{n+1}$ containing the hinge. This set is parameterized by $\mathbb{CP}^n$. So X is diffeomorphic to $\mathbb{CP}^n$. Getting the great circle fibrations to match up

$$\begin{array}{ccccccc} S^1 & \longrightarrow & S^{2n+1} & \longrightarrow & S^{2n+1} & \longleftarrow & S^1 \\ & & \downarrow \pi & & \downarrow \text{Hopf} & & \\ & & X & \longrightarrow & \mathbb{CP}^n & & \end{array}$$

is not much trickier. So my great circle fibration turns out to be diffeomorphic to a Hopf fibration, and that gives

$$M = \bar{B}^{2n+2} \cap X/\pi \cong \bar{B}^{2n+2} \cap \mathbb{CP}^n/\text{Hopf} = M_0.$$

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