

SURFACES WITH FLAT NORMAL BUNDLE IN SPACE FORMS

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ABSTRACT. We consider how much freedom there is to construct isometric immersions of a surface, so that the immersion has flat normal bundle.

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1. INTRODUCTION

1.1. The problem. Given a surface M with a Riemannian metric, and a space form \mathbb{S} , an isometric immersion of M into \mathbb{S} has a Levi-Civita connection on its tangent bundle, and also on its normal bundle. Can we isometrically immerse with flat normal bundle, and if so in how many ways? This problem is not interesting if $\dim \mathbb{S} = 3$, since the normal bundle will be a real line bundle, and therefore flat, and since it is also well known that there is in general no isometric immersion of a surface into a 3-dimensional space form (see [1, 11]). So we assume that $\dim \mathbb{S} \geq 4$. Trivially, taking a surface in a 3-dimensional space form, one embeds the 3-dimensional space form into a higher dimensional space form as a totally geodesic subspace, giving examples of all genera. We will see that most surfaces with flat normal bundle do not come about from this construction.

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1.2. The solution. Our main discovery is that the isometric embedding problem becomes determined if we add the condition of having flat normal bundle. Recall that the theory of isometric embedding is made difficult in high codimension because the problem is underdetermined. Naturally one would like to find geometric conditions to add which lead to determinacy. At least for surfaces, we have found a simple geometric condition: flatness of the normal bundle.

Theorem 1. *Let M be a surface with real analytic Riemannian metric, and \mathbb{S}^{n+2} a Riemannian manifold of constant sectional curvature of dimension $n+2 \geq 4$. Let $C \subset M$ be a real analytic embedded curve. Then there is an isometric immersion of a neighborhood of C in M , which has flat normal bundle. The general isometric embedding of a neighborhood of C with flat normal bundle depends on $2n$ functions of 1 variable, where $n = \dim \mathbb{S} - \dim M$.*

Theorem 2. *Every surface with smooth Riemannian metric admits a local isometric embedding with flat normal bundle into any space form \mathbb{S}^{n+2} (for $n \geq 2$), in a neighborhood of any point.*

I have not succeeded in understanding general topological types, but only cylinders:

Theorem 3. *Let M be a smooth surface, diffeomorphic to a cylinder $S^1 \times [0, 1]$. On one of the boundary components of M , say C , pick a point $m_0 \in C$. Fix a point $s_0 \in \mathbb{S}$, a 2-plane $P_0 \subset T_{s_0}\mathbb{S}$, and a linear immersion $\phi_0 : T_{m_0}M \rightarrow P_0$.*

*To each immersion $\phi : M \rightarrow \mathbb{S}$ satisfying $\phi(m_0) = s_0$ and $\phi'(m_0) = \phi_0$, with trivial normal holonomy along the loop C , define a section $A = A(\phi)$ of the bundle $\text{Sym}^2(T^*M)|_C \otimes P_0^\perp$, by taking*

$$\Pi_C A = a,$$

with a the shape operator of the immersion. Consider the map $\phi \mapsto (g(\phi), A(\phi))$ taking an immersed cylinder to its induced metric g and section A of this vector bundle. This map is a tame bundle map, with section defined on a nonempty open set of the base by asking $\phi(g, A)$ to have flat normal bundle, hyperbolic or mixed shape operator, trivial normal holonomy along the chosen boundary component C , and bicharacteristics passing entirely from one boundary component to the other. If we build a family of space form metrics g_κ on a fixed space \mathbb{S} , with sectional curvature of the metric g_κ being κ , then the same result holds for the map $(\phi, \kappa) \mapsto (g(\phi), A(\phi), \kappa)$.

1.3. Review of the literature. Terng [14] unveils some of the local geometric properties of submanifolds of space forms with flat normal bundle. Ferapontov [7] has a construction using integrable systems techniques which builds surfaces with flat normal bundle in space forms, and every surface with flat normal bundle locally arises from his construction. This construction was recently generalized to all dimensions by Dajczer, Florit & Tojeiro [4]. However, it is unclear how to use his construction to arrive at a surface with a prescribed Riemannian metric, even locally, or how to deform surfaces of prescribed topology. He also gives references for the history of the study of surfaces with flat normal bundle. As Ferapontov points out:

- (1) If a surface $M \subset \mathbb{S}$ has flat normal bundle, then so does the surface obtained by transporting a normal vector in parallel over M , and then applying the

Riemannian exponential map; Terng [14] calls this a *Bonnet transformation*, while Ferapontov calls it a *normal shift*.

- (2) Conformal local diffeomorphisms of one space form to another take surfaces with flat normal bundle to other such surfaces.
- (3) The normal Gauss map, taking a surface with flat normal bundle in Euclidean space \mathbb{E}^{n+2} to the surface in the unit sphere S^{n+1} given by parallel transport of a unit normal vector $v \in S^{n+1}$, gives a surface with flat normal bundle.
- (4) Given a surface $M \subset S^{n+1}$ with flat normal bundle in a sphere, identify the sphere with the unit tangent vectors at a point of a space form \mathbb{S}^{n+2} , and exponentiate to obtain a surface with flat normal bundle.

2. STRUCTURE EQUATIONS

Let $\mathbb{S} = \mathbb{S}^n(\kappa)$ be a space form, i.e. a Riemannian manifold of constant sectional curvature κ , of dimension $n + d$. Let $\pi : F\mathbb{S} \rightarrow \mathbb{S}$ be the bundle of oriented orthonormal frames, a principal right $\mathrm{SO}(n + d)$ -bundle. Let \mathfrak{g} be the Lie algebra of local infinitesimal isometries of \mathbb{S} . We may identify \mathfrak{g} with the Lie algebra of matrices of the form

$$\begin{pmatrix} 0 & -\kappa A_b \\ A_a & A_{ab} \end{pmatrix}$$

(with indices $a, b = 1, \dots, n + d$), for which

$$A_{ab} + A_{ba} = 0.$$

We embed $\mathfrak{so}(n + d)$ into \mathfrak{g} as the lower left corner. Define 1-forms $\omega_a \in \Omega^1(F\mathbb{S})$ by

$$v \lrcorner \omega_a = \langle e_a, \pi'(e)v \rangle$$

for $v \in T_e F\mathbb{S}$, $e = (e_a) \in F\mathbb{S}$. The 1-forms ω_a are semibasic for the map $F\mathbb{S} \rightarrow \mathbb{S}$, and form a basis for the semibasic 1-forms. There is a unique Cartan connection ω satisfying Cartan's equations of structure:

$$d\omega = -\frac{1}{2}[\omega, \omega]$$

with $\omega \in \Omega^1(F\mathbb{S}) \otimes \mathfrak{g}$ and

$$\omega = \begin{pmatrix} 0 & -\kappa \omega_b \\ \omega_a & \omega_{ab} \end{pmatrix}.$$

For $g \in \mathrm{SO}(n + d)$, write $r_g : F\mathbb{S} \rightarrow F\mathbb{S}$ for the right action. Then

$$r_g^* \omega = \mathrm{Ad}_g^{-1} \omega.$$

For any $A \in \mathfrak{g}$, define a vector field \vec{A} on $F\mathbb{S}$ by the equation $\vec{A} \lrcorner \omega = A$.

