

DUAL CURVES AND PSEUDOHOLOMORPHIC CURVES

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ABSTRACT. A notion of dual curve for pseudoholomorphic curves in 4-manifolds turns out to be possible only if the notion of almost complex structure structure is slightly generalized. The resulting structure is as easy (perhaps easier) to work with, and yields many analogues of results in complex surface theory, using a description of the local geometry via Cartan's method of equivalence. Duality then uncovers a new infinite-dimensional family of geometric integrable systems. These are the first steps toward geometry on moduli spaces of pseudoholomorphic curves.

1. INTRODUCTION

Consider the notion of dual curves in the projective plane: the space of lines in the plane forms a plane called the *dual plane*, and the dual of a plane curve is its collection of tangent lines, thought of as a curve in the dual plane. Moreover, the dual of the dual is the origin curve. And the dual of a point is a line. Does this picture work for almost complex structures on the complex projective plane? Mikhael Gromov (see [33]) showed that the dual plane, taken as the space of “complex lines”, i.e. pseudoholomorphic spheres in the homology class of a complex line, is a 4-manifold (if the almost complex structure is tame), but is it equipped with an almost complex structure? I would like to use tangential approximation by “complex lines” to define dual curves, and would like them to be pseudoholomorphic. Naive dimension count suggests that this should determine the almost complex structure on the dual projective plane. However, we will see that there is no such almost complex structure, except in the case we have already considered: the standard complex projective plane. Except in that case, the dual projective plane is not almost complex in any manner which would render the dual curves pseudoholomorphic.

There is a more fundamental question: why think about almost complex structures in the first place? We know that they exist on symplectic manifolds, and that they have pseudoholomorphic curves which are very similar to holomorphic curves in complex manifolds. The curves are usually used to probe symplectic geometry. But what we use about them is that they form a family of surfaces defined by local conditions and depending on several functions of one real variable, and have carefully controlled singularities. In particular, they cannot crease. On top of this, we can use a symplectic structure to “tame” these curves—to provide an a priori estimate preventing all but finitely many singularities of a mild nature.

The largest family of determined first order equations on 4-manifolds with these same properties, a family I will call *pseudocomplex structures*, are described in the same article of Gromov [33] (there called *elliptic systems*). I will use them instead

of almost complex structures, and discover that my construction of dual curves on a dual projective plane above will always work in the category of pseudocomplex structures. Moreover, the analysis of these equations, both local and global, is as easy as the analysis of pseudoholomorphic curves for almost complex structures. It is the author's belief that the pseudocomplex structure is more natural and fundamental than the almost complex structure. To demonstrate this, proofs are provided or sketched of analogues for the basic results of the theory of complex surfaces.

An application of this idea will be given: an explicit geometric description via dual curves of an infinite-dimensional family of integrable systems of pseudocomplex structures first described indirectly by Gaston Darboux in [27].

1.1. Other work on this subject. Much of the material described here began in the author's thesis [41]. The interested reader will find more details of calculations there which I have omitted here for lack of space. The author wishes to acknowledge that some of the results of this paper are nearly identical to those independently discovered by Jean-Claude Sikorav in [44], which appeared in preprint form somewhat earlier than this article, although later than my doctoral thesis. Sikorav's approach treats singularities more fully, elegantly and economically, by employing results of Micallef & White, while my approach has its advantages in organizing the calculations of local invariants, and thereby proving Sikorav's Main Conjecture. No new mathematical material has been added to this article since the appearance of Sikorav's preprint on LANL.

2. LINEAR ALGEBRA AND MICROLOCAL GEOMETRY

In this section we will first present a heuristic discussion of ellipticity, and then provide proofs.

2.1. Rough ideas of ellipticity. Consider a system of differential equations for surfaces in a 4-manifold whose general solution depends on arbitrary functions of one variable, and which has no creasing of solutions, i.e. in any metric, mean curvature of solutions is bounded by a constant on any compact set. Calculation shows that if the system is real analytic, then this system must be elliptic and determined. If we insist further that the system be first order, we call such a system a *pseudocomplex structure* by analogy with the Cauchy–Riemann equations of complex curves in a complex surface or almost complex 4-manifold. I will refer to the solutions of a pseudocomplex structure as its *curves*.

A more formal definition (exactly as in [33]): a *pseudocomplex structure* is a six-dimensional manifold E which is a fiber subbundle of $\widetilde{\text{Gr}}_2(TM) \rightarrow M$, the bundle of oriented 2-planes in the tangent spaces of a 4-manifold M , such that the requirement that a surface $\Sigma \subset M$ should have tangent spaces lying in E is equivalent to an elliptic system of partial differential equations. Call such a surface Σ an E *curve*. If the fibers of $E \rightarrow M$ are compact, then we will say that E is *proper* (because the map $E \rightarrow M$ is a proper map precisely when the fibers are compact). For example, take M a complex surface and let E be the collection of complex lines in the tangent spaces of M .

In a single fiber, $E_m \subset \widetilde{\text{Gr}}_2(T_m M)$ above a point $m \in M$, we have a surface in the Grassmannian of 2-planes. Projectivizing, this is a surface in the Grassmannian

of projective lines in projective 3-space, in other words a *line congruence*. We will now study these in detail.

Take V any four-dimensional vector space, and $X \rightarrow \widetilde{\text{Gr}}_2(V)$ any immersed surface. We will call X a line congruence. A complex structure on V , say $J : V \rightarrow V$, can be represented as a line congruence by its Riemann sphere of complex lines, which form a 2 parameter family of oriented real 2-planes. An arbitrary immersed surface $X \rightarrow \widetilde{\text{Gr}}_2(V)$ will be called *elliptic* if it is tangent at each point to the Riemann sphere of some complex structure. Such a structure is never unique. This notion of ellipticity is exactly the one needed for our definition above of pseudo-complex structure: the fibers $E_m \rightarrow \widetilde{\text{Gr}}_2(T_m M)$ of E are elliptic line congruences precisely when the differential equation for surfaces in M which is determined by E is elliptic. Another definition of ellipticity: thinking of a line congruence as a 2-parameter family X of lines in \mathbb{P}^3 , we want these lines to twist about any transverse line in \mathbb{P}^3 with nowhere vanishing angular momentum. The reader might want to draw a picture of a Riemann sphere as a line congruence to get a geometric feeling for this. A compact elliptic line congruence is precisely a fibration of \mathbb{P}^3 by projective lines.

Yet another approach to define ellipticity: in $\text{GL}(V)$ invariant fashion we can identify

$$T_p \widetilde{\text{Gr}}_2(V) \cong \text{Lin}(p, V/p)$$

so that the determinant of these linear maps, well defined up to scaling, provides a conformal quadratic form on the tangent spaces of the Grassmannian, of signature $(2, 2)$:

$$\det \begin{pmatrix} p+q & r+s \\ r-s & p-q \end{pmatrix} = p^2 + s^2 - q^2 - r^2.$$

This conformal quadratic form restricts to X to be a positive definite conformal structure precisely when X is an elliptic line congruence. The stabilizer of a 2-plane $p \in \widetilde{\text{Gr}}_2(V)$ acts transitively on positive definite 2-planes $\Pi \subset T_p \widetilde{\text{Gr}}_2(V)$, i.e. on positive definite 2-planes in $\text{Lin}(p, V/p)$. Therefore any two elliptic line congruences can be made to match up to first order at any points by linear transformation.

2.2. Making ellipticity more precise. Let us now prove the statements we have just made about ellipticity. We have defined ellipticity of a line congruence $X \rightarrow \widetilde{\text{Gr}}_2(V)$ to be the property that its tangent spaces $T_x X \subset T_P \widetilde{\text{Gr}}_2(V) = \text{Lin}(P, V/P)$ contain no linear maps of rank 1, or equivalently that the conformal quadratic form \det be definite. We can suppose by choice of orientation of V that it is positive definite. For a pair of partial differential equations

$$F^i \left(x^1, x^2, u^1, u^2, \frac{\partial u^j}{\partial x^k} \right) = 0, \quad i = 1, 2,$$

we wish to define ellipticity to be the absence of real points in the characteristic variety. It remains to show that these definitions coincide in the sense outlined above.

Lemma 1. *A pair of independent equations on the first derivatives of two functions of two variables is an elliptic system of partial differential equations (i.e. there are no real points in the characteristic variety of any integral element) precisely when the two equations cut out, for any fixed values of independent and dependent variables, a surface in the “space of first derivatives” (i.e. the Grassmannian of*

2-planes) which is an elliptic line congruence (i.e. the surface is definite for the quadratic form \det on the Grassmannian).

Proof. First let us note that for this pair of equations to be independent equations on the first derivatives of the two functions u^1, u^2 of the two variables x^1, x^2 , we need to ask that, if we write these two equations as

$$F^i(x^1, x^2, u^1, u^2, p_k^j) = 0$$

with fictitious variables p_k^j replacing $\frac{\partial u^j}{\partial x^k}$, then

$$\text{rk} \begin{pmatrix} \frac{\partial F^1}{\partial p_1^1} & \frac{\partial F^1}{\partial p_2^1} & \frac{\partial F^1}{\partial p_1^2} & \frac{\partial F^1}{\partial p_2^2} \\ \frac{\partial F^2}{\partial p_1^1} & \frac{\partial F^2}{\partial p_2^1} & \frac{\partial F^2}{\partial p_1^2} & \frac{\partial F^2}{\partial p_2^2} \end{pmatrix} = 2.$$

The characteristic variety is then defined by the equations

$$0 = \det \begin{pmatrix} \frac{\partial F^1}{\partial p_k^1} \xi^k & \frac{\partial F^1}{\partial p_k^2} \xi^k \\ \frac{\partial F^2}{\partial p_k^1} \xi^k & \frac{\partial F^2}{\partial p_k^2} \xi^k \end{pmatrix}$$

on some new variables ξ_1, ξ_2 (see Bryant et al. [19]). We are asking that these equations have no real solutions except $\xi = 0$. Write

$$DF^i = \begin{pmatrix} \frac{\partial F^1}{\partial p_1^1} & \frac{\partial F^1}{\partial p_1^2} \\ \frac{\partial F^1}{\partial p_2^1} & \frac{\partial F^1}{\partial p_2^2} \end{pmatrix}$$

and write $\xi = (\xi_1, \xi_2)$ a row vector. If we have such a solution ξ , then we obtain a linear relation between the row vectors ξDF^1 and ξDF^2 , which we can suppose, switching subscripts if needed, is $\xi DF^1 = \lambda \xi DF^2$. But then $DF^1 - \lambda DF^2$ has rank one. Conversely, if a linear combination of DF^1 and DF^2 has rank one, then we can pick such a ξ , and the characteristic variety is not empty.

Thinking now of our differential equations as defining a submanifold of the Grassmann bundle, we think of the 2-planes

$$du^i = p_j^i dx^j$$

as our family of 2-planes, with the equations $F^1 = F^2 = 0$ determining 2 constraints on those 2-planes, i.e. as the equations of our subbundle. Therefore the tangent space to this family of 2-planes is given by the equations $dF^1 = dF^2 = 0$. Taking the Grassmannian of 2-planes at a fixed point (x^1, x^2, u^1, u^2) , the tangent plane to our family of 2-planes is precisely cut out by the equations $dF^1 = dF^2 = 0$ with the added conditions that $dx^1 = dx^2 = du^1 = du^2 = 0$ forcing us to stay at a particular point. But then

$$dF^1 = \frac{\partial F^1}{\partial p_j^i} dp_j^i$$

so that the equations on the tangent space are precisely

$$DF^1 dp = DF^2 dp = 0.$$

The identification of tangent vectors $v \in T_P \widetilde{\text{Gr}}_2(V)$ with linear maps is carried out as follows: a curve in $\widetilde{\text{Gr}}_2(V)$ is a one parameter family $P(t)$ of 2-planes: $P(t) \subset V$. Taking any linear map $\phi(t) : V \rightarrow W$ with kernel $P(t)$, assuming that $\phi(t)$ depends smoothly on t , and letting $[\phi(t)] : V/P(t) \rightarrow W$ be the induced linear

map on $V/P(t)$, we can identify the vector $P'(0) \in T_{P(0)}\widetilde{\text{Gr}}_2(V)$ with the linear map

$$[\phi(0)]^{-1} \phi'(0) : P(0) \rightarrow V/P(0).$$

We then have the conformal quadratic form defined by taking determinant, which depends on a choice of volume element in $P(0)$ and in V , so that without choosing a volume element, the quadratic form is only defined up to conformal transformations.

Returning to our partial differential equation, in our local coordinate formulation, we can take V to have coordinates x^1, x^2, u^1, u^2 and let $P(0) = \{(x, 0) | x \in \mathbb{R}^2\}$ and take $\dot{p} = (\dot{p}_j^i)$ any tangent vector to our fiber, i.e. so that $DF^1\dot{p} = DF^2\dot{p} = 0$, and find that using

$$\phi(t) : (x, u) \mapsto u + t\dot{p}x$$

the matrix representing the vector \dot{p} is precisely \dot{p} . Note that we have assumed that the tangent vector is tangent to a point (x, u, p) with $p = 0$; the requirement that our point have $p = 0$ can be arranged by a change of coordinates, or we can use $\phi(t) = (x, u) \mapsto u + (p + t\dot{p})x$.

It is clear that a 2-plane is positive definite precisely when its dual space is positive definite. Note that the dual space of $\text{Lin}(P, V/P)$ is $\text{Lin}(V/P, P)$, and is equipped with the same quadratic form. We have seen that the absence of real points in the characteristic variety is precisely the definiteness of the tangent planes to the fibers of our subbundle. We can assume that they are positive definite by choice of orientation of V . \square

2.3. Properties of line congruences. The only known deep result on line congruences was found by Gluck and Warner [32]. If we impose a positive definite conformal structure on V and an orientation, then we have a splitting

$$\Lambda^2(V) = \Lambda^{2+}(V) \oplus \Lambda^{2-}(V)$$

of 2-forms into self-dual and anti-self-dual 2-forms, and the Plücker embedding

$$\widetilde{\text{Gr}}_2(V) \hookrightarrow \Lambda^2(V)/\mathbb{R}^+$$

which has as image the projectivization of the locus $\{\xi | \xi^2 = 0\}$. This gives a diffeomorphism

$$\widetilde{\text{Gr}}_2(V) \cong \Lambda^{2+}(V)/\mathbb{R}^+ \times \Lambda^{2-}(V)/\mathbb{R}^+ \cong S^{2+} \times S^{2-}$$

into a product of spheres, which is invariant under the orientation preserving conformal group of V .

