DERIVATION OF TWO DIMENSIONAL CONDUCTION SHAPE FACTORS: PART A- SOLVING THE POISSON EQUATION

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ABSTRACT

It is important in many heat transfer applications to use analytical expressions for conduction shape factors. As a first step towards deriving these analytical expressions, an analytical solution is obtained for the 2-D Poisson equation for a disk with two arbitrarily located circular holes. The solution obtained is shown to satisfy all of the three boundary conditions. It also shows good agreement when compared to a numerical solution obtained using the finite element method. It is discussed how the technique presented can be used to obtain an analytical solution to the 2-D Poisson equation for a disk with any number of arbitrarily located circular holes.

INTRODUCTION

Tissue convective energy balance equation or the *TCEBE* is a most recently derived, general bio-heat transfer equation which has a potential to estimate heat transfer from or to the tissue very accurately [1]. However, in order for this equation to implement it is necessary to derive and evaluate expressions for overall and countercurrent heat transfer coefficients. The first step towards deriving these necessary coefficients is to obtain a general analytical solution to the 2-D Poisson equation for a disk with any number of arbitrarily located circular holes.

Since a general analytical solution to the Poisson equation has several other industrial and biomedical heat transfer applications as well [e.g., 2-3], many researchers in the past have obtained exact solution of the Laplace equation for 1) a single hole imbedded in a finite medium [2- 6] and 2) two holes imbedded in an infinite medium [2, 8, 11]. The approximate series solution to the Laplace equation for two holes imbedded in a finite medium have also been obtained [3, 4, 8-11]. However, the exact series solution to the Poisson equation in a finite medium with any number of arbitrarily located holes has never been presented.

To fulfil this gap, an exact series solution is presented of the 2-D energy equation with a uniformly distributed source for a disk with two arbitrarily located holes. The technique developed here to obtain the solution to the 2-D Poisson equation is general and it is discussed how it can be employed to obtain a similar solution to the 2-D energy equation with a uniformly distributed source for a disk with any number of arbitrarily located holes.

Also, in most of the industrial and bio-medical applications, the boundary conditions needed at the disk or hole boundaries are one of the three types: 1) constant temperature, 2) convective boundary condition with uniform convective heat transfer coefficients, or 3) an insulated boundary [3-4, 8-11]. Since the solution obtained using constant temperatures at all of the boundaries can be modified using the convective heat transfer coefficients to include convective boundary conditions or insulation (by taking convective heat transfer coefficients as 0), the solution presented only uses constant temperatures at all of the boundaries as the boundary conditions.

NOMENCLATURE

A_{01}, A_{01}'	constants
A_{02}, A_{02}'	constants
$a_{1,i}$	constants, i = 1, 2
$a_{2,i}$	constants, i = 1, 2
A_{n1}, A_{n1}'	constants
A_{n2}, A_{n2}'	constants
$a_{v,i}$	dimensional distance between the center of the cylinder and the center of the i^{th} hole, i = 1, 2
B_{n1}, B_{n1}'	constants
B_{n2}, B_{n2}'	constants
g'''	uniform power deposited in the cylinder per unit volume
k	conductivity of the cylinder
r	radial distance
$r_{v,i}$	radius of the i^{th} hole, i = 1, 2
r_t	outer radius of the cylinder
t_{bl}	a reference temperature
$t_{v,i}$	temperature at the outer surface of the wall of the i^{th} hole, i = 1, 2
t_t	temperature at the outer boundary of the cylinder

Non-dimensional Parameters

$A_{v,i}$ non-dimensional distance between the center of the cylinder and the center of the i^{th} hole, i = 1, 2

R non-dimensional radius, r/r_t

R_i^* non-dimensional radius in the conformally mapped space

$R_{v,i}$ non-dimensional radius of the i^{th} hole, i = 1, 2

T non-dimensional temperature, $(t - t_{bl})/(t_t - t_{bl})$

Z non-dimensional power deposition, $gr_t^2/[(t_t - t_{bl})k]$

Greek Symbols

θ_i angular position measured from the center of the i^{th} hole in the original coordinate system, i = 1, 2