Given an isometric immersion of a submanifold $\phi : M^d \rightarrow \mathbb{S}$, we define the adapted frame bundle $F_M \mathbb{S}$ to be the bundle of choices of a point $m \in M$, an adapted frame E_i on $T_m M$, and an adapted frame (e_i, e_μ) on $T_{\phi(m)} \mathbb{S}$, for which $\phi'(m)E_i = e_i$. Thus $F_M \mathbb{S} \subset FM \times F\mathbb{S}$. We can write $F_M \mathbb{S}$ as $F_\phi \mathbb{S}$ to emphasise the role of the map ϕ . If we write the structure equations of FM (with indices $i, j, k, l = 1, \dots, d$ and $\mu, \nu, \sigma, \tau = d + 1, \dots, d + n$) as

$$d \begin{pmatrix} \eta_{ij} & \eta_j \\ 0 & 0 \end{pmatrix} = - \begin{pmatrix} \eta_{ik} & \eta_k \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \eta_{kj} & \eta_j \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} r_{ijkl} & 0 \\ 0 & 0 \end{pmatrix} \eta_j \wedge \eta_k,$$

then we see that $F_M\mathbb{S} \subset FM \times F\mathbb{S}$ is an integral manifold of the exterior differential system $\omega_\mu = 0, \omega_i = \eta_i, \omega_{ij} = \eta_{ij}$.

Lemma 1. *Every integral manifold of the exterior differential system*

$$\mathcal{I} = (\omega_\mu = 0, \omega_i = \eta_i, \omega_{ij} = \eta_{ij})$$

on which

$$0 \neq \bigwedge_i \omega_i \wedge \bigwedge_{i < j} \omega_{ij} \wedge \bigwedge_{\mu < \nu} \omega_{\mu\nu}$$

is locally the bundle of adapted frames $F_M\mathbb{S}$ of a unique isometric immersion $\phi : M \rightarrow \mathbb{S}$.

Proof. Clearly for an immersed M , $F_M\mathbb{S}$ is such an integral manifold. Conversely, given such an integral manifold $\Lambda \subset FM \times F\mathbb{S}$, map $\Lambda \rightarrow FM \times F\mathbb{S} \rightarrow M$. The fibers in Λ of this map are the integral manifolds of $\omega_i = 0$. These form a foliation of Λ , since the equations satisfy the Frobenius theorem. Suitably shrinking the choice of Λ , we can arrange that this foliation is a fiber bundle, say $\Lambda \rightarrow M$. The map descends to

$$\begin{array}{ccc} \Lambda & \longrightarrow & F\mathbb{S} \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathbb{S}. \end{array}$$

Since $\bigwedge_i \omega_i \neq 0$ on Λ , and this is a semibasic n -form for $F\mathbb{S} \rightarrow \mathbb{S}$, the map $M \rightarrow \mathbb{S}$ is an immersion. Write $\pi_M : FM \rightarrow M$ and $\pi_{\mathbb{S}} : F\mathbb{S} \rightarrow \mathbb{S}$. At any point $e \in \Lambda$, say $e = (E_i, e_i, e_\mu)$, we have $\omega_\mu = 0$ so any tangent vector $v \in T_e\Lambda$ satisfies $\langle e_\mu, \pi_{\mathbb{S}}'(e)v \rangle = 0$, and $\langle e_i, \pi_{\mathbb{S}}'(e)v \rangle = \langle E_i, \pi_M'(e)v \rangle$. But therefore for any vector $w \in T_mM$, pick any vector $v \in T_e\Lambda$ so that $\pi_M'(e)v = w$, and we find $\langle e_\mu, \phi(m)'w \rangle = 0$, and $\langle e_i, \phi(m)'w \rangle = \langle E_i, w \rangle$, so $e = (E_i, e_i, e_\mu)$ is an adapted frame for T_mM , and thus $\Lambda \subset F_M\mathbb{S}$. \square

Definition 1. The parallel transport of a frame $e \in F_M\mathbb{S}$ along a smooth immersed curve $x : [a, b] \rightarrow M$ is defined by taking the pullback bundle $x^*F_M\mathbb{S}$, and taking the integral curve through e of the exterior differential system $0 = \omega_{ij} = \omega_{\mu\nu}$, defining $\Pi_x e$ to be the end point in $x^*F_M\mathbb{S}$ of that integral curve. It is easy to see that this is well defined (by, for instance, the Frobenius theorem), and that $\Pi_x r_g e = r_g \Pi_x e$, by equivariance of the exterior differential system. Therefore $\Pi_x : T_m\mathbb{S} \rightarrow T_m\mathbb{S}$ is well defined, an orthogonal transformation, and preserves the tangent plane T_mM and the normal plane N_mM . We define the tangent [normal] holonomy group of M at a point $m \in M$ to be the group of transformations of T_mM [N_mM] generated by the Π_x for all loops x starting and ending at m .

By the Sussman orbit theorem [13], $F_M\mathbb{S}$ is foliated by the orbits of the vector fields \vec{A} for

$$A = \begin{pmatrix} 0 & 0 \\ A_a & 0 \end{pmatrix},$$

and clearly the orbits contain the tangent directions to all of the integral manifolds of the exterior differential system, so contain the integral manifolds. Sussmann easily shows that each orbit of that family of vector fields is a principal bundle over M , with fiber at each point the orbit of the holonomy group.

Lemma 2. *An isometric immersion $\phi : M \rightarrow \mathbb{S}$ has flat normal bundle just when the exterior differential system $\omega_{\mu\nu} = 0$ on $F_M\mathbb{S}$ is holonomic (i.e. satisfies the conditions of the Frobenius theorem). Conversely, any integral manifold of the exterior differential system $0 = \omega_\mu = \omega_{\mu\nu}$ on which*

$$0 \neq \bigwedge_i \omega_i \wedge \bigwedge_{i < j} \omega_{ij}$$

is locally a leaf of the exterior differential system $\omega_{\mu\nu} = 0$ on the bundle $F_M\mathbb{S}$ of adapted frames of a unique immersed submanifold M with flat normal bundle.

Proof. Let $\Lambda \subset FM \times FS$ be an integral manifold. Using the same argument as in lemma 1 on the facing page, we find that, after possibly replacing Λ by a nonempty open subset, $\Lambda \subset F_M\mathbb{S}$ for a unique immersed submanifold $M \subset \mathbb{S}$. Thus Λ is an integral manifold of the exterior differential system $0 = \omega_{\mu\nu}$ on $F_M\mathbb{S}$. \square

3. PROLONGATION AND SHAPE OPERATORS

Henceforth we stick with $d = 2$, i.e. we try to immersion only *surfaces* into space forms.

Lemma 3. *The prolongation $\mathcal{I}^{(1)}$ of the exterior differential system \mathcal{I} for isometric immersions of surfaces with flat normal bundle is the system*

$$0 = \omega_i - \eta_i = \omega_\mu = \omega_{\mu\nu} = \omega_{j\mu} - a_{j\mu k} \eta_k = [a_\mu, a_\nu] = \sum_\mu \det a_\mu - (G - \kappa),$$

where

$$a_\mu = \begin{pmatrix} a_{1\mu 1} & a_{1\mu 2} \\ a_{2\mu 1} & a_{2\mu 2} \end{pmatrix}$$

are 2×2 symmetric matrices parameterizing the integral elements.

Proof. Differentiating the equations of the original exterior differential system reveals

$$\omega_{j\mu} \wedge \eta_j = 0,$$

so that on integral elements

$$\omega_{j\mu} = a_{j\mu k} \eta_k,$$

for some unique $a_{j\mu k}$ symmetric in j, k , and immediately reveals the other equations. \square

For any constant G_0 , let $V_{G_0} \subset \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$ be the real algebraic variety of $a = (a_{i\mu j}) \in \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$ for which

$$[a_\mu, a_\nu] = 0 \text{ and } \sum_\mu \det a_\mu = G_0 - \kappa.$$

Now let $Z^{(0)} = FM \times FS$ be the manifold on which the original exterior differential system \mathcal{I} was set, and let $Z' \subset Z^{(0)} \times (\text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n)$ be the set of points (e, a) satisfying the equations of the prolongation. For each $e \in Z^{(0)}$, the fiber of $Z' \rightarrow Z^{(0)}$ is $V_G \subset \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$ (which depends only on the Gauss curvature G of the surface M at the underlying point of M). Let $Z^{(1)}$ be the union of the smooth points of those varieties. Therefore $Z^{(1)}$ is a manifold, bearing the exterior differential system $\mathcal{I}^{(1)}$ consisting in the equations:

$$\omega_{j\mu} = a_{j\mu k} \eta_k, \omega_{j\mu} = a_{j\mu k} \eta_k, \omega_{j\mu} = a_{j\mu k} \eta_k, \omega_{j\mu} = a_{j\mu k} \eta_k, \omega_{j\mu} = a_{j\mu k} \eta_k,$$

(the prolongation), and $Z^{(1)} \rightarrow Z^{(0)}$ is a smooth submersion.