Theorem 1 (Gluck and Warner). *Given an immersed surface $\phi : X \subset \widetilde{\text{Gr}}_2(V)$ (i.e. a line congruence) write*

$$\phi_+ : X \rightarrow S^{2+}$$

and ϕ_- similarly, for the induced projections to the two spheres S^{2+} and S^{2-} . A line congruence is elliptic precisely when $|\phi'_-| > |\phi'_+|$ at all points of X .

Proof. Indeed we will see that the $(2, 2)$ conformal structure on $\widetilde{\text{Gr}}_2(V)$ is just $|\cdot_-|^2 - |\cdot_+|^2$. The result is surprising since the notion of ellipticity of a line congruence is invariant under arbitrary linear transformations of V , while the splitting of $\Lambda^2(V)$ into self-dual and anti-self-dual 2-forms is not. To see the result, note that a 2-plane P can be represented by a 2-vector $u \wedge v$ with u, v a basis for P , and this 2-vector is unique up to scaling. With orientation taken into account, it is unique up to positive rescaling. Conversely, every 2-vector γ with $\gamma^2 = 0$ is of this form for a

unique 2-plane P . Taking a metric on V , we identify 2-vectors and 2-forms, and split our 2-vector into self-dual and anti-self-dual pieces: if x^1, \dots, x^4 are orthogonal coordinates on V , and $dx^{ij} = dx^i \wedge dx^j$, we can express a 2-form uniquely as

$$(1) \quad \begin{aligned} \gamma = & (X_1 + Y_1) dx^{12} + (X_1 - Y_1) dx^{34} + (X_2 + Y_2) dx^{31} \\ & + (X_2 - Y_2) dx^{24} + (X_3 + Y_3) dx^{23} + (X_3 - Y_3) dx^{14} \end{aligned}$$

so that

$$\gamma^2 = 2(X_1^2 + X_2^2 + X_3^2 - Y_1^2 - Y_2^2 - Y_3^2) dx^{1234}.$$

The space of 2-forms satisfying $\gamma^2 = 0$ is just the space of X, Y with $|X| = |Y|$. To mod out by rescaling, we can always take $|X|^2 + |Y|^2 = 2$. Then it is clear that $\widetilde{\text{Gr}}_2(V) = S_+^2 \times S_-^2$ where X provides coordinates on S_+^2 and Y on S_-^2 . We will now see that in these coordinates the conformal quadratic form is $|\dot{Y}|^2 - |\dot{X}|^2$. Take the family of 2-planes $P(t)$ given by

$$\begin{pmatrix} dx^3 \\ dx^4 \end{pmatrix} = -t\dot{p} \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix}$$

and consider the 2-vector

$$u(t) \wedge v(t) = (\partial_1 + t\dot{p}_1^1 \partial_3 + t\dot{p}_1^2 \partial_4) \wedge (\partial_2 + t\dot{p}_2^1 \partial_3 + t\dot{p}_2^2 \partial_4)$$

which represents the 2-plane $P(t)$. Calculate that

$$\left(\left. \frac{d}{dt} u(t) \wedge v(t) \right|_{t=0} \right)^2 = -2 \det \dot{p} dx^{1234}.$$

□

Corollary 1. *If $X \subset \widetilde{\text{Gr}}_2(V)$ is a compact elliptic line congruence, then X is diffeomorphic to S^{2-} , hence a sphere. A compact elliptic line congruence is precisely the graph of a strictly contracting map*

$$S^{2-} \rightarrow S^{2+}$$

For example, the Riemann sphere of a complex structure $J : V \rightarrow V$ will be represented, in complex linear coordinates on V , by a constant map $S^{2-} \rightarrow S^{2+}$. However, after a real linear transformation of V , it can look somewhat different; the picture of the Riemann sphere in arbitrary real linear coordinates is worked out in [30].

2.4. Real curves.

Corollary 2. *Given an elliptic line congruence $X \subset \widetilde{\text{Gr}}_2(V)$, every line through the origin in V is contained in a discrete set of 2-planes belonging to X . Similarly (replacing V by V^*), every 3-plane in V contains a discrete set of 2-planes belonging to X .*

Proof. The set of 2-planes containing the x^1 axis is given in our coordinates by $X_1 = Y_1, X_2 = Y_2, X_3 = -Y_3$ since these 2-planes must be represented as 2-vectors by $\partial_1 \wedge (\dots)$. But this is the graph of an isometry $S_-^2 \rightarrow S_+^2$ and therefore cannot have any tangent vectors in common with a strictly contracting map. □

The set of 2-planes containing a given line sits inside $\widetilde{\text{Gr}}_2(V)$ as the graph of an orientation reversing isometry $S_-^2 \rightarrow S_+^2$, and every orientation reversing isometry occurs this way. To see this, use the action of $\text{SO}(4)$ on $\widetilde{\text{Gr}}_2(V)$. Similarly the orientation preserving isometries are constructed by taking the 2-planes contained in a given 3-plane.

The homology class of the set of 2-planes containing a given line is therefore $[S_-^2] - [S_+^2]$. The intersections with $S_-^2 \times \text{pt}$ are therefore all negative, and so are the intersections with any elliptic line congruence because the intersections do not change sign as we deform elliptic line congruences.

Corollary 3. *A compact elliptic line congruence contains exactly one 2-plane in V containing each line through 0 in V . Moreover, the 2-plane depends smoothly on the choice of line, by transversality.*

Proof. The intersections are all negative, but the intersection number is -1 . \square

We think of this as infinitesimally solving a Cauchy problem, complexifying real curves. Replacing V by V^* , we see that a compact elliptic line congruence contains exactly one 2-plane in V contained in each 3-plane through 0 in V . We think of this as infinitesimal real hypersurface geometry.

Corollary 4. *Any two 2-planes belonging to the same compact elliptic line congruence are transverse.*

2.5. Families of elliptic line congruences.

Proposition 1. *The graph of a strictly contracting map $S^2 \rightarrow S^2$ has image contained in the interior of a single hemisphere.*

Proof. Suppose that there are two antipodal points $x_1, x_2 \in S^2$ that get mapped to the same point $y \in S^2$. Then any point in the image must be the image of a point x lying in a hemisphere about either x_1 or x_2 . So any point in the image lies in the interior of a hemisphere about y . Therefore we need only show that some pair of antipodal points x_1, x_2 get mapped to the same point y .

The image cannot contain two antipodal points, because their preimages could be no further apart than antipodal points, and the map is strictly contracting. Therefore, the image misses some point of S^2 and the result is clear from the following lemma. \square

Lemma 2. *Any continuous map*

$$\phi : S^2 \rightarrow \mathbb{R}^2$$

must map some pair of antipodal points to the same point.

Proof. Suppose otherwise. Let

$$\sigma(x) = \frac{\phi(x) - \phi(-x)}{\|\phi(x) - \phi(-x)\|}.$$

This is a continuous map $\sigma : S^2 \rightarrow S^1$ so that

$$\sigma(-x) = -\sigma(x).$$

Because S^2 is simply connected, there is some continuous map $\tilde{\sigma} : S^2 \rightarrow \mathbb{R}$ so that

$$\sigma = e^{i\tilde{\sigma}}.$$

But then

$$\tilde{\sigma}(-x) = \tilde{\sigma}(x) + (2k + 1)\pi$$

for some integer k . However, plugging in $-x$:

$$\begin{aligned} \tilde{\sigma}(x) &= \tilde{\sigma}(-(-x)) \\ &= \tilde{\sigma}(-x) + (2k + 1)\pi \\ &= \tilde{\sigma}(x) + 2(2k + 1)\pi \end{aligned}$$

which is a contradiction. \square

Corollary 5. *Every compact elliptic line congruence can be smoothly deformed into any other through a family of compact elliptic line congruences.*

Proof. Rotation by $\text{SO}(4)$ and dilation into a map to a point takes us into the Cauchy–Riemann equations. \square

Corollary 6. *Every finite-dimensional family $X \rightarrow B$ of elliptic line congruences in the fibers of a vector bundle $V \rightarrow B$ can be deformed globally and smoothly into a family of Riemann spheres, i.e. a complex vector bundle structure on V .*

Proof. Smoothly pick a positive definite inner product on each fiber of V . The “center of mass” of the image of each strictly contracting map $S_-^2 \rightarrow S_+^2$ provides a target to deform to. \square

2.6. Taming.

Definition 1. *A symplectic form $\Omega \in \Lambda^2(V^*)$ tames an elliptic line congruence $X \subset \widetilde{\text{Gr}}_2(V)$ if Ω is positive on every oriented 2-plane $P \subset V$ such that $P \in X$. An elliptic line congruence is tameable if it is tamed by some symplectic form.*

Corollary 7. *Every compact elliptic line congruence is tameable.*

Proof. Suppose that the line congruence is written as the graph of a map

$$\phi : S^{2-} \rightarrow S^{2+}.$$

Suppose that the image of ϕ is contained in the hemisphere about $\Omega \in S^{2+}$. Suppose also that we have selected on V not just a conformal class of positive definite quadratic form, but an actual quadratic form. Then our Ω can be thought of as a linear function on S^{2+} which is positive on the image of ϕ . At the same time, $\Omega \in \Lambda^{2+}(V) \cong \Lambda^{2+}(V^*)$ is a 2-form. Take any $P \in X$, i.e.

$$P = (v_1 \wedge v_2, w_1 \wedge w_2) \in S^{2+} \times S^{2-}.$$

We have

$$\Omega \in S^{2+} \subset \Lambda^{2+}(V^*)$$

so that

$$\Omega(P) = \Omega(v_1 \wedge v_2) > 0$$

because Ω is positive on the image of ϕ . Therefore Ω is a taming symplectic structure. \square

Lemma 3. *If a 2-form is positive on every 2-plane belonging to a compact elliptic line congruence (i.e. taming), then it is symplectic. The space of symplectic forms taming a compact elliptic line congruence is an open convex cone in the space of symplectic 2-forms.*

Proof. If $\omega > 0$ on every 2-plane in X a compact elliptic line congruence, then pick $P_1, P_2 \in X$ with $P_1 \neq P_2$. Then $P_1 \cap P_2 = 0$ so that $V = P_1 \oplus P_2$ and $\omega > 0$ on both P_1 and P_2 so $\omega^2 > 0$ on V .

If two 2-forms ω_1, ω_2 both tame a compact elliptic line congruence X , then clearly $a\omega_1 + b\omega_2$ does as well for any $a, b > 0$. \square

Definition 2. A pseudocomplex structure $E \subset \widetilde{\text{Gr}}_2(TM)$ is tamed by a symplectic form $\Omega \in \Omega^2(M)$ if $\Omega_m \in \Lambda^2(T_m^*M)$ tames $E_m \subset \widetilde{\text{Gr}}_2(T_mM)$ for every $m \in M$.

Corollary 8. Any proper pseudocomplex structure is locally tameable.

The significance of taming symplectic forms will soon be made clear.

Proposition 2. The space of compact elliptic line congruences tamed by a given symplectic form $\Omega \in \Lambda^2(V^*)$ is contractible.

Proof. We can arrange, by linear transformation, that Ω is an element of S^{2+} . Then the taming condition on an elliptic line congruence is that, thinking of it as a map $\psi : S^{2-} \rightarrow S^{2+}$, its image is contained in the hemisphere of S^{2+} containing Ω . But then we can “shrink” ψ , for example using the Riemannian geometry exponential map about Ω in S^{2+} , which contracts the image and preserves the ellipticity (the strict contractivity) of ψ . \square

A 2-form ω on a manifold M^4 with the property that ω^2 does not vanish at any point is called an *almost symplectic structure*.

Proposition 3. The space of almost symplectic structures taming a fixed proper pseudocomplex structure is an affine space, not empty.

Proof. We can pick these almost symplectic structures locally, and glue them together using convex positive combinations. \square

Proposition 4. A four-dimensional manifold admits a proper pseudocomplex structure precisely when it admits an almost complex structure, which occurs precisely when it admits an almost symplectic structure.

Proof. As is well known, and proven in Steenrod [45], a manifold admits an almost symplectic structure precisely when it admits an almost complex structure, and that almost complex structure can be chosen to be tamed by the almost symplectic structure. Then the Cauchy–Riemann equations of pseudoholomorphic curves for the almost complex structure provide a tamed proper pseudocomplex structure. \square

Corollary 9. Every symplectic 4-manifold admits a proper pseudocomplex structure tamed by its symplectic structure.

Corollary 10. Every proper pseudocomplex structure tamed by a given almost symplectic structure can be deformed through proper pseudocomplex structures tamed by that same almost symplectic structure into an almost complex structure.

Proof. As for Corollaries 6 and 7. \square

Proposition 5. Suppose that $X \subset \widetilde{\text{Gr}}_2(V)$ is a compact elliptic line congruence. Then any two distinct oriented 2-planes belonging to X intersect transversely and positively at the origin.

Proof. The positivity of intersection holds for Riemann spheres, while transversality holds for all compact elliptic line congruences, and we can deform any compact elliptic line congruence into a Riemann sphere. \square

Corollary 11. *Any two transverse E curves of a proper pseudocomplex structure E intersect positively.*

A deeper problem is the positivity of intersections of singular and nontransverse E curves. We will address this problem in Section 7.

2.7. The Grassmannian as homogeneous space. The Grassmannian is a homogeneous space

$$\widetilde{\text{Gr}}_2(\mathbb{R}^4) = \text{GL}^+(4, \mathbb{R}) / H_0$$

where H_0 is the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $\det a, \det d > 0$, i.e. the isotropy subgroup of the 2-plane $P_0 = (x^3 = x^4 = 0)$. (Here, $\text{GL}^+(4, \mathbb{R})$ means the 4×4 matrices with positive determinant.) The tangent space to $\widetilde{\text{Gr}}_2(\mathbb{R}^4)$ is identified with the quotient of Lie algebras, i.e. with matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

modulo those of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

hence with the c component, i.e. the lower left 2×2 matrix. To relate this point of view to the others we have pursued, consider a family of 2-planes $P(t)$ given by the equations

$$\begin{pmatrix} dx^3 \\ dx^4 \end{pmatrix} + t\dot{p} \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix}.$$

Then $P(t) = g(t)P_0$ where

$$g(t) = \begin{pmatrix} 1_2 & 0 \\ -t\dot{p} & 1_2 \end{pmatrix}.$$

Here 1_2 means the 2×2 identity matrix. Thus in terms of the homogeneous space point of view, $c = -\dot{p}$ and the conformal quadratic form is $\det c = \det \dot{p}$.