α_i angular position measured from the center of the i^{th} hole in the transformed coordinate system, i = 1, 2

λ_i constant, i = 1, 2

ψ angular position measured from the center of the disk

ϕ_i angular position of center of the $(i+1)^{th}$ hole from the center of the disk

SOLUTION

To obtain the temperature field in a 2-D disk with two arbitrarily located circular holes, uniform conductivity and uniform source term, we impose equation (1) in the interior and boundary condition equations (2) - (4) (Figure 1).

$$\frac{1}{r}\frac{d}{dr}(r\frac{dt}{dr})+\frac{1}{r^2}\frac{d^2t}{d\psi^2}+\frac{g'''}{k}=0 \tag{1}$$

$$t\big|_{r_1} = t_{v,1} \tag{2}$$

$$t\big|_{r_2} = t_{v,2} \tag{3}$$

$$t\big|_{r_t} = t_t \tag{4}$$

where,

$$r_1 = \{(a_{v,1} + r_{v,1}\cos\theta_1)^2 + (r_{v,1}\sin\theta_1)^2\}^{1/2} \tag{5}$$

$$r_2 = \{(a_{v,2} + r_{v,2}\cos\theta_2)^2 + (r_{v,2}\sin\theta_2)^2\}^{1/2} \tag{6}$$

In non-dimensional form, equations (1) to (6) can be rewritten as,

$$\frac{1}{R}\frac{d}{dR}(R\frac{dT}{dR})+\frac{1}{R^2}\frac{d^2T}{d\psi^2}+Z=0 \quad (7)$$

$$T\big|_{R_1}=T_{v,1} \quad (8)$$

$$T\big|_{R_2}=T_{v,2} \quad (9)$$

$$T\big|_{1}=1 \quad (10)$$

where,

$$R_1=\{(A_{v,1}+R_{v,1}\cos\theta_1)^2+(R_{v,1}\sin\theta_1)^2\}^{1/2} \quad (11)$$

$$R_2=\{(A_{v,2}+R_{v,2}\cos\theta_2)^2+(R_{v,2}\sin\theta_2)^2\}^{1/2} \quad (12)$$

Since Laplace equations can be worked with in conformally mapped space, to convert equation (7) into a Laplace equation, the following substitution is made.

$$T=T_1-ZR^2/4 \quad (13)$$

Equation (13), when substituted into equations (7) - (10) gives equations (14) - (17), respectively.

$$\frac{1}{R}\frac{d}{dR}(R\frac{dT_1}{dR})+\frac{1}{R^2}\frac{d^2T_1}{d\psi^2}=0 \quad (14)$$

$$T_1\big|_{R_1}=T_{v,1}+ZR_1{}^2/4 \quad (15)$$

$$T_1\big|_{R_2}=T_{v,2}+ZR_2{}^2/4 \quad (16)$$

$$T_1\big|_{1}=1+Z/4 \quad (17)$$

The problem defined by equations (14) - (17) can be rewritten as the superposition of the following two sub-problems,

Sub-problem 1-

$$\frac{1}{R}\frac{d}{dR}(R\frac{dT_{11}}{dR})+\frac{1}{R^2}\frac{d^2T_{11}}{d\psi^2}=0 \qquad (18)$$

$$T_{11}\big|_{R_1}=T_{v,1}+ZR_1{}^2/4-c_{21} \qquad (19)$$

$$T_{11}\big|_1=1+Z/4 \qquad (20)$$

Sub-problem 2-

$$\frac{1}{R}\frac{d}{dR}(R\frac{dT_{12}}{dR})+\frac{1}{R^2}\frac{d^2T_{12}}{d\psi^2}=0 \qquad (21)$$

$$T_{12}\big|_{R_2}=T_{v,2}+ZR_2{}^2/4-c_{12} \qquad (22)$$

$$T_{12}\big|_1=0 \qquad (23)$$

and

$$T_1 = T_{11} + T_{12} \qquad (24)$$

where, c_{ij} is the temperature distribution produced on the boundary of the j^{th} hole by the boundary of the i^{th} hole in sub-problem i. This procedure replaces the original problem of a disk with two eccentric holes into two problems, each consisting of a disk with one eccentric hole. The value of c_{ij} is not yet known. To simplify the problem even further, techniques of conformal mapping can be used on both sub-problems to make the holes and the disk concentric [11]. The following transformation is used for sub-problem 1 to make hole 1 and the disk concentric.