Definition 2. An element $a \in V_G$ is called a G -admissible shape operator, and *admissible* if G -admissible for some number G ,¹ G -regular if it is a smooth point of the variety V_G , *regular* if it is admissible, and there is some $\xi \in \mathbb{R}^n$ for which the quadratic form $\langle \xi, a \rangle \in \text{Sym}^2(\mathbb{R}^{2*})$ has two distinct eigenvalues; irregular shape operators will also be called *umbilic*. A shape operator is called *normalized* if

$$a_3 = \begin{pmatrix} a_{131} & 0 \\ 0 & a_{232} \end{pmatrix}, a_4 = \begin{pmatrix} 0 & 0 \\ 0 & a_{242} \end{pmatrix}, a_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots, a_{n+2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Lemma 4. *Every admissible shape operator $a \in \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$ becomes normalized by some element of $\text{SO}(2) \times \text{SO}(n)$.*

Proof. Since the a_μ matrices must commute, they are all simultaneously diagonalizable, and so we can assume that they are all diagonal. There is a 2 dimensional space of diagonal matrices, so at most a 2 dimensional space of vectors $a_{i\mu j} v^i w^j$. We arrange by action of $\text{SO}(n)$ that this space lies in the image of the a_3, a_4 matrices, so that $a_\mu = 0$ for $\mu > 4$. Then we can use the $\text{SO}(n)$ action again to rotate the vector $\eta = (\eta_\mu) = (a_{1\mu 1}) \in \mathbb{R}^{n+2}$ to have $a_{1\mu 1} = 0$ for $\mu > 1$, and $a_{131} \geq 0$. At this stage, we can still rotate the vector $\zeta = (\zeta_\mu) = (a_{2\mu 2})$ in the plane perpendicular to η , to arrange $a_{242} \geq 0$. \square

Lemma 5. *A normalized shape operator is G -regular (i.e. a point of the smooth part of V_G) just when it is regular.*

Proof. Write out the equations of the variety V_G :

$$\begin{aligned} 0 &= \sum_{\mu} \det a_{\mu} - (G - \kappa) \\ 0 &= [a_{\mu}, a_{\nu}] \end{aligned}$$

explicitly, and differentiate. Then plug in the normalization conditions, and see the equations

$$\begin{aligned} 0 &= a_{232} da_{131} + a_{131} da_{232} + a_{242} da_{141} \\ 0 &= a_{242} da_{132} + (a_{131} - a_{232}) da_{142} \\ 0 &= a_{242} da_{1\mu 2} = (a_{131} - a_{232}) da_{1\mu 2}, \end{aligned}$$

for $\mu > 4$. The smooth points of V_G can be normalized by rotations, and will remain smooth since rotations preserve V_G , so the generic normalized point $a \in V_G$ will be smooth. A point is smooth just when these relations on da are maximally independent. But these relations will be maximally independent just when either $a_{242} \neq 0$ or $a_{131} - a_{232} \neq 0$. \square

Corollary 1. *An admissible shape operator $a \in \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$ is regular just when it is G -regular (for $G = \sum_{\mu} \det a_{\mu} + \kappa$, necessarily).*

Proof. Normalize it, and then the condition of nondegeneracy is just that $a_{131} \neq a_{232}$ or $a_{242} \neq 0$. \square

¹The reader who is familiar with Nash's work on isometric embedding [1, 8, 12] will recognize that an admissible shape operator is not *free*. Recall that a shape operator is free when it is a linear embedding $\mathbb{R}^{2*} \rightarrow \mathbb{R}^n$.

Lemma 6. $\dim V_G = 2n$

Proof. $\dim \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n = 3n$. We have seen in the proof of lemma 5 on the preceding page that at a smooth point of V_G , the tangent space $T_a V_G \subset \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$ is cut out by n independent equations on da . \square

Recall that our indices run $i, j, k, l = 1, 2$ and $\mu, \nu, \sigma, \tau = 3, \dots, n+2$.

Lemma 7. *Given any integral element of the exterior differential system on $Z^{(0)}$, there is a change of coframing under which a becomes normalized.*

Proof. We can always change coframing, because an integral element remains an integral element under the action of $\text{SO}(2) \times \text{SO}(n)$ on $Z^{(0)} = FM \times FS$, and consequently under the induced action on $Z^{(1)}$. (Note that we can't use the entire $\text{SO}(2) \times \text{SO}(n+2)$ group, which acts but doesn't preserve the exterior differential system.) \square

Lemma 8. *Define a 1-form $\nabla a \in \Omega^1(Z^{(1)}) \otimes \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$ by*

$$\nabla a_{i\mu j} = da_{i\mu j} + \eta_{ik} a_{k\mu j} + \eta_{jk} a_{i\mu k}.$$

The tableau of the prolongation is ∇a .

Proof. Take exterior derivative of the 1-forms defining the exterior differential system of the prolongation to find $\nabla a_{i\mu j} \wedge \eta_j = 0$. \square

4. INVOLUTION

Theorem 4. *The prolonged exterior differential system $\mathcal{I}^{(1)}$ on $Z^{(1)}$ is involutive, with general integral manifold depending on $2n$ functions of 1 variable.*

Proof. The tableau is ∇a , with relations $\nabla a_{i\mu j} = \nabla a_{j\mu i}$ and the relations coming from admissibility of a , which are given by differentiating the equations

$$\begin{aligned} \sum_{\mu} \det a_{\mu} &= G - \kappa, \\ [a_{\mu}, a_{\nu}] &= 0. \end{aligned}$$

In lemma 5 on the facing page, we differentiated these equations, and found the resulting equations on da at a normalized shape operator, which were

$$\begin{aligned} 0 &= a_{232} da_{131} + a_{131} da_{232} + a_{242} da_{141} - dG \\ 0 &= a_{242} da_{132} + (a_{131} - a_{232}) da_{142} \\ 0 &= a_{242} da_{1\mu 2} = (a_{131} - a_{232}) da_{1\mu 2}, \end{aligned}$$

for $\mu > 4$. A simple calculation shows that we can replace d by ∇ :

$$\begin{aligned} 0 &= a_{232} \nabla a_{131} + a_{131} \nabla a_{232} + a_{242} \nabla a_{141} - dG \\ 0 &= a_{242} \nabla a_{132} + (a_{131} - a_{232}) \nabla a_{142} \\ 0 &= a_{242} \nabla a_{1\mu 2} = (a_{131} - a_{232}) \nabla a_{1\mu 2}, \end{aligned}$$

again for $\mu > 4$. We can now write our tableau as

$$\begin{pmatrix} \nabla a_{131} & \nabla a_{132} \\ \nabla a_{231} & \nabla a_{232} \\ \\ \nabla a_{141} & \nabla a_{142} \\ \nabla a_{241} & \nabla a_{242} \\ \\ \nabla a_{151} & \nabla a_{152} \\ \nabla a_{251} & \nabla a_{252} \\ \\ \vdots & \vdots \\ \\ \nabla a_{1(n+2)1} & \nabla a_{1(n+2)2} \\ \nabla a_{2(n+2)1} & \nabla a_{2(n+2)2} \end{pmatrix} = \begin{pmatrix} \nabla a_{131} & \nabla a_{132} \\ \nabla a_{132} & -\frac{1}{a_{131}}(a_{232}\nabla a_{131} + a_{242}\nabla a_{141}) \\ \\ \nabla a_{141} & \frac{a_{242}}{a_{232}-a_{131}}\nabla a_{132} \\ \frac{a_{242}}{a_{232}-a_{131}}\nabla a_{132} & \nabla a_{242} \\ \\ \nabla a_{151} & 0 \\ 0 & \nabla a_{252} \\ \\ \vdots & \vdots \\ \\ \nabla a_{1(n+2)1} & 0 \\ 0 & \nabla a_{2(n+2)2} \end{pmatrix}$$