The action of H_0 on the tangent space $T_{P_0}\widetilde{\text{Gr}}_2(\mathbb{R}^4)$ is given by quotienting the adjoint action:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} * & * \\ c & * \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ dca^{-1} & * \end{pmatrix}$$

so the action is $c \mapsto dca^{-1}$ which clearly preserves the quadratic form $c \mapsto \det c$ up to a positive factor. Moreover the action is transitive on positive tangent vectors to the Grassmannian, i.e. on matrices c with positive determinant, and on negative vectors, and on nonzero null vectors, i.e. matrices c of rank 1.

Obviously the symmetric bilinear form associated to the quadratic form $c \mapsto \det c$ is

$$\left\langle \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix}, \begin{pmatrix} B_1^1 & B_2^1 \\ B_1^2 & B_2^2 \end{pmatrix} \right\rangle = A_1^1 B_2^2 + B_1^1 A_2^2 - A_2^1 B_1^2 - B_2^1 A_1^2.$$

Lemma 4. *Consider bases of $T_{P_0}\widetilde{\text{Gr}}_2(\mathbb{R}^4)$ which are orthonormal for the quadratic form, up to a positive scaling factor, and are positively oriented. The group H_0 acts on such bases with two orbits.*

Proof. Take such a basis v_1, v_2, v_3, v_4 . We may rescale it to ensure that

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } 2 \\ -1 & \text{if } i = j = 3 \text{ or } 4 \\ 0 & \text{otherwise.} \end{cases}$$

Think of each v_i as a 2×2 matrix. Using the action $v \mapsto dva^{-1}$ we can first arrange that $v_1 = 1_2$. Now we have to work with the subgroup of H_0 fixing 1_2 , which is the group of transformations $v \mapsto ava^{-1}$. Calculate that $\langle v_1, v \rangle = \text{tr } v$ so that v_2, v_3, v_4 are now traceless. But then v_2 must be traceless, have determinant 1, and so has minimal polynomial $t^2 + 1$, i.e. is a complex structure. We can therefore arrange by $v \mapsto ava^{-1}$ that $v_2 = \pm J$ where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let K be complex conjugation, i.e.

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Check that K and JK are perpendicular to $v_1 = 1_2$ and $v_2 = \pm J$, and to one another. So v_2 and v_3 must be obtained from K and JK by an orthogonal transformation of the plane they span. But we still have the freedom to employ the subgroup of H_0 fixing $v_1 = 1_2$ and $v_2 = \pm J$, i.e. the transformations $v \mapsto ava^{-1}$ where a is complex linear. This enables us to rotate v_3 into K and then get $v_4 = \pm JK$.

But now the orientation of the basis forces the two \pm signs to be equal. \square

Corollary 12. *The general linear group $\text{GL}^+(V)$ acts transitively on $\widetilde{\text{Gr}}_2(V)$ and moreover acts transitively on the positive definite 2-planes in the tangent spaces of $\widetilde{\text{Gr}}_2(V)$.*

Proof. Given any positive definite 2-plane $\Pi \subset T_{P_0}\widetilde{\text{Gr}}_2(V)$ we can take a basis v_1, v_2, v_3, v_4 for $T_{P_0}\widetilde{\text{Gr}}_2(V)$ which is orthonormal, up to a positive scaling factor, with v_1, v_2 spanning Π and we can use the group $H_0 \subset \text{GL}^+(V)$ to get it to the form $1_2, \pm J, K, \pm JK$. \square

Corollary 13. *A line congruence is tangent to a Riemann sphere at a point precisely when it is elliptic near that point.*

2.8. Totally real surfaces.

Definition 3. *Take $X \subset \widetilde{\text{Gr}}_2(V)$ an elliptic line congruence, and $R \subset V$ a 2-plane, so that with either orientation, R does not belong to X . Call such an R a totally real 2-plane for X .*

Lemma 5. *A vector w belongs to a 2-plane P represented by a 2-vector $u \wedge v$ precisely when $w \wedge u \wedge v = 0$.*

Lemma 6. *A pair of 2-planes P_0 and P_1 represented by 2-vectors $u_0 \wedge v_0$ and $u_1 \wedge v_1$ intersect in a line or coincide (modulo orientation) precisely when*

$$u_0 \wedge v_0 \wedge u_1 \wedge v_1 = 0.$$

Proof. If the 2-planes P_0 and P_1 contain a common nonzero vector, we can take u_0 and u_1 to be that vector. Conversely, if $u_0 \wedge v_0 \wedge u_1 \wedge v_1 = 0$, then by Cartan's lemma

$$u_1 \wedge v_1 = u_0 \wedge x + v_0 \wedge y$$

for some vectors x and y . Squaring both sides

$$0 = u_0 \wedge v_0 \wedge x \wedge y.$$

Without loss of generality, we take $u_0 = e_1, v_0 = e_2$ in a basis e_1, e_2, e_3, e_4 . We find then that the matrix

$$\begin{pmatrix} x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}$$

has vanishing determinant, so has kernel. Then take $v = v_1 e_1 + v_2 e_2$ and

$$\begin{aligned} v \wedge u_1 \wedge v_1 &= v \wedge (e_1 \wedge x + e_2 \wedge y) \\ &= e_1 \wedge e_2 \wedge (v_1 y - v_2 x) \end{aligned}$$

so we can pick v to make this vanish. \square

Using equation 1 on page 6 to write $u_0 \wedge v_0$ in terms of X_0 and Y_0 , one finds

$$u_0 \wedge v_0 \wedge u_1 \wedge v_1 = 2(\langle Y_0, Y_1 \rangle - \langle X_0, X_1 \rangle) dx^{1234}.$$

Lemma 7. *A pair of 2-planes P_0 and P_1 represented by points (X_0, Y_0) and (X_1, Y_1) in $S_+^2 \times S_-^2$ intersect in a line or coincide (modulo orientation) precisely when*

$$\langle X_0, X_1 \rangle = \langle Y_0, Y_1 \rangle.$$

Proposition 6. *Let $X \rightarrow \widetilde{\text{Gr}}_2(V)$ be an elliptic line congruence and R a totally real 2-plane for X . Define $X(R)$ to be the elements of X which, as 2-planes in V , intersect the 2-plane R in a line. Then $X(R) \subset X$ is a smooth curve (called the R real points of X).*

Proof. Suppose that $X(R)$ has a singular point. Pick a complex structure on V tangent to X at that singular point. Suppose that R is represented in the splitting

$$\widetilde{\text{Gr}}_2(V) = S^{2+} \times S^{2-}$$

by $R = (X_0, Y_0)$, while the line congruence X is represented, at least locally, by a strictly contracting map $\psi : S_-^2 \rightarrow S_+^2$. Then a point $(\psi(Y), Y)$ on the graph of ψ represents a 2-plane intersecting the 2-plane R in at least a line precisely when

$$0 = -\langle X_0, \psi(Y) \rangle + \langle Y_0, Y \rangle.$$

We can arrange that the Riemann sphere tangent to X at this point be $X_0 \times S_-^2$. Taking the gradient with respect to x , we find that singular points will be precisely those where

$$Y_0 - \psi'(Y)^t X_0 \perp T_Y S^{2-}$$

where $()^t$ indicates transpose. But by the tangency condition, $\psi'(Y) = 0$. So therefore $Y_0 = \pm X_0$. But then

$$\begin{aligned} 0 &= -\langle X_0, \psi(Y) \rangle + \langle Y_0, Y \rangle \\ &= -\langle X_0, \psi(Y) \rangle \pm 1 \end{aligned}$$

so that $X_0 = \pm\psi(Y)$. This shows that R is $(\psi(Y), Y)$ up to reorientation, so that intersection occurs on more than a line. Hence R is not totally real. \square

Corollary 14. *If $X \subset \widetilde{\text{Gr}}_2(V)$ is a compact elliptic line congruence and $R \subset V$ is a totally real 2-plane for X , then $X(R) \subset X$ is a smooth embedded circle.*

Proof. We can deform X to any of its osculating complex structures while keeping R totally real. We may have to move R while we move X , but it is easy to arrange that R , with either orientation, never belongs to X during the deformation of X since R is just a point in the Grassmannian, while X is a surface. In the process, we never generate a singular point on $X(R)$. The result is now a calculation for the standard Riemann sphere on $V = \mathbb{C}^2$. \square

2.9. The moving frame. We will now employ Cartan's method of the moving frame to uncover differential invariants of elliptic line congruences, invariant under orientation preserving linear transformations of V . Let B_0 be the set of orientation preserving linear maps identifying V with \mathbb{R}^4 . Pick a subspace $\mathbb{R}^2 = \mathbb{R}^2 \oplus 0 \subset \mathbb{R}^4$. Consider the map $B_0 \rightarrow \widetilde{\text{Gr}}_2(V)$ given by taking for each $\lambda \in B_0$ the subspace $\lambda^{-1}\mathbb{R}^2 \subset V$, and we orient this subspace so that λ is orientation preserving. Then $B_0 \rightarrow \widetilde{\text{Gr}}_2(V)$ is a principal left H_0 bundle, where H_0 is the group of invertible matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with a, b, c representing 2×2 matrices, and $\det a > 0$ and $\det c > 0$.

On B_0 we have the Maurer–Cartan 1-form

$$\mu = d\lambda\lambda^{-1} \in \Omega^1(B_0) \otimes \mathfrak{gl}(4, \mathbb{R})$$

which satisfies

$$(2) \quad d\mu = \mu \wedge \mu$$

and under left action of H_0

$$L_h^* \mu = \text{Ad}_h \mu.$$

We will split μ into complex linear and complex conjugate linear parts:

$$\mu \cdot v = \begin{pmatrix} \xi & \eta \\ \vartheta & \zeta \end{pmatrix} v + \begin{pmatrix} \xi' & \eta' \\ \vartheta' & \zeta' \end{pmatrix} \bar{v}$$

for $v \in \mathbb{R}^4 = \mathbb{C}^2$. Then we calculate that equation (2) becomes

$$d \begin{pmatrix} \xi \\ \xi' \\ \eta \\ \eta' \\ \vartheta \\ \vartheta' \\ \zeta \\ \zeta' \end{pmatrix} = \begin{pmatrix} \xi' \wedge \bar{\xi}' + \eta \wedge \vartheta + \eta' \wedge \bar{\vartheta}' \\ \xi \wedge \xi' + \xi' \wedge \bar{\xi} + \eta \wedge \vartheta' + \eta' \wedge \bar{\vartheta} \\ \xi \wedge \eta + \xi' \wedge \bar{\eta}' + \eta \wedge \zeta + \eta' \wedge \bar{\zeta}' \\ \xi \wedge \eta' + \xi' \wedge \bar{\eta} + \eta \wedge \zeta' + \eta' \wedge \bar{\zeta} \\ (\zeta - \xi) \wedge \vartheta - \xi' \wedge \vartheta' + \zeta' \wedge \bar{\vartheta}' \\ -\xi' \wedge \vartheta + \zeta' \wedge \bar{\vartheta} - (\bar{\xi} - \zeta) \wedge \vartheta' \\ \vartheta \wedge \eta + \vartheta' \wedge \bar{\eta}' + \zeta' \wedge \bar{\zeta}' \\ \vartheta \wedge \eta' + \vartheta' \wedge \bar{\eta} + \zeta \wedge \zeta' + \zeta' \wedge \bar{\zeta} \end{pmatrix}.$$

On each fiber of $B_0 \rightarrow \widetilde{\text{Gr}}_2(V)$, we have $\vartheta = \vartheta' = 0$, i.e. ϑ, ϑ' are semibasic.

Now suppose that we have an elliptic line congruence $X \subset \widetilde{\text{Gr}}_2(V)$. First, we can form the pullback bundle

$$\begin{array}{ccc} B_1(X) & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & \widetilde{\text{Gr}}_2(V). \end{array}$$

This is also a principal left H_0 subbundle. The fibers of $B_0 \rightarrow \widetilde{\text{Gr}}_2(V)$ are cut out by the equations

$$\vartheta = \bar{\vartheta} = \vartheta' = \bar{\vartheta}' = 0$$

(four independent equations) so that $\vartheta, \bar{\vartheta}, \vartheta', \bar{\vartheta}'$ are semibasic for $B_0 \rightarrow \widetilde{\text{Gr}}_2(V)$. The fibers of $B_1(X)$ are cut out by the same equations, but on $B_1(X)$. Since X has only two dimensions, we must have at least two relations among these four 1-forms on $B_1(X)$. Let us try to normalize these relations.

To do this, let us examine X near some point $p_0 \in X$. This surface X is tangent to a Riemann sphere at each point, so pick a Riemann sphere X_0 tangent to X at p_0 . Suppose that X_0 is the Riemann sphere of the complex structure $J_0 : V \rightarrow V$. Let H_2 be the subgroup of H_0 consisting of matrices

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where $a, c \in \mathbb{C}^\times$ are complex linear, and b is still a real linear 2×2 matrix. The tangent space $T_{p_0}X_0$ to a Riemann sphere associated to a complex structure $J_0 : V \rightarrow V$ is canonically identified with the J_0 complex linear maps $p_0 \rightarrow V/p_0$. Above each point p in our Riemann sphere, we have a distinguished subset of $B_1(X_0)$: those elements $\lambda \in B_1(X_0)$ which identify $p = \lambda^{-1}\mathbb{R}^2$ with $\mathbb{R}^2 = \mathbb{C}$ by a complex linear map. Call this $B_2(X_0)$. It is easy to show by homogeneity under $\text{GL}(2, \mathbb{C})$ that $B_2(X_0)$ is a manifold and that $B_2(X_0) \rightarrow X_0$ is a principal left H_2 bundle. If X_0 and X_1 are the Riemann spheres of complex structures J_0 and J_1 , and X_0 is tangent to X_1 at a point $p \in \widetilde{\text{Gr}}_2(V)$, then we have canonical identifications:

$$\{J_0 \text{ linear maps } p \rightarrow V/p\} \cong T_pX_0 = T_pX_1 \cong \{J_1 \text{ linear maps } p \rightarrow V/p\}.$$

Lemma 8. *If X_0 and X_1 are the Riemann spheres of complex structures J_0 and J_1 on a four-dimensional vector space V , and X_0 is tangent to X_1 at a point $p \in \widetilde{\text{Gr}}_2(V)$, then J_0 and J_1 agree on p and on V/p .*

Proof. The characteristic variety of the tableaux

$$A_k = \{J_k \text{ linear maps } p \rightarrow V/p\}$$

for $k = 0, 1$ is the same for $k = 0$ as for $k = 1$ because $A_0 = A_1$. But by definition,

$$\Xi_{\mathbb{C}}(A_k) = \{v \in p \otimes \mathbb{C} \mid \exists \xi \in ((V/p) \otimes \mathbb{C})^*, \xi \otimes v \in A_k\}.$$

Calculating in complex coordinates, we find that the characteristic variety is the union of the two eigenspaces of J_0 , with eigenvalues $\pm\sqrt{-1}$. This splitting of the characteristic variety (as a real algebraic variety) determines J_0 up to sign on p , and so $J_1 = \pm J_0$ on p . If we had a minus sign here, that would give p the opposite orientation. Similarly, by taking transposes, we find $J_1 = J_0$ on V/p . \square

This shows that the fibers match:

$$B_2(X_0)_p = B_2(X_1)_p.$$

We can now unambiguously define the set

$$B_2(X)$$

for any elliptic line congruence X by the equation

$$B_2(X)_p = B_2(X_0)_p$$

whenever X_0 is a Riemann sphere tangent to X at p . It is still not clear that this set $B_2(X)$ is a principal left H_2 subbundle of $B_1(X) \rightarrow X$.