$$w_1 = (Z_1 - \lambda_1)/(1- \lambda_1 Z_1) \qquad (25)$$

The mapping defined by equation (25) relates x and y coordinates of the original coordinate system to the new coordinates U_1 and V_1 as follows,

$$U_1 = \{(x - \lambda_1)\,(1- \lambda_1 x) - \lambda_1 y^2\}/\{(1- \lambda_1 x)^2 - (\lambda_1 y)^2\} \qquad (26)$$

$$V_1 = \{(1 - \lambda_1{}^2)\, y\}/\{(1- \lambda_1 x)^2 - (\lambda_1 y)^2\} \qquad (27)$$

where,

$$\lambda_1 = \{1 + a_{11}a_{21} - ((1 - a_{11}^2)(1 - a_{21}^2))^{1/2}\}/(a_{11} + a_{21}) \tag{28}$$

$$a_{11} = A_{v,1} - R_{v,1} \tag{29}$$

$$a_{21} = A_{v,1} + R_{v,1} \tag{30}$$

After the conformal mapping, the modified sub-problem 1 can be written as,

$$\frac{1}{R_1^*}\frac{d}{dR_1^*}(R_1^*\frac{dT_{11}}{dR_1^*}) + \frac{1}{(R_1^*)^2}\frac{d^2T_{11}}{d\alpha_1^2} = 0 \tag{31}$$

with,

$$T_{11}\Big|_1 = 1 + Z/4 \tag{32}$$

The general solution to equation (31) is:

$$T_{11} = A_{01} + A_{01}' \ln(R_1^*) + \sum_{n=1}^{\infty}\{A_{n1}(R_1^*)^n + A_{n1}'(R_1^*)^{-n}\}\sin(n\alpha_1) + \sum_{n=1}^{\infty}\{B_{n1}(R_1^*)^n + B_{n1}'(R_1^*)^{-n}\}\cos(n\alpha_1) \tag{33}$$

where,

$$R_1^* = (U_1^2 + V_1^2)^{1/2} \tag{34}$$

$$\alpha_1 = \tan^{-1}(V_1/U_1) \tag{35}$$

Using equations (32) and (33) and the orthogonality of sine and cosine functions, the following relations can easily be derived.

$$A_{01} = 1 + Z/4 \tag{36}$$

$$A_{n1} = - A_{n1}' \tag{37}$$

$$B_{n1} = - B_{n1}' \tag{38}$$

Using equations (36), (37) and (38), equation (33) can be rewritten as follows.

$$T_{11} = 1 + Z/4 + A_{01}{}' \ln(R_1{}^*) + \sum_{n=1}^{\infty} A_{n1}\{(R_1{}^*)^n - (R_1{}^*)^{-n}\}\sin(n\alpha_1) +$$
$$\sum_{n=1}^{\infty} B_{n1}\{(R_1{}^*)^n - (R_1{}^*)^{-n}\}\cos(n\alpha_1) \tag{39}$$

Similarly, for sub-problem 2, the following solution can be obtained after a conformal transformation similar to equation (25) and by applying the second boundary condition (equation (23)) in the transformed plane.