as long as $a_{232} \neq a_{131} \neq 0$. Clearly we can “borrow from the second column” to arrange that each element in the first column be linearly independent. There are $2n$ rows, 2 for each value of $\mu = 1, \dots, n$, and therefore the Cartan characters are $s_1 = 2n$, and $s_2 = 0$. Our tableau is that of a system of linear equations of the form

$$(1) \quad \begin{aligned} \frac{\partial^2 u_3}{\partial x_2^2} &= -\frac{1}{a_{131}} \left(a_{232} \frac{\partial^2 u_3}{\partial x_1^2} + a_{242} \frac{\partial^2 u_4}{\partial x_1^2} \right) \\ \frac{\partial^2 u_4}{\partial x_1 \partial x_2} &= \frac{a_{242}}{a_{232} - a_{131}} \frac{\partial^2 u_3}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u_5}{\partial x_1 \partial x_2} &= 0 \\ &\vdots \\ \frac{\partial^2 u_{n+2}}{\partial x_1 \partial x_2} &= 0. \end{aligned}$$

In particular, one immediately sees that the free derivatives are

$$\frac{\partial^2 u_3}{\partial x_1^2}, \frac{\partial^2 u_3}{\partial x_1 \partial x_2}, \frac{\partial^2 u_\mu}{\partial x_1^2}, \frac{\partial^2 u_\mu}{\partial x_2^2},$$

for $\mu > 3$, so 2 free derivatives for each of $\mu = 3, \dots, n+2$. Therefore the space of integral elements is the space of these free derivatives, and has dimension $S = 2n$. Thus $S = 2n = s_1 + 2s_2$. The apparent torsion in the exterior differential system comes from the dG term above. It can be absorbed into ∇a_{132} and ∇a_{131} . Consequently, the system is in involution with general solution depending on $2n$ functions of 1 variable.

Should $a_{131} = 0$ or $a_{131} = a_{232}$, we can use the hypothesis that $a_{242} \neq 0$ or the hypothesis that $a_{232} \neq 0$ to ensure our result, in the same vein. Should these fail us as well, then the shape operator is umbilic, and therefore not a smooth point of V_G , and therefore not a point of $Z^{(1)}$. \square

Corollary 2. *The generic surface with flat normal bundle in a space form does not lie inside any totally geodesic subspace.*

Corollary 4. *A closed surface whose shape operator is everywhere regular is a torus or Klein bottle.*

Proof. The characteristic variety determines a pair of orthonormal line fields, which forces the Euler characteristic to vanish. \square

Corollary 5. *The generic integral curve of $\mathcal{I}^{(1)}$ is noncharacteristic, and therefore is contained in a unique integral manifold.*

6. CONSTRUCTING CYLINDERS WITH FLAT NORMAL BUNDLE

Taking well known immersions of surfaces into 3-dimensional space forms, and embedding those into higher dimensional space forms as totally geodesic submanifolds, we obtain examples with various topologies, but they have very degenerate shape operators. The Cartan–Kähler theorem builds local solutions with quite general shape operators, but we need to combine it with another trick to build solutions with some topology. We will only aim to construct solutions which are cylinders, but we will want to be able to impose fairly general shape operators.

Proposition 1. *Any regular shape operator occurs as the shape operator at some point of a real analytic cylinder in Euclidean space \mathbb{E}^{n+2} , for any $n > 1$, with flat normal bundle and everywhere regular shape operator and trivial normal holonomy along an embedded closed curve.*

Proof. We will need some lemmata:

Lemma 10. *Fix any shape operator $a = (a_{i\mu j}) \in \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$, and a constant k . Consider the $(n+3) \times (n+3)$ matrix*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & k & a_{1\mu 1} \\ 0 & -k & 0 & a_{1\mu 2} \\ 0 & -a_{1\mu 1} & -a_{1\mu 2} & 0 \end{pmatrix}.$$

For dense set of values of constant $k \neq 0$, A has all of its eigenvalues imaginary with rational ratios.

Proof. Calculating the determinant $\det(\lambda - A)$, by computing along the top row, one finds that the spectrum consists in $\lambda = 0$ together with the spectrum of

$$A' = \begin{pmatrix} 0 & k & a_{1\mu 1} \\ -k & 0 & a_{1\mu 2} \\ -a_{1\mu 1} & -a_{1\mu 2} & 0 \end{pmatrix}.$$

Since A' is skew symmetric, it is orthogonally diagonalizable with imaginary eigenvalues. Let $b = (a_{1\mu 1}), c = (a_{1\mu 2}) \in \mathbb{R}^n$. Looking at the equation $A'v = \lambda v$, one easily computes that the spectrum consists in $\lambda = 0$ or

$$\lambda^2 = -\frac{(k^2 + |b|^2 + |c|^2) \pm \sqrt{D}}{2},$$

where the discriminant is

$$D = k^4 + 2(|b|^2 + |c|^2)k^2 + (|b|^2 - |c|^2)^2 + 4\langle b, c \rangle^2.$$

It suffices to ensure that the ratio of eigenvalues

$$\sqrt{\frac{-(k^2 + |b|^2 + |c|^2) + \sqrt{D}}{-(k^2 + |b|^2 + |c|^2) - \sqrt{D}}}$$

is rational. One can check that the second derivative of this ratio at $k = 0$ is nonzero, unless c is a constant multiple of b or vice versa. But in that case, $D = 0$, and the ratio of eigenvalues is rational. Therefore a dense set of values of k ensure that A has all its eigenvalues imaginary and rational multiples of a single eigenvalue. \square

Lemma 11. *Fix a choice of regular shape operator a , and a constant k , and consider the exterior differential system on $F\mathbb{E}^{n+2}$:*

$$(3) \quad \omega_\mu = 0, \omega_2 = 0, \omega_{\mu\nu} = 0, \omega_{12} = k\omega_1, \omega_{1\mu} = a_{1\mu 1}\omega_1, \omega_{2\mu} = a_{2\mu 1}\omega_1.$$

The manifold $F\mathbb{S}$ is foliated by the integral curves of this exterior differential system. For a dense set of choices of the constant k , the integral manifolds are closed curves.

Proof. The integral manifolds are the integral curves of

$$\omega = A\omega_1.$$

If we write $\omega = g^{-1}dg$, then this equation is the flow of

$$\frac{dg}{ds} = gA, \omega_1 = ds.$$

\square

Now we prove proposition 1 on the facing page. Take a any regular shape operator, and let $G_0 = \sum_\mu \det a_\mu$. By possibly rotating coordinates on \mathbb{R}^2 , arrange that a is not characteristic in the direction of $(1, 0) \in \mathbb{R}^2$, i.e. the characteristic variety of the prolonged exterior differential system $\mathcal{I}^{(1)}$ for isometric immersions of surfaces with flat normal bundle is not characteristic in that direction. Take a constant k for which the integral curves of the exterior differential system 3 are closed. Pick one such integral curve \tilde{C} . We need one more lemma:

Lemma 12. *For any length $L > 0$ and constants $k \neq 0$ and G_0 , there is a real analytic surface $M \subset \mathbb{E}^3$ diffeomorphic to a cylinder, with a boundary circle C , so that M has constant Gauss curvature $G = G_0$ at every point of C , C has length L and constant geodesic curvature k .*

Proof. Draw a circle C of the required length L . Form a surface of revolution by revolving a curve in a plane perpendicular to C around C . For instance if the curve is $z = f(r)$ in cylindrical coordinates, then the Gauss and geodesic curvatures along C are

$$G = \frac{f'(r)f''(r)}{r\sqrt{1+f'(r)^2}}, \quad k = \frac{f'(r)}{r\sqrt{1+f'(r)^2}}.$$