Lemma 9. $B_2(X) \subset B_0$ is a smooth submanifold satisfying the equation $\vartheta' = 0$.

Proof. Recall that we defined B_0 to be the real linear isomorphisms

$$V \rightarrow \mathbb{R}^4$$

matching orientation. There is a map

$$\lambda : B_0 \rightarrow \text{Lin}(V, \mathbb{R}^4)$$

defined by inclusion. The Maurer–Cartan 1-form is just $\mu = d\lambda\lambda^{-1}$. Take X_0 a Riemann sphere tangent to X at p , and J_0 the associated complex structure. Let us assume that $V = \mathbb{R}^4 = \mathbb{C}^2$, with complex linear coordinates z, w for simplicity of notation, so that J_0 is the standard complex structure, and consider the quantity

$$Q = i\lambda^{-1}(\partial_z \wedge \partial_{\bar{z}})$$

so that

$$Q : B_0 \rightarrow \Lambda^{1,1}(V^*).$$

We calculate that

$$J_0 Q = Q \text{ at } \lambda = I$$

and that again at $\lambda = I$

$$d(J_0 Q - Q) = 2i\vartheta' \partial_z \wedge \partial_w + 2i\bar{\vartheta}' \partial_{\bar{w}} \wedge \partial_{\bar{z}}$$

so that indeed $\vartheta' = 0$ precisely along the directions of $T_\lambda B_2(X_0)$, where X_0 is the Riemann sphere of our complex structure.

Consequently, every Riemann sphere X_0 tangent to X at a point λ must satisfy

$$T_\lambda B_2(X_0) = \{\vartheta' = 0\} \subset T_\lambda B_1(X_0) = T_\lambda B_1(X).$$

This shows us that the relations among $\vartheta, \bar{\vartheta}, \vartheta', \bar{\vartheta}'$ near the locus $B_2(X) \subset B_0(X)$ cannot be

$$\vartheta = \bar{\vartheta} = 0$$

since these relations would also have to hold on

$$T_\lambda B_2(X_0) = T_\lambda B_2(X).$$

Indeed, on $B_2(X_0)$ we still have $\vartheta \wedge \bar{\vartheta} \neq 0$, since $\vartheta' = \bar{\vartheta}' = 0$ there, but

$$B_2(X_0) \rightarrow X_0$$

is a submersion, and $\vartheta \wedge \bar{\vartheta}$ is semibasic for it, providing a basis for the semibasic forms (check this in complex coordinates). Therefore, near $B_2(X)$ we must find that the relations among $\vartheta, \bar{\vartheta}, \vartheta', \bar{\vartheta}'$ can be written

$$\begin{pmatrix} \vartheta' \\ \bar{\vartheta}' \end{pmatrix} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \vartheta \\ \bar{\vartheta} \end{pmatrix}.$$

We can follow how these relations behave as we travel up the fibers of $B_1(X)$. We find that the coefficients a, b can be made to vanish, on a principal left H_2 bundle. Moreover, it is easy to see that $B_2(X)$ is precisely this bundle, following the general formalism presented in [35]. \square

Now we find that on $B_2(X)$, our equations are $0 = \vartheta'$ which implies from $d\mu = \mu \wedge \mu$ that

$$\begin{pmatrix} \xi' \\ \zeta' \end{pmatrix} = \begin{pmatrix} f & h \\ -h & g \end{pmatrix} \begin{pmatrix} \vartheta \\ \bar{\vartheta} \end{pmatrix}$$

for some uniquely determined complex valued functions

$$f, g, h : B_1 \rightarrow \mathbb{C}.$$

Under the action of H_2 we find that traveling up the fibers of $B_2(X) \rightarrow X$,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} a^2 \bar{a}^{-1} c^{-1} f \\ \bar{a} c \bar{c}^{-2} g \\ (ah + b') \bar{c}^{-1} \end{pmatrix}$$

where b' is the complex conjugate linear part of b . Therefore, letting H_3 be the group of matrices

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with a, b, c complex numbers, and $a, c \neq 0$, we find a principal left H_3 subbundle of $B_2(X)$, call it $B_3(X)$, on which $h = 0$.

Theorem 2. *Let V be a four-dimensional real vector space, and B_0 the set of all linear isomorphisms $V \rightarrow \mathbb{R}^4$. Let $B_0 \rightarrow \widetilde{\text{Gr}}_2(V)$ be the map*

$$\lambda \mapsto \lambda^{-1} \mathbb{R}^2 \oplus 0.$$

Let H be the group of complex 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

Each elliptic line congruence

$$X^2 \subset \widetilde{\text{Gr}}_2(V)$$

determines invariantly a principal left H subbundle $B(X) \subset B_0$ so that

$$\begin{array}{ccc} B(X) & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & \widetilde{\text{Gr}}_2(V) \end{array}$$

and so that the Maurer–Cartan 1-form $\mu = d\lambda \lambda^{-1}$ on B_0 , when written out as

$$\mu \cdot v = \begin{pmatrix} \xi & \eta \\ \vartheta & \zeta \end{pmatrix} v + \begin{pmatrix} \xi' & \eta' \\ \vartheta' & \zeta' \end{pmatrix} \bar{v}$$

for $v \in \mathbb{R}^4 = \mathbb{C}^2$, satisfies on $B(X)$ the equations

$$\begin{aligned}\vartheta' &= 0 \\ \xi' &= f\vartheta \\ \zeta' &= g\bar{\vartheta} \\ \eta' &= -s\vartheta - t\bar{\vartheta}\end{aligned}$$

for uniquely determined complex valued functions

$$f, g, s, t : B(X) \rightarrow \mathbb{C},$$

and so that the covariant derivatives

$$\begin{aligned}\nabla f &= df - f(2\xi - \bar{\xi} - \zeta) \\ \nabla g &= dg - g(\zeta - 2\bar{\zeta} + \bar{\xi})\end{aligned}$$

satisfy

$$\nabla \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} u & s \\ t & v \end{pmatrix} \begin{pmatrix} \vartheta \\ \bar{\vartheta} \end{pmatrix}$$

for uniquely determined complex valued functions

$$u, v : B(X) \rightarrow \mathbb{C}.$$

At each point p of X , we therefore have a well-defined family of maps $\lambda : V \rightarrow \mathbb{R}^4$, forming the fiber of $B(X)$, determined up to H action. By complex linearity of the elements of the group H , this determines a complex structure $J(p)$ on V , for which the real 2-plane p is a complex line, and for which V/p is also a complex line. We call this the *osculating complex structure* to X at p . It determines a map

$$X \rightarrow \mathcal{J}(V).$$

The osculating complex structure determines a complex structure on X itself in the obvious manner: each tangent space to X is identified with the complex linear maps

$$p \rightarrow V/p$$

for the osculating complex structure, hence a one-dimensional complex vector space. Consider the map $X \rightarrow \mathcal{J}(V)$ from our surface X to the space of complex structures on the vector space V . Recall that the inclusion $\mathcal{J}(V) \subset \mathfrak{sl}(V)$ is a coadjoint orbit of $\mathrm{SL}(V)$ under the identification

$$\mathfrak{sl}(V) \cong \mathfrak{sl}(V)^*$$

given by the Killing form. Thus $\mathcal{J}(V)$ is a $\mathrm{SL}(V)$ homogeneous space, preserving the symplectic structure given by the Kirillov symplectic structure on a coadjoint orbit, and preserving the pseudo-Kähler structure given by the Killing form. This provides invariants of X , given by pulling back these structures. Using the complex structures on X and $\mathcal{J}(V)$, and the map $X \rightarrow \mathcal{J}(V)$, we can take ∂ and $\bar{\partial}$ of this map as invariants of an elliptic line congruence. Essentially, these are f and g . It turns out that if X is compact, then $X \rightarrow \mathcal{J}(V)$ is never holomorphic, unless it is constant, so that X is the Riemann sphere of a single complex structure $J : V \rightarrow V$. The symplectic form on $\mathcal{J}(V)$ pulls back to

$$i(|f|^2 - |g|^2) \vartheta \wedge \bar{\vartheta}$$

which consequently integrates to zero, giving

$$(3) \quad \int i|f|^2\vartheta \wedge \bar{\vartheta} = \int i|g|^2\vartheta \wedge \bar{\vartheta}$$

if X is compact, so that, roughly speaking, the map

$$X \rightarrow \mathcal{J}(V)$$

is equal parts holomorphic and conjugate holomorphic. Vanishing of the f and g invariants of an elliptic line congruence occurs precisely for Riemann spheres of complex structures.

Proposition 7. *There are no compact homogeneous elliptic line congruences except Riemann spheres.*

Proof. Let X be a compact homogeneous elliptic line congruence. Then by homogeneity, the 2-forms $if\vartheta \wedge \bar{\vartheta}$ and $ig\vartheta \wedge \bar{\vartheta}$ (which one can calculate are invariantly defined on X itself) must be either everywhere vanishing or everywhere not vanishing. Moreover, if one vanishes everywhere, then by equation (3) they both do. But if neither vanishes anywhere, then there is a distinguished subbundle of $B(X)$ on which $f = 1$. Using the Maurer–Cartan equations, we see that this renders ϑ well defined on X , and thereby forces X to have a globally defined nowhere vanishing 1-form, in fact a parallelism, so that X cannot be topologically a sphere. \square

2.10. Elliptic line congruences as nonlinear complex structures.

Proposition 8. *Given $X \subset \widetilde{\text{Gr}}_2(V)$ a compact elliptic line congruence, we can define a map $J_X : V \setminus 0 \rightarrow V \setminus 0$ by asking that on each 2-plane $P \subset V$ belonging to X , J_X acts as the osculating complex structure on P . This defines a smooth map, which satisfies*

$$(1) \quad J_X^2 = -1$$

$$(2) \quad J_X(tv) = tJ_X(v)$$

for $v \in V \setminus 0$ and $t \in \mathbb{R} \setminus 0$ and

$$(3) \quad J_X \text{ restricts to each 2-plane } \text{span}\{v, J_X v\} \text{ to be a linear map.}$$

Conversely, given such a map, define X to be the set of 2-planes invariant under the map, and orient these 2-planes so that $v \wedge Jv$ is positive. Then X is an elliptic line congruence.

Proof. We only have to prove that starting from some such J we can define X as above, and it turns out to be an elliptic line congruence. It is clear that X is compact, since it is a subset of a Grassmannian (which is compact) and defined by a closed condition. Define a vector field Z on the sphere V/\mathbb{R}^+ by

$$Z(v) = Jv \pmod{\text{span}\{v\}}.$$

This is a nowhere vanishing vector field defining a circle action on the sphere. Moreover, on the 2-plane P containing a vector v and the vector Jv , this vector field is just $Z(v) = J_P v$, where J_P is the osculating complex structure. So the flow of $Z(v)$ on P is $t \mapsto e^{tJ_P} v$, and in particular the flow curves on the sphere V/\mathbb{R}^+ are great circles. The quotient space is clearly X . By properness and freedom of the action, X is a smooth compact surface.

We have to show that X is elliptic. This follows from the result of Gluck and Warner [32] that great circle fibrations of the 3-sphere correspond precisely to elliptic line congruences. We won't need this result, so the reader will forgive the author for not providing a complete proof of the results of Gluck and Warner. \square

3. APPLYING CARTAN'S METHOD OF EQUIVALENCE

So far, we have only studied the microlocal geometry. The full geometry of a pseudocomplex structure $E^6 \subset \widetilde{\text{Gr}}_2(TM)$ requires Cartan's equivalence method. The algorithm for this method is explained thoroughly in [31]; the calculations for this specific equivalence problem are given in great detail in my thesis [41]; therefore only the result will be presented here. First, to define a G structure associated to a pseudocomplex structure $E^6 \subset \widetilde{\text{Gr}}_2(TM)$ on a 4-manifold M , we work on E rather than working on M . Already we know that the fibers of $E \rightarrow M$ are elliptic line congruences, so they have complex structures, and each point of E_m imposes a complex structure on T_mM , the osculating complex structure. Also, E is equipped with a 4-plane field: if we write $p : E \rightarrow M$ for the projection $E \subset \widetilde{\text{Gr}}_2(TM) \rightarrow M$, then we have 4-planes:

$$\Theta_e = p'(e)^{-1}(e) \subset T_eE$$

where $e \in E$ is thought of as a 2-plane $e \subset T_mM$. Contained in these 4-planes are the 2-planes

$$V_e = \ker p'(e) \subset \Theta_e$$

which are the tangent spaces to the fibers of $E \rightarrow M$, so are tangent spaces of elliptic line congruences, and hence have complex structure. But also

$$\Theta_e/V_e \cong T_mM$$

has the osculating complex structure.

Thus at each point of E , we have a 2-plane V_e contained in a 4-plane Θ_e , and complex structures on V_e and on Θ_e/V_e . We let W_0 be any six-dimensional vector space, equipped with a 4 dimensional subspace $\Theta_0 \subset W_0$ and a 2 dimensional subspace $V_0 \subset \Theta_0$, with complex structures on V_0 and Θ_0/V_0 , and let G be the group of linear transformations $W \rightarrow W$ preserving this structure. Then any pseudocomplex structure $E^6 \subset \widetilde{\text{Gr}}_2(TM)$ is canonically equipped with a G structure. Moreover, this G structure encodes completely the pseudocomplex structure.