$$T_{12} = A_{02}{}' \ln(R_2{}^*) + \sum_{n=1}^{\infty} A_{n2}\{(R_2{}^*)^n - (R_2{}^*)^{-n}\}\sin(n\alpha_2) +$$
$$\sum_{n=1}^{\infty} B_{n2}\{(R_2{}^*)^n - (R_2{}^*)^{-n}\}\cos(n\alpha_2) \tag{40}$$

where,

$$R_2{}^* = (U_2{}^2 + V_2{}^2)^{1/2} \tag{41}$$

$$\alpha_2 = \tan^{-1}(V_2/U_2) \tag{42}$$

$$U_2 = \{(x' - \lambda_2)(1 - \lambda_2 x') - \lambda_2(y')^2\}/\{(1 - \lambda_2 x')^2 - (\lambda_2 y')^2\} \tag{43}$$

$$V_2 = \{(1 - \lambda_2{}^2)\, y'\}/\{(1 - \lambda_2 x')^2 - (\lambda_2 y')^2\} \tag{44}$$

$$\lambda_2 = \{1 + a_{12}a_{22} - \mathrm{sqrt}((1 - a_{12}{}^2)(1 - a_{22}{}^2))\}/(a_{12} + a_{22}) \tag{45}$$

$$a_{12} = A_{v,2} - R_{v,2} \tag{46}$$

$$a_{22} = A_{v,2} + R_{v,2} \tag{47}$$

$$x' = x\, cos(\phi_1) + y\, sin(\phi_1) \tag{48}$$

$$y' = y\, cos(\phi_1) - x\, sin(\phi_1) \tag{49}$$

The complete solution using equation (24) can be written as,

$$T_1 = 1 + Z/4 + A_{01}{}' \ln(R_1{}^*) + A_{02}{}' \ln(R_2{}^*) + \sum_{n=1}^{\infty} A_{n1}\{(R_1{}^*)^n - (R_1{}^*)^{-n}\}\sin(n\alpha_1) +$$
$$\sum_{n=1}^{\infty} A_{n2}\{(R_2{}^*)^n - (R_2{}^*)^{-n}\}\sin(n\alpha_2) + \sum_{n=1}^{\infty} B_{n1}\{(R_1{}^*)^n - (R_1{}^*)^{-n}\}\cos(n\alpha_1) +$$
$$\sum_{n=1}^{\infty} B_{n2}\{(R_2{}^*)^n - (R_2{}^*)^{-n}\}\cos(n\alpha_2) \tag{50}$$

Since the values of c_{ij} in equations (19) and (22) are not known, the expression shown in equation (50) should be transformed back into the original coordinates using equations (26-30), (34-35) and (41-49) to evaluate the complete set of 2(2n+1) constants. It can be seen that a system of 2(2n+1) independent linear equations consisting of 2(2n+1) constants can be formed by using equations (15) and (11) together with equations (16) and (12). It is easily found using the orthogonality of sine and cosine functions that all of the coefficients attached to sine terms are zero. This leaves us a system of 2(n+1) equations with the same number of unknowns. The general solution takes the following form.

$$T_1 = 1 + Z/4 + A_{01}{}' \ln(R_1{}^*) + A_{02}{}' \ln(R_2{}^*) + \sum_{n=1}^{\infty} B_{n1}\{(R_1{}^*)^n - (R_1{}^*)^{-n}\}\cos(n\alpha_1) +$$
$$\sum_{n=1}^{\infty} B_{n2}\{(R_2{}^*)^n - (R_2{}^*)^{-n}\}\cos(n\alpha_2) \tag{51}$$

The obtained system of 2(n+1) independent equations can easily be solved for 2(n+1) unknown coefficients using the available numerical linear equation solvers.

RESULTS

An example problem is solved below.

Given: r_t = 1 cm, $r_{v,1}$ = 0.2 cm, $r_{v,2}$ = 0.3 cm, $a_{v,1}$ = -0.6 cm, $a_{v,2}$ = 0.4 cm, $t_{v,1}$ = 37.25 ^{0}C, $t_{v,2}$ = 37.50 ^{0}C, t_t = 38 ^{0}C, t_{bl} = 37 ^{0}C, g''' = 2 watts/cm^3, ϕ_1 = 0.