So it is easy to construct lots of cylinders M with the required behaviour, for example with f a cubic polynomial. \square

Let M be a real analytic surface with Riemannian metric, and $C \subset M$ a closed curve on M of constant geodesic curvature k . Suppose that the Gauss curvature of M is constant equal to G_0 at every point of C , and that C has length equal to the length of \tilde{C} . Map C to \tilde{C} by arclength parameterization. The graph of this map is a noncharacteristic curve for the prolonged exterior differential system $\mathcal{I}^{(1)}$

for isometric immersions of surfaces with flat normal bundle. Therefore there is a unique integral surface of the system through this curve. Cutting off all of the integral surface except near the curve, we can ensure that it is a cylinder. \square

Corollary 6. *There are real analytic cylinders in any space form $\mathbb{S}^{n+2}(\kappa)$, for any $n > 1$, with flat normal bundle and everywhere regular shape operator, and trivial normal holonomy along some boundary component.*

Proof. Slightly altering the metric on Euclidean space by a conformal map will preserve regularity, and then dilating the metric by a constant will preserve regularity as well. \square

7. CAUCHY CHARACTERISTICS

Lemma 13. *In terms of the basis of 1-forms*

$$\omega_i, \omega_\mu, \omega_{ij}, \omega_{\mu\nu}, \eta_i, \eta_{ij}, \nabla a_{i\mu j},$$

on $Z^{(1)}$, the Cauchy characteristics of the prolonged exterior differential system $\mathcal{I}^{(1)}$ for isometric immersions with flat normal bundle are precisely the integral curves of the exterior differential system

$$D\mathcal{I}^{(1)} = (0 = \omega_i = \omega_\mu = \omega_{ij} - \eta_{ij} = \omega_{\mu\nu} = \eta_i = \nabla a_{i\mu j}).$$

The space of Cauchy characteristics is the space $\bar{Z}^{(1)}$ of choices of points $m \in M$, $s \in \mathbb{S}$, linear isometry $\phi_0 : T_m M \rightarrow T_s \mathbb{S}$, basis for the normal space $N_0 = \phi_0(T_m M)^\perp$, and regular G -admissible shape operator $A : \text{Sym}^2(T_m^* M) \rightarrow N_0$; in particular the space of Cauchy characteristics is a smooth manifold and $Z^{(1)} \rightarrow \bar{Z}^{(1)}$ is a smooth circle bundle. The prolonged exterior differential system $\mathcal{I}^{(1)}$ is defined on the space of Cauchy characteristics.

Proof. The integral curves of $D\mathcal{I}^{(1)}$ are those which head in the direction “dual to” $\omega_{ij} - \eta_{ij}$. Clearly these are Cauchy characteristics. Taking exterior derivatives of the differential forms in the prolonged system, we find that Cauchy characteristics must be precisely these. Moreover, the map $Z^{(1)} \rightarrow \bar{Z}^{(1)}$ is clearly well defined, and constant along Cauchy characteristics. Clearly a choice of a point of $\bar{Z}^{(1)}$ determines a choice of a point of $Z^{(1)}$ up to the action of $\text{SO}(2)$, which acts precisely along the Cauchy characteristics. Exterior differential systems always drop down to their space of Cauchy characteristics. \square

N.B. the integral manifolds of the prolonged exterior differential system on $Z^{(1)}$ are 3-manifolds, while those of the exterior differential system induced on $\bar{Z}^{(1)}$ are surfaces.

Proposition 2. *Pick a surface M with an analytic Riemannian metric, and an embedded real analytic curve $C \subset M$ diffeomorphic to an open interval $[0, 1]$, and a complete space form $\mathbb{S} = \mathbb{S}^{n+2}(\kappa)$ for $n \geq 2$ and any choice of constant $\kappa \in \mathbb{R}$. At each point $m \in C$, pick analytically a choice of $A \in \text{Sym}^2(T_m M^*) \otimes \mathbb{R}^n$, G -admissible, with characteristic variety not containing the tangent direction to C . Pick a point $s_0 \in \mathbb{S}$, and a linear isometry $\phi_0 : T_{m_0} M \rightarrow T_0 \subset T_{s_0} \mathbb{S} = T_0 + N_0$, and linear isometry $N_0 = \mathbb{R}^n$. Then there is a unique isometric immersion $U \rightarrow \mathbb{S}$ of a neighborhood U of C , say ϕ , so that $\phi'(m_0) = \phi_0$, and the shape operator a of ϕ satisfies*

$$A(u, v) = \Pi_C a(u, v)$$

where Π_C refers to parallel transport of normal vectors along the curve C between the appropriate points, $u, v \in T_m M$ any tangent vectors.

Proof. Let $Z_C^{(1)}$ be the pullback bundle

$$\begin{array}{ccc} Z_C^{(1)} & \longrightarrow & Z^{(1)} \\ \downarrow & & \downarrow \\ F_C M & \longrightarrow & F M. \end{array}$$

Write each point of $Z_C^{(1)}$ as (c, e, E, a) where $c \in C, E \in F M_c, e \in F \mathbb{S}, a \in \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$. Then consider the subset $X \subset Z_C^{(1)}$ given by the equation

$$\langle e_\mu, \Pi_C A(u, v) \rangle = a_{i\mu j} \langle u, e_i \rangle \langle v, e_j \rangle.$$

Since these are linear equations, of full rank in $a_{i\mu j}$, X is a smooth submanifold. On X consider the exterior differential system

$$\omega_1 = \eta_1, \omega_2 = 0, \omega_\mu = 0, \omega_{1\mu} = a_{1\mu 1} \eta_1, \omega_{2\mu} = a_{2\mu 1} \eta_2, \omega_{ij} = \eta_{ij}, \omega_{\mu\nu} = 0.$$

It is easy to see that this system is holonomic, i.e. satisfies the conditions of the Frobenius theorem, and that its solution is a noncharacteristic integral curve of the prolonged exterior differential system $\mathfrak{J}^{(1)}$ restricted to X . Therefore there is an integral manifold through each point of that integral curve. \square

8. LINEARIZATION OF EXTERIOR DIFFERENTIAL SYSTEMS

We will state without proof some well known results in the theory of exterior differential systems.

Definition 3. Suppose that Z is a manifold bearing an involutive C^k exterior differential system. Suppose that $\phi : \Lambda \rightarrow Z$ is an integral manifold. Let $Z^{(1)} \subset \text{Gr}(n, TZ)$ be the space of regular integral elements, which is a submanifold of the Grassmannian of n -planes in TZ ; the map $Z^{(1)} \rightarrow Z$ is a smooth submersion. Let $\phi^{(1)} : \Lambda \rightarrow Z^{(1)}$ be the prolongation of ϕ , i.e. $\phi^{(1)}(\lambda) = \phi'(\lambda)T_\lambda \Lambda$. The *linearization* of the exterior differential system about ϕ is the linear operator

$$L_\phi : J^1 \phi^* TZ / T\Lambda \rightarrow \phi^{(1)*} T \text{Gr}(n, TZ) / \phi^{(1)*} TZ^{(1)},$$

defined as follows: for any map $\Phi : \Lambda \times \mathbb{R} \rightarrow Z$ for which each $\Phi_t : \Lambda \rightarrow Z$ is an immersion, and for which Φ_0 is an integral manifold, let

$$L_{\Phi_0} \frac{\partial}{\partial t} = \Phi_* \frac{\partial}{\partial t} + TZ^{(1)}.$$

Lemma 14. *Representing the exterior differential system locally in any system of local coordinates as a system of partial differential equations, the linearization of the exterior differential system is identified with the linearization of the system of partial differential equations in the usual sense. In particular, the linearization is well-defined, and is a first order differential operator. If the integral manifold is C^k , then the linearization is a C^{k-1} operator (i.e. has C^{k-1} “coefficients”), mapping C^k sections to C^{k-1} sections.*