This G structure is input to Cartan's method of equivalence, and after a single step (without making any prolongations of structure equations) we obtain an H structure, where $H \subset G$ is the group of matrices preserving $V_0 \subset \Theta_0 \subset W$ and preserving a complex structure on W for which Θ_0 and V_0 are complex subspaces, and preserving a certain 2-form from $\Lambda^2(\Theta_0) \otimes W/\Theta_0$ so that H , in complex coordinates z_1, z_2, z_3 , is the collection of matrices:

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & ac^{-1} \end{pmatrix}$$

with $V_0 = \{z_1 = z_2 = 0\}$ and $\Theta_0 = \{z_1 = 0\}$ and the 2-form is $dz_2 \wedge dz_3 \pmod{\Theta_0}$. Write the total space of this H structure as $B \rightarrow E$, so that B is a collection of linear maps identifying tangent spaces of E with W , and is a principal right H

bundle. The Cartan structure equations are

$$(4) \quad d \begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix} = - \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ \delta & \varepsilon & \alpha - \gamma \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix} - \pi \wedge \begin{pmatrix} \omega \\ \sigma \\ 0 \end{pmatrix} + \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \wedge \bar{\theta}$$

where $\theta, \omega, \pi, \tau_1, \tau_2, \tau_3, \sigma$ are complex valued 1-forms on B , uniquely determined, and $\alpha, \beta, \gamma, \delta, \varepsilon$ are complex valued 1-forms, not uniquely determined, and

$$\begin{aligned} 0 &= \sigma \wedge \bar{\theta} \wedge \bar{\omega} \\ 0 &= \tau_j \wedge \bar{\omega} \wedge \bar{\pi} \end{aligned}$$

and $\theta \wedge \bar{\theta} \wedge \omega \wedge \bar{\omega} \wedge \pi \wedge \bar{\pi} \neq 0$. Consequently, we can write

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} T_2 & T_3 \\ U_2 & U_3 \\ V_2 & V_3 \end{pmatrix} \begin{pmatrix} \bar{\omega} \\ \bar{\pi} \end{pmatrix}$$

and

$$\sigma = S_1 \bar{\theta} + S_2 \bar{\omega}$$

and these S, T, U, V are the lowest order invariants of pseudocomplex structures.

Let $(\theta_0, \omega_0, \pi_0)$ be a local section of the H bundle, so that

$$\theta_0, \omega_0, \pi_0 \in \Omega^1(E) \otimes \mathbb{C}.$$

Then the 4-plane field Θ is cut out by the equation $\theta_0 = \bar{\theta}_0 = 0$, while V is $\theta_0 = \bar{\theta}_0 = \omega_0 = \bar{\omega}_0 = 0$. The H structure preserves an almost complex structure J on E , for which $\theta_0, \omega_0, \pi_0$ are $(1, 0)$ forms. The 4-plane Θ is a field of J -complex 2-planes, while V is a field of J -complex lines. Every E curve $\Sigma \subset M$ has tangent planes belonging to E , forming a surface called the *prolongation* of Σ .

Proposition 9. *The prolongation $\hat{\Sigma} \subset E$ of an E curve $\Sigma \subset M$ is a J -holomorphic curve $\hat{\Sigma} \subset E$ tangent to the 4-plane field Θ . Conversely, a surface $\hat{\Sigma} \subset E$ which is tangent to the 4-plane field Θ and transverse to the fibers of $E \rightarrow M$ is locally the prolongation of an E curve $\Sigma \subset M$.*

Proof. First, take $\Sigma \subset M$ an E curve. The prolongation $\hat{\Sigma} \subset E$ is the set of tangent planes of $\Sigma \subset M$. We find that $p : E \rightarrow M$ restricts to $p : \hat{\Sigma} \rightarrow \Sigma$. So $p'(e) : T_e \hat{\Sigma} \rightarrow T_p \Sigma$ and thus

$$T_e \hat{\Sigma} \subset p'(e)^{-1} T_p \Sigma.$$

Therefore if $x \in \Sigma$ and $e \in \hat{\Sigma}$ with $p(e) = x$, then $e = T_p \Sigma$. We have

$$\Theta_e = p'(e)^{-1} e = p'(e)^{-1} T_p \Sigma$$

so $T_e \hat{\Sigma} \subset \Theta_e$. We have to show that $\hat{\Sigma}$ is J -holomorphic. Choose (locally) a coframing

$$\theta_0, \omega_0, \pi_0 \in \Omega^{(1,0)}(E)$$

from our H structure. Since Θ is $\theta_0 = \bar{\theta}_0 = 0$, we have $\theta_0 = \bar{\theta}_0 = 0$ on $\hat{\Sigma}$, and so $0 = d\theta_0 = d\bar{\theta}_0$. But from the structure equations, this gives

$$0 = \pi_0 \wedge \omega_0 = \bar{\pi}_0 \wedge \bar{\omega}_0.$$

Therefore the $(1, 0)$ forms $\theta_0, \omega_0, \pi_0$ restrict to $\hat{\Sigma}$ to satisfy some complex linear relations, and thus $\hat{\Sigma}$ is J -pseudoholomorphic.

Now take $\hat{\Sigma} \subset E$ any Θ tangent surface transverse to the fibers of $p : E \rightarrow M$. Since our desired result is local, we can suppose that

$$p : \hat{\Sigma} \rightarrow \Sigma = p(\hat{\Sigma})$$

is a diffeomorphism. Let $e \in \hat{\Sigma}$, $m = p(e) \in \Sigma$. By definition,

$$T_e \hat{\Sigma} \subset \Theta_e = p'(e)^{-1}e$$

so that

$$p'(e)T_e \hat{\Sigma} \subset e \subset T_m M$$

and since $p : \hat{\Sigma} \rightarrow \Sigma$ is a diffeomorphism:

$$p'(e)T_e \hat{\Sigma} = T_p \Sigma.$$

Therefore, $e = T_p \Sigma$, and so $\hat{\Sigma}$ is the prolongation of Σ . \square

Proposition 10. *Suppose that we have some 1-forms $\theta, \omega, \pi, \alpha, \dots$ and functions on a six-dimensional manifold satisfying the structure equations 4 on the preceding page, and that*

$$\theta \wedge \bar{\theta} \wedge \omega \wedge \bar{\omega} \wedge \pi \wedge \bar{\pi} \neq 0.$$

Then locally these define a pseudocomplex structure on a 4-manifold.

Proof. The 4-manifold is constructed by taking the foliation

$$\theta = \bar{\theta} = \omega = \bar{\omega} = 0$$

which is locally a fiber bundle, and producing a base space for that fiber bundle. The rest is elementary, following the general pattern of the equivalence method. \square

We can prolong the exterior differential system on E given by the 4-plane Θ , to form $E^{(1)}$, and prolong that to $E^{(2)}$, and so on. Each of these $E^{(k)}$ has an almost complex structure, easily seen from the structure equations on the $E^{(k)}$, which can be easily calculated (although the calculation is somewhat lengthy). The prolongations of an E curve to all orders will be pseudoholomorphic curves in each of the $E^{(k)}$.

3.1. Microlocal invariants.

Proposition 11. *Take E a pseudocomplex structure on a 4-manifold M , and let $B \rightarrow E$ be the induced G structure constructed above. Call this bundle B the pseudocomplex structure bundle of E . Now take a point $m \in M$ and look at the elliptic line congruence $E_m \subset \widetilde{\text{Gr}}_2(T_m M)$. We are faced with two principal bundles over E_m : (1) $B(E_m) \rightarrow E_m$ the bundle associated to the elliptic line congruence, following our results on line congruences above (call it the line congruence bundle of E_m), and (2) the bundle $B_m \rightarrow E_m$ given as the part of the pseudocomplex bundle that lies above E_m . Given a coframe*

$$(\theta_0, \omega_0, \pi_0 \in \Lambda^1(T_e E))$$

from the pseudocomplex bundle, we can map it to a linear map

$$u : T_m M \rightarrow \mathbb{R}^4$$

belonging to the line congruence bundle as follows: since the fiber of $E \rightarrow M$ is given by the equations

$$\theta_0 = \bar{\theta}_0 = \omega_0 = \bar{\omega}_0 = 0$$

we can treat θ_0, ω_0 as well-defined complex valued 1-forms on $T_m M$. Then we can use the map

$$u(w) = \begin{pmatrix} \omega_0 \\ \theta_0 \end{pmatrix}.$$

This gives an equivariant bundle map

$$\begin{array}{ccc} B_m & \xrightarrow{\quad} & B(E_m) \\ & \searrow & \swarrow \\ & E_m & \end{array}$$

over each line congruence E_m . Under this map

$$\begin{pmatrix} \xi & \eta \\ \vartheta & \zeta \end{pmatrix} = \begin{pmatrix} -\gamma & -\beta \\ -\pi & -\alpha \end{pmatrix} \quad \begin{pmatrix} \xi' & \eta' \\ \vartheta' & \zeta' \end{pmatrix} = \begin{pmatrix} -S_2\pi & U_3\bar{\pi} - S_1\pi \\ 0 & T_3\bar{\pi} \end{pmatrix}.$$

Moreover

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -S_2 \\ T_3 \end{pmatrix}$$

3.2. Identifying almost complex structures. From our identification of the microlocal invariants we have:

Proposition 12. *The following are equivalent:*

- (1) $E \subset \widetilde{\text{Gr}}_2(TM)$ is an almost complex structure.
- (2) $0 = \sigma \wedge \theta = \tau_1 \wedge \bar{\omega}$
- (3) $0 = \sigma = \tau_1 \wedge \bar{\omega} = \tau_3 \wedge \bar{\omega}$
- (4) The projection $E \rightarrow M$ is holomorphic for some almost complex structure on M (which is then necessarily given by E itself, as a subset of $\widetilde{\text{Gr}}_2(TM)$).

Proposition 13. *A proper pseudocomplex structure is an almost complex structure precisely when either $0 = \sigma \wedge \theta$ or $0 = \tau_1 \wedge \bar{\omega}$.*

Proof. We have seen in equation 3 that the vanishing of the microlocal invariant f forces the vanishing of g , and vice-versa, and that vanishing of both forces an elliptic line congruence to be a Riemann sphere (in other words, flat). Applying our identification of the elliptic line congruence invariants on a pseudocomplex structure, the result is immediate. \square

4. APPROXIMATION BY COMPLEX STRUCTURES

Proposition 14. *Suppose that $E \subset \widetilde{\text{Gr}}_2(TM)$ is a pseudocomplex structure. Take $E_0 \subset \widetilde{\text{Gr}}_2(TM_0)$ a complex structure (to be thought of as a flat pseudocomplex structure). Pick points $e \in E$ and $e_0 \in E_0$, and let $m \in M$ and $m_0 \in M_0$ be the projections of e, e_0 to M, M_0 . There is a local diffeomorphism $f : U \rightarrow U_0$ of a neighborhood of m to a neighborhood of m_0 so that $f_*e = e_0$ and so that near e*

$$\begin{pmatrix} \theta_0 \\ \bar{\theta}_0 \\ \omega_0 \\ \bar{\omega}_0 \\ \pi_0 \\ \bar{\pi}_0 \end{pmatrix} = \begin{pmatrix} 1 & a_1 & 0 & a_2 & 0 & 0 \\ \bar{a}_1 & 1 & \bar{a}_2 & 0 & 0 & 0 \\ 0 & b_1 & 1 & b_2 & 0 & 0 \\ \bar{b}_1 & 0 & \bar{b}_2 & 1 & 0 & 0 \\ 0 & c_1 & 0 & c_2 & 1 & a_2\bar{S}_2 \\ \bar{c}_1 & 0 & \bar{c}_2 & 0 & \bar{a}_2S_2 & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \bar{\theta} \\ \omega \\ \bar{\omega} \\ \pi \\ \bar{\pi} \end{pmatrix}$$

where θ_0 , etc. are the soldering 1-forms of E_0 , and so that we have 1-forms satisfying the structure equations (i.e. sections of the adapted coframe bundles), which at e satisfy

$$\begin{pmatrix} \alpha_0 & 0 & 0 \\ \beta_0 & \gamma_0 & 0 \\ \delta_0 & \varepsilon_0 & \alpha_0 - \gamma_0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ \delta & \varepsilon & \alpha - \gamma \end{pmatrix}$$

and

$$d \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\tau_1 \\ 0 \\ -\tau_2 + S_1\pi \\ S_2\pi \\ -\tau_3 \\ 0 \end{pmatrix}.$$

Proof. The diffeomorphism f is constructed first by taking any complex valued coordinates z, w on M , for which e is a complex line, and then using elementary coordinate changes and changes of θ, ω, π adapted coframing to arrange the required equations.

To arrange the remaining equations, which are only required to hold at e , we write out an exterior differential system for the first equations (which we have already solved), prolong it, check that it is torsion free, and see thereby that there is no obstruction to solving the prolongations of all orders, at least at any required order (but not solving it as a PDE system—we are only solving at, say, second order). The system will look like the above equations together with equations like

$$\alpha_0 - \alpha + a_1\bar{\tau}_1 + a_2(\bar{\tau}_2 - \bar{S}_1\bar{\pi}) = p_{11}\theta + p_{1\bar{1}}\bar{\theta} + p_{1\bar{2}}\bar{\omega}$$

and so on (quite complicated), describing relations between the various 1-forms. These coefficients p_{ij} can be chosen at will, and of course we choose them to vanish. Then we take a solution at whichever order we like, arranging it to solve all the required equations at e , arranging all of these arbitrary coefficients of the prolongations to vanish at e , and then repeat the manipulations we used previously to obtain new equations which do not disturb the equations at e , as is easy to see. \square

Corollary 15. *Every point e of E lies in a neighborhood on which there are local complex coordinates z, w on M and p on E and adapted 1-forms θ, ω, π so that*

$$\begin{aligned} dw - p dz - f d\bar{z} &= \theta + a\bar{\theta} \\ dz &= b\bar{\theta} + \omega + c\bar{\omega} \\ dp &= \pi + e_1\theta + e_{\bar{1}}\bar{\theta} + e_2\omega + e_{\bar{2}}\bar{\omega} + e_3\pi + e_{\bar{3}}\bar{\pi} \end{aligned}$$

where $a = b = c = e_\bullet = f = 0$ at e_0 .

In particular, the adapted coframes of E match those of the complex structure E_0 given taking the complex coordinates z, w, p to be holomorphic, and thus E and E_0 have the same almost complex structure J and same 4-plane field Θ at the point $e = e_0$.