Therefore, R_t = 1, $R_{v,1}$ = 0.2, $R_{v,2}$ = 0.3, $A_{v,1}$ = -0.6, $A_{v,2}$ = 0.4, $T_{v,1}$ = 0.25, $T_{v,2}$ = 0.50, T_t = 1, Z = 2. A matlab script is written which uses the mentioned values to calculate all the constants for equation (51). The values obtained using the analytical expression at the three boundaries are plotted in Figures 2-4 to show how well the boundary conditions are matched. The results obtained using the analytical expression are also compared to the Ansys results (commercial finite element software) at five locations (Figure 1). The relative errors found are presented in Table 1.

It can be realized from equation (51) that every other hole adds (n+1) constants to the series solution. However, as may be evident from the presented solution technique that the evaluation of these extra constants is not difficult. The main problem of one disk with N holes can be divided into N sub-problems of one disk and one hole each and superposition of the boundary condition at the outer cylinder can be used to obtain the relationship among various constants. The complete solution then can be inverted back into the original coordinates and the boundary condition at each of the N holes can be used to obtain a set of N(n+1) independent equations for the same number of unknown coefficients. This set of N(n+1) independent equations can easily be solved for the N(n+1) unknown coefficients by any available linear equation solver.

CONCLUSION

A new and simple procedure to obtain the solution of Poisson equation in a disk with two arbitrary holes is demonstrated. The solution obtained is shown to match all of the three boundary conditions very closely. The solution obtained when compared to the corresponding finite element solution is also found to be accurate. It is discussed how the same analysis can be extended to obtain solutions to the 2-D Poisson equation for a disk with N arbitrarily located holes.

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List of Tables

Table 1 Comparison between the values obtained using the solution and Ansys for n = 10

Region	x	y	Ansys result	Analytical solution	% error
I	-0.08990	0.52035	0.87692	0.87691	0.00114
II	-0.91695	0.00284	0.74265	0.74266	-0.00135
III	0.81942	0.01936	0.75934	0.75936	-0.00263
IV	-0.05448	-0.57512	0.90957	0.90957	0
V	-0.08132	0.01678	0.58739	0.58743	-0.00681

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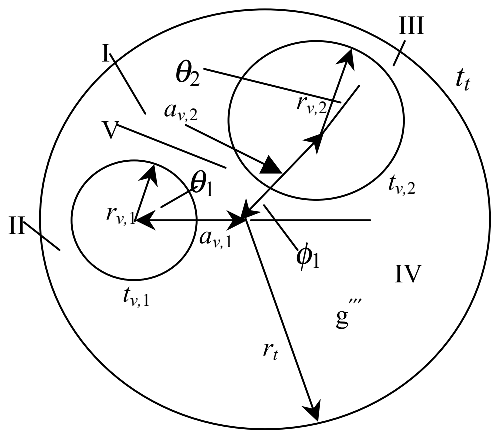

Figure 1 Schematic of the disk with two unequal arbitrarily located holes

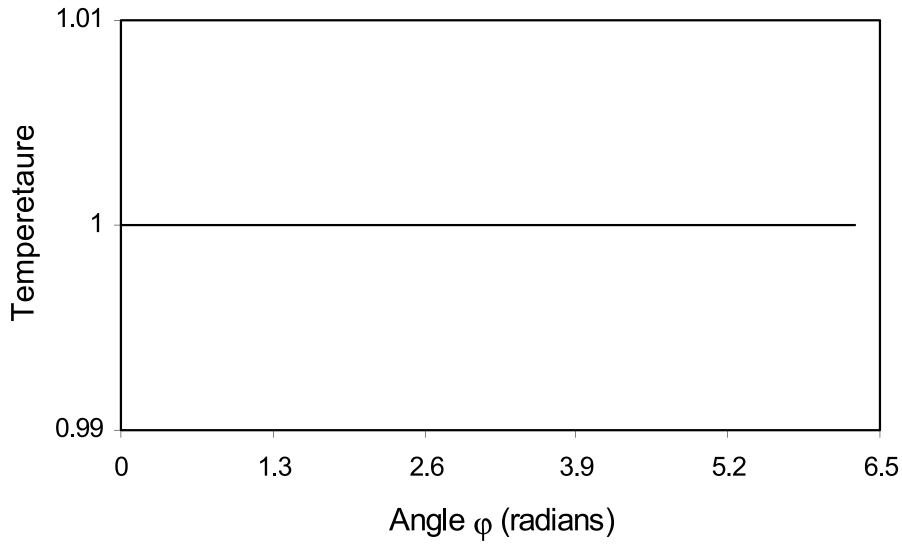

Figure 2 Comparison between the specified and determined boundary conditions at the outer wall for n = 5