Proof. See step B, pg. 922 of [3]. \square

Lemma 15. *The tableau of the linearization about an integral element is the tableau of the integral element.*

Proof. See [2]. □

Remark 1. We can consider any exterior differential system, at least locally, as a nonlinear partial differential operator, as follows. Consider an open subset $Y \subset \text{Gr}(d, TZ)$ which is an open neighborhood of $Z^{(1)}$; construct any fiber bundle map $p : Y \rightarrow X$ for which $Z^{(1)}$ is a fiber, say $Z^{(1)} = p^{-1}(0)$, some point $0 \in X$. Any submanifold $\phi : M \rightarrow Z$ whose lift $\phi^{(1)} : M \rightarrow \text{Gr}(d, TZ)$ lies in Y can be plugged into the differential operator $D\phi = p\phi^{(1)}$, representing the exterior differential system as the system of partial differential equations $D\phi^{(1)} = 0$. Thus the linearization is the map $\xi \mapsto L\xi$ taking 1-jets of sections of the normal bundle ϕ^*TZ/TM to an immersed surface $\phi : M \rightarrow Z$ and mapping them to tangent vectors to X : $L : J^1\phi^*TZ/TM \rightarrow (p\phi^{(1)})^*TX$.

Remark 2. In the example of interest to us, the discussion is mildly complicated by wanting to work with the involutive prolonged exterior differential system on $Z^{(1)}$, and linearize it, rather than following the discussion we have just given which linearizes on Z . We trust that the reader can work through these minor details.

9. HYPERBOLICITY OF THE LINEARIZATION

Corollary 7. *The prolonged exterior differential system $\mathcal{I}^{(1)}$ for isometric immersion with flat normal bundle is determined.*

Proof. The linear equations 1 on page 8 in u which we wrote in the proof of theorem 4 on page 7 are determined, and have the same tableau. See [3] for the definition of a determined exterior differential system. □

Definition 4. The *discriminant* of a normalized shape operator a is

$$\Delta = \Delta(a) = a_{131}(a_{232} - a_{131})(a_{232}^2 + a_{242}^2 - a_{131}a_{232}).$$

A regular shape operator is called *hyperbolic* if after normalization $\Delta < 0$. It is called *mixed* if as a map $\text{Sym}^2(\mathbb{R}^{2*}) \rightarrow \mathbb{R}^n$ it is a linear embedding and after normalization $\Delta > 0$.

Example 1. A surface immersed in a 3-dimensional totally geodesic subspace of \mathbb{S} has hyperbolic shape operator everywhere where its Gauss curvature is less than the ambient sectional curvature; we see this directly from the normalized shape operator. On the other hand, such a surface can never be mixed, since the shape operator has rank 1.

Lemma 16. *The linearization of the exterior differential system around an integral manifold is a hyperbolic linear partial differential operator just when the shape operator is hyperbolic at every point. In particular, this ensures that hyperbolicity of a shape operator is actually well defined, independent of the choice of normalization, which is not obvious from the definition.*

Proof. Consider a normalized shape operator. Looking at equation 2 on page 9, all of the roots of the characteristic variety are real just when the final quadratic factor

$$a_{131}(a_{232} - a_{131})\xi_1^2 + (a_{242}^2 + a_{232}^2 - a_{131}a_{232})\xi_2^2$$

has negative discriminant as a quadratic form in ξ_1, ξ_2 . But its discriminant is just precisely Δ . □

Theorem 5. *Any hyperbolic [mixed] shape operator occurs as the shape operator of a cylinder in Euclidean space \mathbb{E}^{n+2} (for any $n \geq 2$), with everywhere hyperbolic shape operator, flat normal bundle, and bicharacteristics passing entirely from one component to the other.*

Proof. We have constructed the cylinder in proposition 1 on page 10. In case the shape operator a which we choose was hyperbolic [mixed], we can cut a sufficiently small piece of the resulting cylinder, and ensure that the bicharacteristics travel from one component of the boundary to the other, because the bicharacteristics on the surface are transverse to the curve. \square

Definition 7. Let $H_p(\mathbb{R}^n)$ be the completion of the smooth compactly supported functions $C^\infty(\mathbb{R}^n)$ in the norm

$$|u|_p = \sqrt{\int |\hat{u}|^2 (1 + |\xi|^2)^p d\xi},$$

where \hat{u} is the Fourier transform of u . For any vector bundle $V \rightarrow X$ on any manifold, we define $H_p^{loc}(V)$ to be the space of sections of V which, in any local system of coordinates on X , when restricted to any relatively open compact subset of the coordinate region $\Omega \subset \mathbb{R}^n$, equal the restriction of some element of $H_p(\mathbb{R}^n)$. In this way, the norm $|u|_p$ of a section is not globally defined, but is defined on any compact set up to Lipschitz equivalence. For X compact, we can define similarly the space $H_p(V)$.

Definition 8. On a Riemannian manifold X , for a vector bundle $V \rightarrow X$ equipped with an inner product, we let $G \rightarrow P \rightarrow X$ be the associated right principal bundle, G a compact Lie group, and we identify each section of $V \rightarrow X$ with a function $P \rightarrow V_0$, which is equivariant for the structure group G , with V_0 a fixed G -module. Equip P with a connection, whose connection 1-form we denote $\phi \in \Omega^1(P) \otimes \mathfrak{g}$. Given a section σ of $V \rightarrow X$, identify it with a map $s : P \rightarrow V_0$. On $P \times_X FX$, write $ds = \nabla_i s \omega_i$, $d\nabla_i s = \nabla_{ij} s \omega_j - \phi s$, etc. So $\nabla_{Ij} s \omega_j$ is the semibasic part of $d\nabla s_I$. For p an integer, define $|s|_p$ by

$$|s|_p = \sqrt{\sum_{|I| \leq p} \int_{P \times_X FX} \nabla_I s^2},$$

(where if $I = (i_1, \dots, i_q)$ then $|I| = q$).

Theorem 6. *Suppose that Λ is an integral manifold of the prolonged exterior differential system $\mathcal{I}^{(1)}$ for isometric immersion with flat normal bundle, with everywhere hyperbolic [mixed] shape operator. Then the linearized operator L is locally solvable, i.e. around each point of Λ , there are neighborhoods in which for any p and any $f \in H_p^{loc}$ supported in the larger neighborhood, there is some $u \in H_{p+1}^{loc}$, supported in the same neighborhood, so that $Lu = f$ on the smaller neighborhood, and*

$$|u|_{p+1} \leq C |Lu|_p$$

on the larger open set, for constant C independent of f , depending only on the neighborhoods and value of $s \geq 1$. Moreover, if the bicharacteristic curves are not trapped, then there is a finite codimension subspace $S \subset C^\infty$ so that $Lu = f$ is solvable just for $f \in H_p \cap S$, with solution $u \in H_{p+1}$, and again

$$|u|_{p+1} \leq C |Lu|_p$$

for constant C independent of f , depending only on p (which is assumed a positive integer here). If the surface is a cylinder, and all of the bicharacteristics travel from one component to the other, then the space S is $S = C^\infty$, i.e. L is surjective.