Corollary 16. *Take $E \subset \widetilde{\text{Gr}}_2(TM)$ a pseudocomplex structure. Take $e \in E$ and let $m \in M$ be the projection of e to M . There are complex valued coordinates $z, w : M \rightarrow \mathbb{C}$ defined in a neighborhood of m , vanishing at m , and coordinates z, w, p, q on $\widetilde{\text{Gr}}_2(TM)$ near e defined by*

$$dw = p dz + q d\bar{z}$$

with e represented by $z = w = p = q = 0$, and so that E is defined by the equation

$$q = Q(z, \bar{z}, w, \bar{w}, p, \bar{p})$$

where Q is a smooth function of three complex variables so that $Q = dQ = 0$ at e . In particular, E curves near m are precisely solutions of the equation

$$\frac{\partial w}{\partial \bar{z}} = Q \left(z, \bar{z}, w, \bar{w}, \frac{\partial w}{\partial z}, \overline{\left(\frac{\partial w}{\partial z} \right)} \right).$$

5. ANALYSIS OF E CURVES AND COMPACTNESS

To allow singularities in E curves, and study their limiting behavior, it is natural to consider them as parameterized rather than as submanifolds. This is because any singular Riemann surface can be parameterized by a (canonically chosen) smooth Riemann surface. We will say that a smooth map $\phi : C \rightarrow E$ from a Riemann surface C (possibly with boundary) to a pseudocomplex structure $E \subset \widetilde{\text{Gr}}_2(TM)$ is a *parameterized E curve* in M if (1) it is J -holomorphic and (2) every 1-form on E which vanishes on Θ pulls back through ϕ to 0. Usually, but not always, we ask our Riemann surface to be compact. We will say that our parameterized E curve is *basic* if ϕ is injective on a dense open set and intersects each fiber of $E \rightarrow M$ on a discrete set of points. Henceforth, when we use the term *E curve* we will mean a parameterized E curve, unless we say otherwise.

We will also allow the selection of distinct marked points on the Riemann surface C , and let the Riemann surface have ordinary double points. (The definition of parameterized E curve from a singular Riemann surface is just a continuous map which lifts to a parameterized E curve from the universal cover.) If $S \subset M$ is a totally real surface, we can also consider parameterized E curves with boundary in S , by which we mean that $p \circ \phi$ takes ∂C to S . A *symmetry* of a parameterized E curve is a map $C \rightarrow C$ under which all marked points are fixed and ϕ is invariant. A parameterized E curve is called *stable* if its symmetry group is finite.

We have to be careful about parameterized E curves: they should be thought of as curves in M , not in E , despite the definition. However, there is one subtlety we should keep in mind: consider the example of $M = \mathbb{CP}^2$, with E the standard complex structure. Take the E curve C_ε defined in affine coordinates z, w on \mathbb{CP}^2 and an affine coordinate σ on \mathbb{CP}^1 by

$$\begin{aligned} z &= \sigma + \frac{\varepsilon}{\sigma} \\ w &= \sigma - \frac{\varepsilon}{\sigma} \end{aligned}$$

which satisfies

$$z^2 - w^2 = 4\varepsilon.$$

We are naturally led to suspect that the limit as $\varepsilon \rightarrow 0$ should be a pair of transverse lines. The resulting curve in E is

$$\begin{aligned} z &= \sigma + \frac{\varepsilon}{\sigma} \\ w &= \sigma - \frac{\varepsilon}{\sigma} \\ p &= \frac{\sigma^2 + \varepsilon}{\sigma^2 - \varepsilon} \end{aligned}$$

or

$$p = \frac{z}{w}.$$

Naively treating z, w, p as affine coordinates on $\mathbb{C}\mathbb{P}^3$, using the equations

$$z^2 - w^2 = 4\varepsilon, wp = z$$

we find that the limiting object as $\varepsilon \rightarrow 0$ must be composed of four lines, not two. This is because we traverse the exceptional divisor $z = w = 0$ twice. So E curves considered inside E are slightly different from what their images look like in M .

First we will develop the local analysis.

Theorem 3 (Elliptic regularity). *Let $E \subset \widetilde{\text{Gr}}_2(TM)$ be a pseudocomplex structure. Define a weak E curve to be a parameterized E curve*

$$\phi : C \rightarrow E$$

so that ϕ belongs to the Sobolev space of maps whose 1-jet is locally square integrable. Then ϕ is smooth in the interior of C .

Proof. This is identical to the proof in [51]. □

Corollary 17. *A weak E curve has smooth prolongations to all orders, forming pseudoholomorphic curves in the almost complex manifolds $E^{(k)}$.*

Proposition 15. *Every E curve has a prolongation to some pseudoholomorphic curve in $E^{(k)}$ with no branch points—an immersed pseudoholomorphic curve.*

Proof. As in the theory of almost complex manifolds, we have holomorphic polynomials as lowest order terms, in suitable complex coordinates (see [39]). Moreover, the prolongations force the order of these polynomials down at each step, eventually reaching order one. □

Theorem 4 (Uniqueness of continuation). *Any two parameterized E curves $\phi, \psi : C \rightarrow E$ with the same infinite jet at a point are identical throughout the component of C containing that point.*

Proof. A simple application of Aronszajn’s lemma. □

Theorem 5. *Take a parameterized E curve $\phi : C \rightarrow E$ and a point $p_0 \in C$. Let z, w, p be coordinates as in proposition 14 so that $\phi(p_0)$ is the origin of these coordinates. Then there is a neighborhood U of p_0 in C and a holomorphic map $f = (Z, W, P) : U \rightarrow \mathbb{C}^3$ so that in the coordinates z, w, p the maps f and ϕ have the same leading order terms in their Taylor expansions. In particular, the projection to M in these coordinates is holomorphic up to leading order terms.*

Proof. See [39]. The basic idea is as follows. First write out the condition on ϕ being J -holomorphic in a local coordinate σ on C and adapted coordinates (z, w, p) on E in the form

$$\frac{\partial \phi}{\partial \bar{\sigma}} = M \overline{\left(\frac{\partial \phi}{\partial \sigma} \right)}$$

where M is a complex 3×3 matrix, vanishing at e_0 . This is possible because the adapted complex coordinates impose a complex structure which agrees at $e_0 =$

$\phi(p_0)$ with the almost complex structure J on E . We apply Cauchy's theorem to see that in any disk about $p_0 = 0$ in C :

$$\phi + \frac{1}{2\pi\sqrt{-1}} \int_D \frac{1}{\zeta - \sigma} M \left(\frac{\partial \phi}{\partial \sigma} \right) d\zeta \wedge d\bar{\zeta} = f$$

a triple of holomorphic functions. These intuitively represent the holomorphic part of ϕ in these coordinates. One needs to show that the map $\phi \mapsto f$ given in this way is a smooth map of appropriate Banach spaces if the disk D is made small enough since $\phi - f$ is quite small in appropriate norms. \square

Proposition 16. *Take $E \subset \widetilde{\text{Gr}}_2(TM)$ a pseudocomplex structure and $\phi : C \rightarrow E$ a basic E curve. Define the critical points of ϕ to be the points $z \in C$ where $(p \circ \phi)'(z) = 0$. The set of critical points is a discrete subset of C , except on components of C that are mapped to a single point.*

Proof. Choose any adapted coframing θ, ω, π . Now pull them back with ϕ . By definition, you get $\theta = 0$ and ω and π are $(1, 0)$ forms, say

$$\begin{pmatrix} \omega \\ \pi \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} d\zeta$$

where ζ is a local holomorphic coordinate on C . Then taking differential, we find

$$0 = \frac{\partial f}{\partial \bar{\zeta}} + f\gamma^{(0,1)} - gS_2\bar{f}$$

where

$$\begin{aligned} \gamma &= \gamma^{(1,0)}d\zeta + \gamma^{(0,1)}d\bar{\zeta} \\ \sigma &= S_2\bar{\omega} + S_3\bar{\pi}. \end{aligned}$$

Therefore f satisfies a linear first order determined elliptic equation with smooth coefficients, and we can apply Aronszajn's lemma to show that if f vanishes to infinite order at a point, then it vanishes everywhere. But points where $f = 0$ are precisely points where ϕ is tangent to the fibers of $E \rightarrow M$ since $\theta, \bar{\theta}, \omega, \bar{\omega}$ span the semibasic 1-forms for this map and $\theta = \bar{\theta} = 0$. Thus f can only have a discrete set of zeros, and the set of critical points is discrete. \square

Corollary 18. *Suppose that two parameterized basic E curves $\phi_0, \phi_1 : C \rightarrow E$ have projections $p \circ \phi_0, p \circ \phi_1 : C \rightarrow M$ with the same infinite jet at a noncritical point $z \in C$. Then $\phi_0 = \phi_1$ on the connected component of z in C .*

Proof. The equation for the lift of the projection

$$(p \circ \phi)^\wedge = \phi$$

is easy to prove, and holds except at critical points. Therefore, matching of $p \circ \phi_0$ and $p \circ \phi_1$ to all orders at z implies matching of ϕ_0 and ϕ_1 to all orders at z , and therefore we can apply uniqueness of jets of pseudoholomorphic curves in almost complex manifolds. \square

Corollary 19. *The same conclusion holds for basic parameterized E curves even at a critical point.*

Proof. The projections must be asymptotic to all orders near the critical point z . This implies that the prolongations of any order of the projections must be asymptotic to all orders as we approach the critical point. As above

$$(p \circ \phi_j)^\wedge = \phi_j$$

at noncritical points. This implies, because the critical points are discrete, that the ϕ_j must be asymptotic at z to all orders. Therefore, they must agree by Aronszajn's lemma. \square

Lemma 10. *Suppose that $C \subset M$ is an embedded smooth E curve, and $m \in C$ is a point of C . There are local coordinates z, w, p on E near m , with z, w defined on M , so that C is cut out near m by the equations*

$$w = p = 0 .$$

Moreover there is an adapted coframing θ, ω, π defined near $T_m C$ in E so that

$$\begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix} = \begin{pmatrix} dw \\ dz \\ dp \end{pmatrix}$$

at all points of C near m .

Proof. The relevant equivalence problem concerns the pullback of $B \rightarrow E$ to the lift of C , $\tilde{C} \subset E$. We can further adapt the coframes $\theta_0, \omega_0, \pi_0$ of this pullback by asking that $\pi_0, \bar{\pi}_0$ vanish on tangent spaces of C . This forces structure equations

$$\theta = \bar{\theta} = \pi = \bar{\pi} = 0$$

which gives us as structure equations on this bundle

$$d\omega = -\gamma \wedge \omega$$

which are the structure equations of a complex structure on a surface, and so local equivalence with the flat example follows by the Newlander–Nirenberg theorem. Then z, w, p can be arbitrarily extended off of \tilde{C} so that z, w are functions on E . \square

Proposition 17. *Let $E \subset \widetilde{\text{Gr}}_2(TM)$ be a proper pseudocomplex structure. Suppose that $\phi_0 : C_0 \rightarrow E$ and $\phi_1 : C_1 \rightarrow E$ are E curves, and that there are convergent sequences of points*

$$s_j \rightarrow s \in C_0, \quad t_j \rightarrow t \in C_1$$

so that

$$p \circ \phi_0(s_j) = p \circ \phi_1(t_j) .$$

Also suppose that $\phi_0'(s) \neq 0$. Then there is a holomorphic map f of a neighborhood of s to a neighborhood of t so that

$$\phi_1 = \phi_0 \circ f .$$

Proof. If we were to try to prove this result by imitating the almost complex case, we would simply look in coordinates of the type guaranteed by the previous lemma, so that C_0 is cut out by $w = p = 0$. We can use z as a local holomorphic coordinate on C_0 , and arrange that $z = 0$ is the point s . Take ζ a local holomorphic coordinate on C_1 , so that $\zeta = 0$ is the point t . Now we have C_1 described by functions

$$z(\zeta), w(\zeta), p(\zeta)$$

with $w(\zeta)$ having infinitely many zeros near $\zeta = 0$. The lowest order terms of

$$(z(\zeta), w(\zeta), p(\zeta))$$

at $\zeta = 0$ must be holomorphic polynomials, unless they vanish to all orders. In the almost complex case, this actually determines that $w(\zeta)$ itself has holomorphic lowest order term. But then, $w(\zeta)$ cannot vanish at infinitely many points approaching $\zeta = 0$, unless this lowest order term vanishes, because it will dominate:

$$w(\zeta) = w_0 \zeta^k + \dots$$

would give $w(\zeta) \neq 0$ near $\zeta = 0$. Therefore $w(\zeta)$ vanishes to infinite order at $\zeta = 0$. Because of the equation

$$dw = p dz + q(z, w, p) d\bar{z}$$

we have $p(\zeta)$ and $q(z(\zeta), w(\zeta), p(\zeta))$ vanishing to all orders in ζ . Now this gives us

$$\omega = dz, \pi = dp$$

at $\zeta = 0$, up to infinite order, and since ω is a $(1, 0)$ form, this tells us that $z(\zeta)$ is holomorphic to all orders. We can $\omega \wedge d\zeta = 0$ to show that the formal holomorphic series given by the Taylor expansion of $z(\zeta)$ must converge (because the terms are controlled, by differentiating this equation, in terms of values of z, w, p). The result then follows by uniqueness of continuation, using Aronszajn's lemma.

However, in the pseudocomplex case, it is unclear that $w(\zeta)$ must have holomorphic lowest order term, simply because $(z(\zeta), w(\zeta), p(\zeta))$ does; it might look like

$$p(\zeta) = p_0 + \dots$$

with $p_0 \neq 0$, and then we would have lowest order terms (constant terms) of $(z, w, p)(\zeta)$ holomorphic. The problem is essentially to show that $p(0) = 0$, so that in our coordinates the two curves strike in E .

But this is not too hard: each real line at a point of m is contained in a unique E plane in $E_m \subset T_m M$. Now take each pair of points $\phi_0(s_j), \phi_0(s_k)$ and draw a line between them in some local coordinates. Then the E planes containing these lines must converge to $(p \circ \phi_0)'(s) \cdot T_s C_0$, the E plane tangent to C_0 at s . But then, we could have used points close to the s_j instead of the actual s_j , or points close to the t_j as well. We can pick points of C_1 which are not critical for ϕ_1 , and carry out the same construction. This shows that the values of $p(\zeta)$ must tend to 0 as $\zeta \rightarrow 0$. Therefore C_0 and C_1 have the same osculating complex geometry at m , so their lowest order terms are holomorphic there. The story is now the same as in the almost complex case. \square

Theorem 6. *Two E curves have a discrete set of intersection points in M (finite, if they are compact). Each intersection point, after small perturbation, becomes a positive number of transverse intersections. Moreover, the local intersection number at an intersection is one precisely when the intersection is transverse.*

Proof. The proof is essentially as in [40]. \square

Definition 4. *Take M a 4-manifold with Riemannian metric g and $E \subset \widetilde{\text{Gr}}_2(TM)$ a pseudocomplex structure for M . Let $\Omega \in \Omega^2(M)$ be a symplectic form on M . The weight of E with respect to g and Ω is the smallest constant a so that*

$$a \cdot \Omega(u, v) \geq 1$$

for $u, v \in T_m M$ any oriented g orthonormal basis of a 2-plane $\text{span}\{u, v\}$ belonging to E . The weight of Ω with respect to g is the smallest constant b so that

$$\Omega(u, v) \leq b$$

whenever $u, v \in T_m M$ are orthogonal g -unit vectors. The width of E with respect to g and Ω is the smallest constant c so that all of the osculating complex structures $J_e : T_m M \rightarrow T_m M$ for $e \in E$ satisfy

$$|J_e v| \leq c$$

for $v \in T_m M$ any g -unit vector.