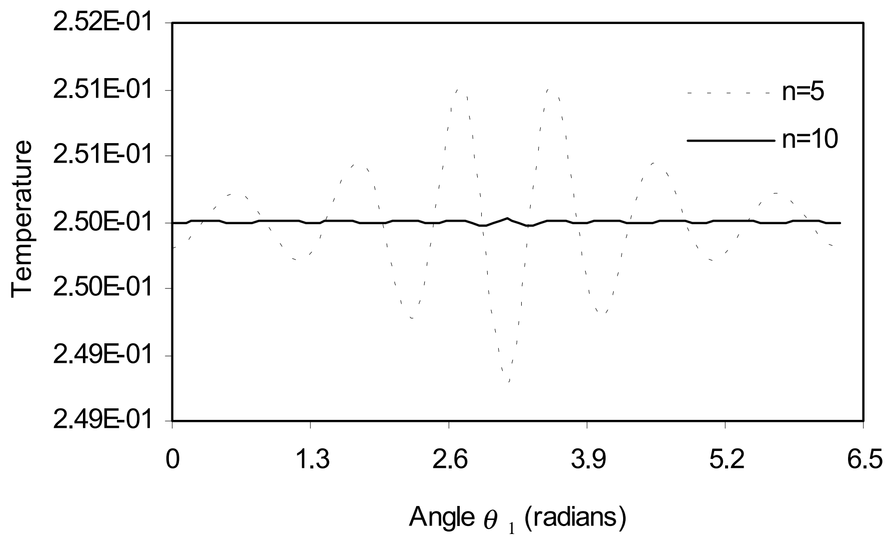

Figure 3 Comparison between the specified and determined boundary condition for hole 1 for n = 5 and 10

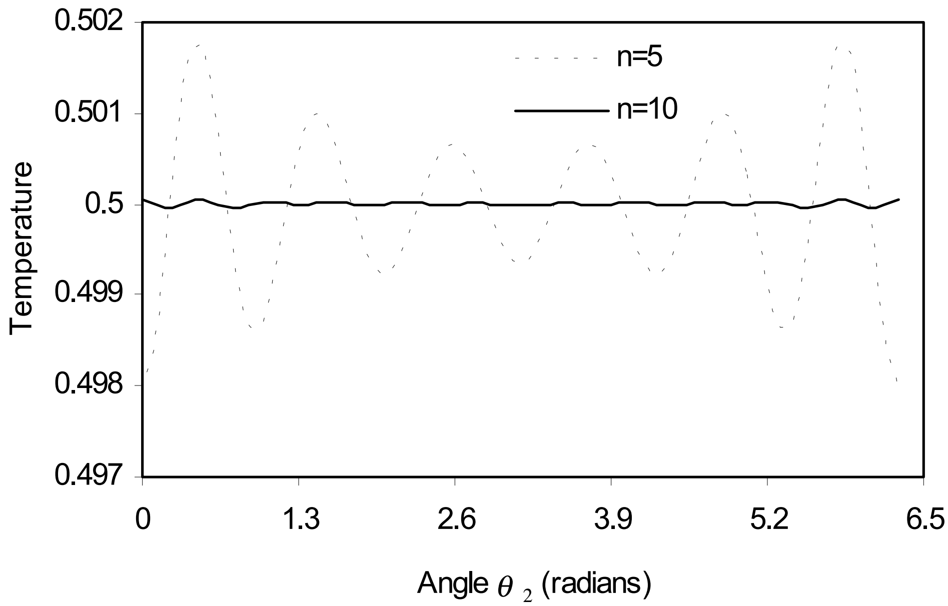

Figure 4 Comparison between the specified and determined boundary condition for hole 2 for n = 5 and 10