Proof. The bicharacteristics are clearly not locally trapped, i.e. not by any very small compact sets. The result follows in the hyperbolic case from Duistermaat & Hörmander [6], and in mixed case the elliptic part follows from well-known elliptic theory (see for example Duistermaat [5]). The semiglobal solvability is proven by Hörmander [10] section 23.2, p. 394, assuming the existence of a time function, i.e. a function which strictly increases along bicharacteristics and is constant on boundary components. The existence of such a function when bicharacteristics travel from one side of a cylinder to the other is proven in Hamilton [9], pg. 208. The semiglobal solvability of the elliptic problem in the mixed case is proven in Duistermaat [5]. \square

Lemma 19. *Let $R \subset \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$ be the set of regular shape operators (not necessarily admissible). Let $Y = (FM \times FS \times R) / \text{SO}(2)$, so that $\bar{Z}^{(1)} \subset Y$. The map associating to any choice of immersion $\phi : M \rightarrow \mathbb{S}$ with no umbilic points and framing e_μ for the normal bundle of M , the lift $M \rightarrow Y$ is tame in the sense of Hamilton [9], of degree 1 and base 1.*

Proof. The lift of such data is defined in the obvious manner, and clearly only depends on first derivatives. \square

Corollary 8. *The family of inverses R of the linearizations L of integral manifolds is tame.*

Proof. See Bryant, Griffiths & Yang [3] for the strictly hyperbolic theory generalizing Hamilton [9], and the rest is proven by Hamilton. \square

Theorem 7. *The set of smooth Riemannian metrics on a given compact disk or cylinder which can be realized by an isometric immersion with flat normal bundle is open in the fine topology.*

Proof. Follows from local surjectivity by Nash–Moser theorem; see Hamilton [9]. \square

Corollary 9. *For every smooth Riemannian metric on a surface, and every point of that surface, there is an isometric embedding of a neighborhood of that point as a surface in \mathbb{S} , with flat normal bundle, and arbitrarily chosen admissible shape operator at that point.*

Proof. See [3], p. 993. \square

Theorem 8. *Let M be a smooth surface, diffeomorphic to a cylinder $S^1 \times [0, 1]$. On one of the boundary components of M , say C , pick a point $m_0 \in C$. Fix a point $s_0 \in \mathbb{S}$, a 2-plane $P_0 \subset T_{s_0}\mathbb{S}$, and a linear immersion $\phi_0 : T_{m_0}M \rightarrow P_0$.*

*To each immersion $\phi : M \rightarrow \mathbb{S}$ satisfying $\phi(m_0) = s_0$ and $\phi'(m_0) = \phi_0$, with trivial normal holonomy along C , define a section $A = A(\phi)$ of the bundle $\text{Sym}^2(T^*M)|_C \otimes P_0^\perp$, by taking*

$$\Pi_C A = a,$$

with a the shape operator of the immersion. Consider the map $\phi \mapsto (g(\phi), A(\phi))$ taking an immersed cylinder to its induced metric g and section A of this vector

bundle. This map is a tame bundle map, with section defined on a nonempty open set of the base by asking $\phi(g, A)$ to have flat normal bundle, hyperbolic or mixed shape operator, and bicharacteristics passing entirely from one boundary component to the other. If we build a family of space form metrics g_κ on a fixed space \mathbb{S} , with sectional curvature of the metric g_κ being κ , then the same result holds for the map $(\phi, \kappa) \mapsto (g(\phi), A(\phi), \kappa)$.

Proof. Proof is identical to Hamilton's proof [9] p. 207 of a very similar statement on negatively curved surfaces in Euclidean space. \square

Corollary 10. *Any cylinder with flat normal bundle and hyperbolic [or mixed] shape operator and bicharacteristics travelling from one boundary component to the other, and sitting in Euclidean space (for example, those constructed in proposition 1 on page 10), with trivial normal holonomy around a boundary loop, can be deformed to have any required smooth metric near enough to its initial metric, flat normal bundle, and hyperbolic [or mixed] shape operator, in a sphere or hyperbolic space of small enough curvature.*

11. COUNTING SURFACES WITH FLAT NORMAL BUNDLE

Rather than fixing a metric on a surface, and looking for an isometric immersion with flat normal bundle, we can just look for immersed surfaces with flat normal bundle, with any metric on them. This is the problem addressed by Ferapontov [7]. We will only look briefly at the relevant exterior differential system.

Definition 9. Given an immersed submanifold of dimension d with flat normal bundle $\phi : M \rightarrow \mathbb{S} = \mathbb{S}^{n+d}(\kappa)$, let $E_M\mathbb{S}$ be the set of frames $e \in F\mathbb{S}$ for which $e_\mu \perp \phi'(m)T_mM$ for $\mu > d$. Thus $E_M\mathbb{S} \rightarrow M$ is a principal right $\text{SO}(d) \times \text{SO}(n)$ -bundle. We will write $E_M\mathbb{S}$ as $E_\phi\mathbb{S}$ when we want to emphasis the immersion ϕ .

Lemma 20. *The adapted frame bundles $E_M\mathbb{S}$ of immersed surfaces with flat normal bundle in \mathbb{S} are precisely the integral submanifolds of dimension $d + d(d-1)/2 + n(n-1)/2$ of the exterior differential system*

$$\mathcal{E} = (\omega_\mu = 0, \omega_{\mu\nu} = 0)$$

on $F\mathbb{S}$.

Proof. The proof is essentially the same as that of lemma 1 on page 4. \square

Lemma 21. *The Cauchy characteristics of \mathcal{E} are the integral manifolds of the exterior differential system*

$$D\mathcal{E} = (0 = \omega_i = \omega_\mu = \omega_{\mu\nu}),$$

so that the space of Cauchy characteristics \bar{W} is the space of tuples (s, P, e_μ) of a point $s \in \mathbb{S}$, a d -plane $P \subset T_s\mathbb{S}$, and an orthonormal basis e_μ for $P^\perp \subset T_s\mathbb{S}$.

Proof. Just differentiate. \square

Definition 10. Let $V \subset \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$ be the set of $a = (a_{i\mu j})$ for which $[a_\mu, a_\nu] = 0$ where once again a_μ is the symmetric matrix $a_\mu = (a_{i\mu j})$. Let V_{sm} be the set of smooth real points of V as real algebraic variety. Say that a shape operator $a \in V$ is *regular* if it is a smooth point of V .

Lemma 22. *A shape operator is regular just when it is not umbilic, i.e. regularity in this new sense agrees with regularity in the old sense. Moreover, $\dim V = 2n + 1$.*

Proof. Just as the proof of lemma 5 on page 6. \square

Lemma 23. *Let $W = FS$. The prolongation of \mathcal{E} is the exterior differential system $\mathcal{E}^{(1)}$ on $W^{(1)} = W \times V_{sm}$:*

$$\mathcal{E}^{(1)} = (0 = \omega_\mu = \omega_{\mu\nu} = \omega_{i\mu} - a_{i\mu j}\omega_j).$$

Proof. Differentiate. \square

Definition 11. On $W^{(1)}$, let

$$\nabla a_{i\mu j} = da_{i\mu j} + a_{k\mu j}\omega_{ik} + a_{i\mu k}\omega_{jk}.$$

Lemma 24. *The Cauchy characteristics of $\mathcal{E}^{(1)}$ are the integral manifolds of the exterior differential system*

$$D\mathcal{E}^{(1)} = (0 = \omega_i = \omega_\mu = \omega_{\mu\nu} = \omega_{i\mu} - a_{i\mu j}\omega_j = \nabla a_{i\mu j}),$$

so that the space of Cauchy characteristics $\bar{W}^{(1)}$ is the space of tuples (s, P, e_μ, a) of a point $s \in \mathbb{S}$, a d -plane $P \subset T_s\mathbb{S}$, an orthonormal basis e_μ for $P^\perp \subset T_s\mathbb{S}$, and a shape operator $a \in \text{Sym}^2(P^) \otimes P^\perp$.*

Proof. Differentiate. \square

Proposition 3. *The exterior differential system $\mathcal{E}^{(1)}$ is involutive. The general immersed surface with flat normal bundle in a space form depends on 1 function of 2 variables.*

Proof. The proof is the same as theorem 4 on page 7. \square

Remark 3. This certainly agrees with Ferapontov [7], although his results are much stronger. The characteristic variety is \mathbb{P}^1 . There are many solutions through a given ribbon.