Proposition 18. *Suppose that M is a 4-manifold, that $\Omega \in \Omega^2(M)$ is a symplectic form, and that $E \subset \widetilde{\text{Gr}}_2(TM)$ is a pseudocomplex structure. Let g be a flat Riemannian metric. Let a be the weight of E , b the weight of Ω and c the width of E . Suppose that $a, b, c > 0$ and that $\Omega = d\vartheta$ is exact, for some 1-form ϑ . Take any parameterized E curve $\phi : C \rightarrow E$ and any Borel set $B \subset M$ and rectifiable current X on M with*

$$\begin{aligned} \partial X &= \partial(p \circ \phi(C) \cdot B) \\ \text{Mass}(p \circ \phi(C) \llcorner B) &\leq abc \text{Mass}(X \llcorner B). \end{aligned}$$

Proof. See [3]. □

Proposition 19. *Suppose that E, Ω, g and $\phi : C \rightarrow E$ are as in the last proposition, with $a, b, c > 0$. Suppose further that g is a flat Euclidean metric, and that C is compact with boundary, and that $p \circ \phi(\partial C)$ has support (as a current) contained in a ball of radius r . Then for any integer $j > 0$ and real number $\varepsilon > 0$, the support (as a current) of $p \circ \phi(C)$ is contained in a ball of radius*

$$\varepsilon^j r + \left(\frac{1 - \frac{1}{abc}}{1 - \varepsilon^{-2}} \right)^{j/2} \sqrt{\frac{abc}{4\pi} \text{Mass}(S \llcorner (\mathbb{R}^{2n} \setminus B(r)))}.$$

Proof. See [3]. □

Theorem 7 (Removable singularities). *Let $E \subset \widetilde{\text{Gr}}_2(TM)$ be a proper pseudocomplex structure. Suppose that $\phi : C \rightarrow E$ is an E curve, with C a punctured Riemann surface, and the image of $p \circ \phi$ is contained in a compact subset of M . Then ϕ extends across the puncture precisely when the area of the image of ϕ is finite.*

Proof. As in [33]. □

Theorem 8 (Gromov compactness). *Take $E_j \subset \widetilde{\text{Gr}}_2(TM)$ a sequence of proper pseudocomplex structures, $S_j \subset M$ a sequence of surfaces, with S_j totally real for E_j and a sequence of stable parameterized E curves $\phi_j : (C_j, \partial C_j) \rightarrow (E, p^{-1}S_j)$. Suppose that $E_j \rightarrow E$ converges to a proper pseudocomplex structure, and $S_j \rightarrow S \subset M$ converges to a totally real surface for E . Suppose that there are Riemannian metrics on the E_j converging to one on E for which the area of the image of any of the ϕ_j is less than some bound A , and that the $p \circ \phi_j$ have images contained inside some compact set $K \subset M$. Then there is some subsequence ϕ_{j_k} and a unique stable parameterized E curve $\phi : C \rightarrow E$ so that ϕ_{j_k} converges to ϕ in the Gromov-Hausdorff sense.*

Proof. This follows from any of the myriad proofs of Gromov compactness, by applying the canonical almost complex structure on the E_j . \square

Corollary 20. *Uniform convergence on compact sets for a sequence of parameterized E curves implies uniform convergence on compact sets for all derivatives.*

Proof. Use prolongation to all orders. \square

Proposition 20. *Uniform convergence on compact sets of a sequence of symmetries of a pseudocomplex 4-manifold implies uniform convergence on compact sets of all derivatives.*

Proof. Apply the symmetry to various E curves. \square

Consider a pseudocomplex structure in local adapted coordinates (z, w, p) . It looks like a function

$$q(z, w, p)$$

which is smooth, and satisfies

$$0 = q(0) = dq(0).$$

A parameterized E curve with local holomorphic parameter σ must satisfy

$$dw = p dz + q(z, w, p) d\bar{z}$$

and also have

$$\omega \wedge d\sigma = \pi \wedge d\sigma = 0.$$

This can be written in our local coordinates as equations

$$(5) \quad \begin{pmatrix} \frac{\partial z}{\partial \sigma} \\ \frac{\partial w}{\partial \sigma} \\ \frac{\partial p}{\partial \sigma} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial \bar{\sigma}} \\ \frac{\partial w}{\partial \bar{\sigma}} \\ \frac{\partial p}{\partial \bar{\sigma}} \end{pmatrix}^\dagger$$

where the \dagger indicates complex conjugation, and a_j, b_k depend on z, w, p smoothly, algebraically determined from $q(z, w, p)$ and its first derivatives. We can try to produce holomorphic approximations to z, p by writing

$$\begin{pmatrix} Z \\ P \end{pmatrix}(\sigma) = S_U \begin{bmatrix} z \\ p \end{bmatrix}(\sigma) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial U} \begin{pmatrix} z(\zeta) \\ p(\zeta) \end{pmatrix} \frac{d\zeta}{\zeta - \sigma}$$

for some small region U around the origin, so that

$$\begin{pmatrix} z \\ p \end{pmatrix} = \begin{pmatrix} Z \\ P \end{pmatrix} + T_U \begin{pmatrix} \bar{\partial} z \\ \bar{\partial} p \end{pmatrix}$$

where

$$T_U [g(\zeta)d\bar{\zeta}] (\sigma) = \frac{1}{2\pi\sqrt{-1}} \int_U \frac{g(\zeta)d\bar{\zeta} \wedge d\zeta}{\sigma - \zeta}.$$

Now suppose that we start with some arbitrary complex valued functions z, p of σ , and determine a function w by solving the ODE system

$$dw = p \cdot dz + q(z, w, p) d\bar{z}$$

radially away from the origin, in a small open set in the σ complex plane, and then construct Z, P by

$$\begin{pmatrix} Z \\ P \end{pmatrix} = \begin{pmatrix} z \\ p \end{pmatrix} - T_U \begin{pmatrix} \bar{\partial} z \\ \bar{\partial} p \end{pmatrix}$$

but we plug in the expressions from the right-hand side of equation 5 instead of the partials $\bar{\partial}z, \bar{\partial}p$. This gives a map from arbitrary functions z, p defined near 0, satisfying some appropriate smoothness condition, to functions Z, P satisfying a similar condition. Using the same arguments as in [39], we find that this map is invertible near 0, on a suitable function space, for small enough open set U , and that the holomorphic choices of $Z(\sigma), P(\sigma)$ correspond precisely to the $z(\sigma), p(\sigma)$ so that the triple $z(\sigma), w(\sigma), p(\sigma)$ is an E curve.

Proposition 21. *Let $E_t \subset \widetilde{\text{Gr}}_2(TM)$ be a family of pseudocomplex structures, parameterized by a real variable t . For any E_0 curve $\phi : C \rightarrow E_0$, there is a neighborhood $U \subset C$ of any point and a deformation $\phi_t : U \rightarrow E_t$ of ϕ which is smooth in t , defined for some open set of t values near 0, giving an E_t curve for each t .*

Proof. The proof is due to McDuff [39]. The idea is to carry out the above construction of Z, P for the initial curve, which is governed by the function $q_0(z, w, p)$, say. Then we invert the construction for the functions $q_t(z, w, p)$ corresponding to each E_t . This can only be done locally. \square

Proposition 22. *Given any E curve $\phi : C \rightarrow E$ we can make a small deformation of E , say E_t , and of ϕ , say ϕ_t , an E_t curve, so that $E_0 = E$ and E_1 is flat (complex) near all singular values of ϕ_1 .*

6. DUAL CURVES

Theorem 9. *Let E be a pseudocomplex structure on $\mathbb{C}\mathbb{P}^2$, tamed by a symplectic structure. Then the moduli space of E spheres representing the hyperplane homology class is smooth and compact and diffeomorphic to $\mathbb{C}\mathbb{P}^2$.*

Proof. By results of Taubes, there is only one symplectic structure on $\mathbb{C}\mathbb{P}^2$ up to rescaling, so that we can suppose that our symplectic structure is the usual one on $\mathbb{C}\mathbb{P}^2$, and from here the rest of the arguments are the same as in Gromov's [33]. \square

Now suppose that E is a pseudocomplex structure on a 4-manifold X , and that Y is a 4 parameter family of E curves. Let Z be the set of pointed E curves whose underlying E curves belong to the family Y . We have maps

$$\begin{array}{ccc} & Z & \\ \lambda \swarrow & & \searrow \rho \\ X & & Y \end{array}$$

with λ given by forgetting the E curve, and ρ by forgetting the point. We also have a map

$$(p, C) \in Z \rightarrow T_p C \in E .$$

Suppose that this map is a local diffeomorphism (as it is in the case of $\mathbb{C}\mathbb{P}^2$, essentially proved in [33]). Then we can pull our G structure from E back to Z , and reduce the structure group by taking only the adapted coframings $\theta_0, \omega_0, \pi_0$ for which the fibers of $Z \rightarrow Y$ satisfy

$$\theta = \bar{\theta} = \pi = \bar{\pi} = 0 .$$

This reduces the structure group to the group of complex matrices

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & 0 & ac^{-1} \end{pmatrix}$$

so that $a, c \neq 0$. This reduction forces the 1-form ϵ to be semibasic, and we can arrange

$$\epsilon = E_1 \bar{\theta} + E_2 \bar{\pi} .$$

Now define 1-forms on this principal bundle by

$$\begin{pmatrix} \theta' \\ \omega' \\ \pi' \end{pmatrix} = \begin{pmatrix} \theta \\ \pi \\ -\omega \end{pmatrix} \quad \begin{pmatrix} \alpha' & 0 & 0 \\ \beta' & \gamma' & 0 \\ \delta' & \epsilon' & \alpha' - \gamma' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ \delta & \alpha - \gamma & 0 \\ -\beta & \sigma & \gamma \end{pmatrix}$$

and define complex valued functions according to

$$\begin{pmatrix} S'_1 \\ S'_2 \end{pmatrix} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \quad \begin{pmatrix} T'_2 & T'_3 \\ U'_2 & U'_3 \\ V'_2 & V'_3 \end{pmatrix} = \begin{pmatrix} -T_3 & T_2 \\ V_3 & -V_2 \\ -U_3 & U_2 \end{pmatrix}$$

Then these new 1-forms and functions satisfy the structure equations 4. Therefore, by Proposition 10, this is a pseudocomplex structure, at least locally, and the map $Z \rightarrow Y$ presents Y as the base space of this structure, so that this is locally a pseudocomplex structure on Y (it could be “multivalued”). Moreover, the E curves for $Z \rightarrow Y$ are the same (as surfaces in Z) as the E curves of $Z \rightarrow X$.

Theorem 10. *Suppose that the pseudocomplex structure given (locally) by Z is an almost complex structure (locally) both on X and on Y . Then it is a complex structure on both.*

Proof. This follows immediately from the identification of the equations on microlocal invariants of almost complex structures: we would need

$$S_1 = S_2 = T_3 = U_3 = S'_1 = S'_2 = T'_3 = U'_3 = 0$$

which (when we unwind these equations and differentiate the structure equations) forces all of our invariants to vanish; hence local equivalence with the flat case. \square

7. GROMOV–MCDUFF–YE INTERSECTION THEORY

Intersections behave as they do in complex geometry, because of Proposition 14.

Proposition 23. *Let $\phi : C \rightarrow E$ be a continuous map which is an E curve in a neighborhood of a point $p_0 \in C$. Define the local self-intersection number of C at p_0 to be the local self-intersection number of the map $p \circ \phi$. This number is always positive unless $(p \circ \phi)'(p_0)$ is an injection. The local intersections in M of two E curves are always positive. The local intersection number is 1 only if the curves intersect transversely.*

7.1. Blowup and adjunction. Take $E \subset \widetilde{\text{Gr}}_2(TM)$ a proper pseudocomplex structure. To define blowup of a point, say $m \in M$, first we need the analogues of lines through that point. Any family of E curves will do, as long as it contains one line through each point in each “direction”, i.e. each E tangent line. First, by cutting out a neighborhood of $m \in M$, and pasting in $\mathbb{C}\mathbb{P}^2$, following arguments of Bangert in [3], we can arrange that the resulting pseudocomplex structure be tamed by the usual symplectic structure on $\mathbb{C}\mathbb{P}^2$ and that it agree with the usual complex structure outside some small neighborhood, and we can even get it to be as close as we like to the usual complex structure on $\mathbb{C}\mathbb{P}^2$ using our arguments for approximation by complex structure. Now we can employ Gromov’s compactness theorem on E to ensure that the small deformation of the usual complex structure to E preserves the family of lines. Using the intersection theory arguments of McDuff, given in Section 7, we find that these curves must remain smooth under such deformation, and that they intersection in M in only one place, transversely. The problem is to get one of them through each point in each direction. This is not difficult: we find that any pair of distinct points, are joined by a unique E curve in our family, by arguments of Gromov [33]. This E curve varies smoothly with the choice of points.

Now take a point of one of these E curves, and look at the linearization of the E operator on that curve. One finds (for example, by index theory of $\mathbb{C}\mathbb{P}^1$) that its kernel has real dimension 4, which matches the index, and that the space of solutions of the linearization vanishing at a point is precisely two-dimensional, again matching the index for that problem. So at least locally, the space of E curves from our family, passing through a given point, is a smooth surface, and no two such curves sufficiently near one another can have the same tangent plane at that point. Therefore, the family of curves is locally identified with their tangent planes. Properness ensures that this is true globally as well.

The end result then is that locally, we can construct such a family of E curves, looking “like lines”. We can then define blowup imitating the usual definition: away from $m \in M$ define the blowup of M at m to be just identified with M itself; but in a small neighborhood of m , we define it to consist of pairs (n, C) where $n \in M$ is a point belonging to an E curve C which belongs to our family. The exceptional divisor is then just E_m .