12. THE BONNET TRANSFORMATION

Take $\phi : M \rightarrow \mathbb{S}$ an immersed surface with flat normal bundle. Let ξ be a normal vector at a point of M , and consider parallel transports of ξ around M . As long as we keep v inside a simply connected open subset of M , or the normal bundle of M has trivial holonomy, these parallel transports of ξ are well defined independently of path. Lets assume that this is the case, and call them all ξ . The *Bonnet transformation* or *normal shift* is the application of the exponential map to this parallel transported vector. We can apply the Bonnet transformation not only to the surface itself, but by parallel transport along the geodesics $s \mapsto \exp s\xi$, we can take the entire frame bundle $E_M\mathbb{S}$ and transport it. Of course, under parallel transport the tangent spaces of M remain perpendicular to the geodesics along which they are transported, so the tangent planes of the normal shift are perpendicular to the transport of ξ , and therefore $\exp \xi_* E_\phi \mathbb{S} = E_{\exp \xi \phi} \mathbb{S}$.

Definition 12. Suppose that $\phi : M \times \mathbb{R} \rightarrow \mathbb{S}$, written $\phi_s(m) = \phi(m, s)$, is a smooth family of immersions of a surface, all with flat normal bundle. Will will not ask that they be isometric immersions. Let $F \rightarrow M \times \mathbb{R}$ be the bundle whose fiber over (m, s) is the fiber of $E_{\phi_s}\mathbb{S}$ over m , i.e. the orthonormal frames e for \mathbb{S} with

$e_\mu \perp \phi'(m)T_m M$. Thus F is fibered by leaves, each of which is $E_{\phi_s} \mathbb{S}$, and so on F , we have the equations

$$0 = \omega_\mu - a_\mu ds = \omega_{\mu\nu} - a_{\mu\nu} ds.$$

Lemma 25. *A one parameter family of isometric immersions of a surface with flat normal bundle is a family of Bonnet transformations*

$$\phi_s(m) = \exp s\xi,$$

just when

$$0 = \omega_\mu - v_\mu ds = \omega_{\mu\nu}.$$

Proof. The equation $\omega_\mu = v_\mu ds$ just says that the motion is along v , while $\omega_{\mu\nu} = 0$ parallel transports along the v direction. \square

Definition 13. Suppose that $a \in \text{Sym}^2(\mathbb{R}^{2*}) \otimes \mathbb{R}^n$. Write a_η for the linear transformation $\langle a_\eta(u), v \rangle = \langle \eta, a(u, v) \rangle$ for any $\eta \in \mathbb{R}^n$ and $u, v \in \mathbb{R}^2$.

Definition 14. Consider the linear fractional transformations

$$T_{\kappa, s} = \begin{cases} \begin{bmatrix} \cos(s\sqrt{\kappa}) & -\sqrt{\kappa} \sin(s\sqrt{\kappa}) \\ \cos(s\sqrt{\kappa}) & \frac{\sin(s\sqrt{\kappa})}{\sqrt{\kappa}} \end{bmatrix} & \text{if } \kappa > 0, \\ \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} & \text{if } \kappa = 0, \\ \begin{bmatrix} \cosh(s\sqrt{|\kappa|}) & -\sqrt{|\kappa|} \sinh(s\sqrt{|\kappa|}) \\ \cosh(s\sqrt{|\kappa|}) & \frac{\sinh(s\sqrt{|\kappa|})}{\sqrt{|\kappa|}} \end{bmatrix} & \text{if } \kappa < 0, \end{cases}$$

$$S_{\kappa, s} = \begin{cases} \begin{bmatrix} 0 & 1 \\ \frac{\sin(s\sqrt{\kappa})}{\sqrt{\kappa}} & \cos(s\sqrt{|\kappa|}) \end{bmatrix} & \text{if } \kappa > 0, \\ \begin{bmatrix} 0 & 1 \\ s & 1 \end{bmatrix} & \text{if } \kappa = 0, \\ \begin{bmatrix} 0 & 1 \\ \frac{\sinh(s\sqrt{|\kappa|})}{\sqrt{|\kappa|}} & \cosh(s\sqrt{|\kappa|}) \end{bmatrix} & \text{if } \kappa < 0. \end{cases}$$

Lemma 26. *Under the Bonnet transformation $\exp \xi$, the shape operator transformations according to*

$$a_{\exp \xi_* \xi} = \Pi T_{\kappa, s}(a_\xi) \Pi^{-1}$$

$$a_{\exp \xi_* \eta} = \Pi a_\eta S_{\kappa, s}(a_\xi) \Pi^{-1},$$

with Π the parallel transport along the geodesic $t \mapsto \exp(t\xi)$, $s = |\xi|$ and $\eta \perp \xi$ any perpendicular normal vectors.

Proof. Differentiate our equations to find that $\omega_{i\mu} = a_{i\mu j} \omega_j$ for some functions $a_{i\mu j} = a_{j\mu i}$. Differentiate once more to find

$$\nabla a_{i\mu j} \wedge \omega_j = 0,$$

where

$$\begin{aligned}\nabla a_{i\mu j} &= da_{i\mu j} + a_{k\mu j}\omega_{ik} + a_{i\mu k}\omega_{kj} \\ &= \nabla_k a_{i\mu j} \omega_k - (\kappa v_\mu \delta_{ij} + a_{i\mu k} a_{k\nu j} v_\nu) ds.\end{aligned}$$

So if we move only along the v direction, by carrying our frame e_i, e_μ in parallel,

$$\frac{da_{i\mu j}}{ds} = -(\kappa v_\mu \delta_{ij} + a_{i\mu k} a_{k\nu j} v_\nu).$$

Assuming that $v_\mu = \delta_{\mu 3}$, which we can arrange for simplicity by change of coframing in the normal direction at one point, we find the differential equations

$$\begin{aligned}\frac{da_3}{ds} &= -(\kappa + a_3^2) \\ \frac{da_\mu}{ds} &= -a_3 a_\mu.\end{aligned}$$

We can solve these equations explicitly:

$$a_3(s) = \begin{cases} -\sqrt{\kappa} \tan(c_3 + \sqrt{\kappa}s), & \text{if } \kappa > 0, \\ (c_3 + s)^{-1}, & \text{if } \kappa = 0, \\ \sqrt{|\kappa|} \tanh(c_3 + \sqrt{|\kappa|}s), & \text{if } \kappa < 0. \end{cases}$$

$$a_\mu(s) = c_\mu \begin{cases} \sec(c_3 + \sqrt{\kappa}s), & \text{if } \kappa > 0, \\ (c_3 + s)^{-1}, & \text{if } \kappa = 0, \\ \operatorname{sech}(c_3 + \sqrt{|\kappa|}s), & \text{if } \kappa < 0. \end{cases}$$

□

Remark 4. Terng [14] calculated the transformation of the shape operator under Bonnet transformations, but only in the direction ξ . One can easily see that $a_\xi = 0$ just when the Bonnet transformations $\exp(t\xi)$ are isometries of the Riemannian metric on the surface (for all t), and a_ξ is a multiple of the identity just when the Bonnet transformations are conformal. So the Bonnet transformations match up the flat normal bundles, but generally don't match the Riemannian metrics.

Corollary 11. *In Euclidean space, a compact surface (perhaps with boundary) with hyperbolic shape operator of rank 2 is taken to one with mixed shape operator by some Bonnet transformation, and vice versa.*

Proof. Check that under the Bonnet transformation, the discriminant Δ of a normalized shape operator is taken to

$$\Delta' = \frac{\Delta + sa_{131}^2 a_{242}^2 (a_{232} - a_{131})}{(1 + sa_{131})^3 (1 + sa_{232})^3}.$$

The leading term in s will have a different sign for large positive s than for large negative s . We need to take s large enough so that the Bonnet transformation is smooth at all points. □

Comment 1. There is a subtle issue here: I can't carry out the construct of A out of a , i.e. trivialize by parallel transport, $\Pi A = a$, unless the parallel transport along the loop C has trivial normal holonomy. So I need to add that condition.

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