Unfortunately, there is in general no pseudocomplex structure on the blowup which (1) agrees with the usual structure away from the exceptional divisor (where we can identify the blowup with E) and which (2) will extend continuously across the exceptional divisor. However, we can define the lift of a curve C_0 through m to the blowup: to each point $n \in C_0$ we associate the point n in the blowup if n is away from m , and for n nearby m associate the point of the form (n, C) . This will extend to a smooth real surface in the blowup, which we see easily as follows: the blowup near the exceptional divisor is embedded into E by sending a point (n, C) of the blowup to $T_n C \in E$. This map

$$(n, C) \mapsto T_n C \in E$$

is easily seen to be an embedding of a neighborhood of the exceptional divisor (for example, in adapted coordinates). Now we can see easily that the blown up curve inside the blowup of M at m sits in E as a surface asymptotic to the lift \hat{C}_0 of C (the set of tangent lines to C). This allows us to extend the blowup curve across the exceptional divisor smoothly. Intersections with the exceptional divisor correspond

(including multiplicities) with intersections of the lift $\hat{C}_0 \subset E$ and the exceptional divisor E_m , which we can see again by deforming to the complex case.

Since the blowup does not have a pseudocomplex structure (the obvious choice pulled back from M does not extend differentiably across the exceptional divisor, in general) it is probably not natural to approach the study of E curves via blowup. In particular, repeated blowup is not possible, while repeated prolongation is. However, results on prolongations always require differential geometry and are not topological.

The notion of blowdown is particularly intriguing here, but even in the presence of an exceptional divisor, blowdowns will not generally be well defined.

Totally real surfaces can be treated in two ways: first, the way I approached them in my thesis, where I simply took as the lift of a surface all of the 2-planes in E that live at that point of M . But to treat totally real surfaces, one can look at each 2-plane belonging to E above a point of my surface S , look at each $e \in E$ above a point $m \in S$. Then ask if $T_m S$ is totally real for $J_e : T_m M \rightarrow T_m M$, i.e. $T_m S$ is not J_e invariant, where J_e is the osculating complex structure. Then I ask further if e is invariant (up to reorientation) under the complex conjugation across $T_m S$ in $T_m M$. Recall that every totally real 2-plane in a complex two-dimensional vector space has a conjugation operator. These e are precisely the complexifications of real lines (they strike $T_m S$ in a real line). So they can be identified with a circle bundle (the real projectivization of TS) above S in the case of an almost complex manifold. Call the total space S'' . But for a pseudocomplex manifold, there is more to it since the osculating Riemann sphere may not be constant. S'' is a circle bundle over S , in the proper case. Therefore, a 3-manifold. This allows much tighter adaptation of frames. If C is an E curve with boundary in S , then its lift is a pseudoholomorphic curve in E with boundary in S'' .

By carrying out infinite prolongation, we will find that any two E curves with boundary, whose boundaries have the same infinite jet at a point, must be identical. The reasoning: we will find that it is possible to complexify a real curve to each order, by analogy with the case of complex surfaces. No finite order obstruction emerges since we easily see involutivity. This determines the infinite jet of a pseudoholomorphic curve. Now we apply Aronszajn's lemma.

The possibility of slightly extending smooth E curves with boundary is immediate from the continuity method.

If $e \in S''$, then we have $T_m S$ totally real for $J_e : V \rightarrow V$, so that $T_m S$ is also totally real for any approximating complex structure which is tangent to our pseudocomplex structure. We can therefore use holomorphic coordinates for the complex structure in which S matches \mathbb{R}^2 to any desired order at a single point.

The three manifold S'' is totally real inside E . This S'' is the first step in an infinite prolongation of S .

8. THE CANONICAL BUNDLE

Define the canonical bundle K of a pseudocomplex structure E on a 4-manifold M to be the complex line bundle associated to the representation

$$\rho \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & ac^{-1} \end{pmatrix} = ac$$

of the structure group G . Note that this line bundle lives on E and not on M . For a proper pseudocomplex structure, it follows from local deformation to a complex structure that K restricts to each fiber of $E \rightarrow M$ to be a trivial holomorphic line bundle.

We can define a $\bar{\partial}$ operator on K by the following apparatus: a section of K is identified with a function $f : B \rightarrow \mathbb{C}$ satisfying equivariance

$$r_g^* f = \frac{f}{\rho(g)}.$$

We define $\bar{\nabla} f$ to be the $(0, 1)$ part of the 1-form

$$df - f(\alpha + \gamma).$$

However, there is no reason to believe that local holomorphic sections of K exist. The reader might try to determine when such sections exist, as an exercise in using the structure equations.

Now we have a canonical bundle for any pseudocomplex structure and a canonical bundle on any complex curve, so we can take a parameterized E curve

$$\phi : C \rightarrow E$$

and pull back the canonical bundle K_M from E ,

$$\phi^* K_M$$

and look for an adjunction formula, comparing

$$\phi^* K_M \text{ and } K_C.$$

Theorem 11 (Adjunction). *Let $E \subset \widetilde{\text{Gr}}_2(TM)$ be a proper pseudocomplex structure. Let C be a compact smooth Riemann surface, and $\phi : C \rightarrow E$ a parameterized E curve. Then*

$$-\chi_C = K_M \cdot C + C \cdot C - 2\delta$$

where δ , the embedding defect is a nonnegative integer, vanishing only if $p \circ \phi : C \rightarrow M$ is an embedding.

Proof. Suppose that $p \circ \phi : C \rightarrow M$ is an embedding. Then we can see from the structure equations that the normal bundle to C in M , call it ν_C , is the pullback by ϕ of the vector bundle associated to the principal bundle $B \rightarrow E$ and the representation

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & ac^{-1} \end{pmatrix} \mapsto a^{-1}$$

while the canonical bundle of M with respect to E is associated to the representation

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & ac^{-1} \end{pmatrix} \mapsto ac$$

and the canonical bundle of C is associated to

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & ac^{-1} \end{pmatrix} \mapsto c.$$

Therefore

$$K_M \otimes C = K_C$$

which gives the result immediately:

$$\begin{aligned}
-\chi_C &= \int_C c_1(K_C) && \text{by Riemann–Roch} \\
&= \int_C c_1(K_M \otimes \nu_C) \\
&= \int_C c_1(K_M) + \int_C c_1(\nu_C) \\
&= K_M \cdot C + C \cdot C.
\end{aligned}$$

The problem in the nonembedded case comes from $C \cdot C$, the self intersection number, not being identical to $\int_C c_1(\nu_C)$. We might imitate the complex proof and carry out blowup, but the blowup is not a pseudocomplex manifold. To deal with this, we can follow McDuff’s arguments, making a deformation of pseudocomplex structure, say as E_t , near each singular value of $p \circ \phi$, and deforming ϕ as we do, say as ϕ_t , so that $E_0 = E$ and E_1 is complex near each singular point of ϕ_1 . Then we can employ blowup at each singular point. \square

9. DARBOUX’S METHOD OF INTEGRATION

Definition 5. A pseudocomplex structure $E \subset \widetilde{\text{Gr}}_2(TM)$ is called Darboux integrable if near any point of E there are two independent holomorphic functions.

We can use such functions as follows: take $f, g : E \rightarrow \mathbb{C}$ to be such functions. Then any holomorphic curve in \mathbb{C}^2 has preimage under (f, g) some almost complex 4-manifold in E . The equations $\theta = \bar{\theta} = 0$ restrict to this 4-manifold to form a holonomic plane field, i.e. a system of ordinary differential equations, to find all E curves which map under (f, g) to the chosen curve. Conversely, if we have any E curve, then (f, g) must restrict to it to form a pair of holomorphic functions on the E curve so that for generic E curves this will trace out a holomorphic curve in \mathbb{C}^2 . Consequently, E curves can be found by solving ordinary differential equations only.

Indeed, we need not require f, g to be holomorphic, but only to be holomorphic on all E curves; however this turns out to force f and g to be holomorphic on E . Indeed, plugging in to the structure equations, using the equation

$$d^2 f = 0$$

and writing out

$$df = f_1\theta + f_2\omega + f_3\pi + f_{\bar{1}}\bar{\theta} + f_{\bar{2}}\bar{\omega} + f_{\bar{3}}\bar{\pi},$$

we find that if such f and g exist, then we have linear equations among the rows of

$$\begin{pmatrix} T_2 & T_3 \\ U_2 & U_3 \\ V_2 & V_3 \end{pmatrix}.$$

All of these invariants vanish precisely when E is a complex structure on a complex surface M . Darboux integrability requires two functions f and g which have independent complex linear differentials modulo $\bar{\theta}$. So this implies that, in some adapted coframing

$$U_2 = U_3 = V_2 = V_3 = 0.$$

There is no complete classification of the possibilities, but if we assume that either (1) $T_3 \neq 0$ everywhere, or (2) that both $T_3 = 0$ everywhere and $T_2 \neq 0$ everywhere,

or (3) that $T_2 = T_3 = 0$ everywhere, then we can split up our study into each of these cases. Similarly, in case (2), we have to split up into possibilities depending on whether some other invariant vanishes; the reader will find details and computations in [41]. The resulting pseudocomplex structures can be tabulated as follows (at least locally):

- (1) Complex surfaces. Locally, up to diffeomorphism there is a unique example. The symmetry pseudogroup is the group of local biholomorphisms, depending on 4 real functions of 2 real variables.
- (2) Take C a complex curve, and \mathcal{K} its canonical bundle. Then the equation

$$d\xi = \bar{\xi} \wedge \xi$$

defines an almost complex structure on the total space of the canonical bundle. If we take a local holomorphic coordinate z on C , this differential equation can be written

$$\frac{\partial w}{\partial \bar{z}} = |w|^2$$

in local holomorphic coordinates. Locally, up to diffeomorphism, there is a unique example. The symmetry pseudogroup is the group of local biholomorphisms of the complex curve C , depending on 2 real functions of 1 real variable.

- (3) An almost complex example, which has coordinates Z, W and can be described as the almost complex structure for which the 1-forms

$$dW - \frac{W\bar{Z}}{1-|Z|^2} dZ - \frac{\bar{W}}{1-|Z|^2} d\bar{Z}$$

and

$$dZ$$

are holomorphic. To find the E curves, take $Z(P)$ any holomorphic function of one complex variable, and integrate the ordinary differential equation

$$0 = dW - \left(\frac{W\bar{Z}}{1-|Z|^2} + P(Z) \right) dZ - \frac{\bar{W}}{1-|Z|^2} d\bar{Z}.$$

The manifold M is a bundle of complex curves over a complex curve. The base has parameter Z and a hyperbolic metric. The fibers $Z = \text{constant}$ are complex curves, so this is an almost complex structure canonically defined on some line bundle over a hyperbolic complex curve. Its geometric meaning is unclear to the author. Locally, up to diffeomorphism, there is a unique example. These Z, W coordinates are determined up to the following transformations:

- Any orientation preserving hyperbolic metric isometry of the curve parameterized by the Z variable. Under these transformations, the W variable behaves as a spinor.

- Take $f(Z)$ any holomorphic function, and write down the complex valued functions $a(Z, \bar{Z}), r(Z, \bar{Z})$ determined by the equations

$$r(Z, \bar{Z}) = Z^2 f(Z) + \overline{\frac{d}{dZ} Z^2 f(Z)}$$

$$a(Z, \bar{Z}) = \frac{\bar{r} + r\bar{Z}}{1 - |Z|^2}.$$

Then define new coordinates \hat{Z}, \hat{W} by

$$\hat{Z} = Z$$

$$\hat{W} = W + a(Z, \bar{Z}).$$

- (4) Take $F(z, \bar{z}, w, \bar{w}, p, \bar{p})$ any solution of the elliptic determined quasilinear system of partial differential equations

$$0 = \frac{\partial F}{\partial \bar{p}} + \bar{F} \frac{\partial F}{\partial \bar{z}} + \frac{\bar{w}}{1 - |p|^2} \frac{\partial F}{\partial w} + \frac{p\bar{w}}{1 - |p|^2} \frac{\partial F}{\partial \bar{w}}.$$

Define complex valued 1-forms

$$\theta_0 = dw - \left(\frac{w\bar{p}}{1 - |p|^2} - z \right) dp - \frac{\bar{w}}{1 - |p|^2} d\bar{p}$$

$$\omega_0 = dz - F dp$$

$$\pi_0 = dp.$$

These define a section of a unique G structure, and moreover the foliation

$$\theta = \bar{\theta} = \omega = \bar{\omega} = 0$$

if it is a fibration (which is the case locally) will have as base a 4-manifold M , so that these $\theta_0, \omega_0, \pi_0$ are an adapted coframing for a pseudocomplex structure on M . It is remarkable that this structure is dual to the previous one via

$$(W, Z, P) = (w, p, -z)$$

so that all of these examples are in fact examples of 4-parameter families of J curves in the previous example. Moreover, the generic 4-parameter family of J curves in that example will give rise to an example of this kind.

Locally, up to diffeomorphism, these examples are described by the choice of function F , depending on 2 real functions of 5 real variables. The possible symmetry pseudogroups are unknown, but there are several homogeneous cases (including some depending on at least two parameters).

10. KOBAYASHI AND BRODY HYPERBOLICITY

An E line in a manifold M with a pseudocomplex structure $E \subset \widetilde{\text{Gr}}_2(TM)$ is a nonconstant basic parameterized E curve $\mathbb{C} \rightarrow E$. A pseudocomplex 4-manifold is called Brody hyperbolic if it admits no E lines.

It is clear that in the category of complex manifolds, a holomorphic fiber bundle with Brody hyperbolic base is Brody hyperbolic precisely if its fibers are. Conversely, for a complex surface which is a holomorphic fiber bundle with hyperbolic fibers, the base is hyperbolic precisely when the surface is Brody hyperbolic. This is not true for pseudocomplex manifolds, as we see from the Darboux integrable almost complex structure 2.

We now define the Kobayashi pseudometric. A *graph of disks* is a connected Riemann surface with two marked points and finitely many ordinary double points, whose finitely many irreducible components are disks. We impose the Poincaré metric on each disk, so that it is complete with constant negative curvature -1 . This induces a metric on the entire graph of disks. We define the *length* of a graph of disks to be the distance between the two marked points.

A *Kobayashi chain* (or simply a *chain*) is an E curve

$$\phi : C \rightarrow E$$

where C is a graph of disks. We say that this chain joins $p \circ \phi(z) \in M$ to $p \circ \phi(w) \in M$ where z and w are the two marked points. Define the *Kobayashi metric*

$$\text{dis}_K(x, y) = \inf \text{length} \{ \phi_j \}$$

with the infimum taken over all chains joining x to y . The existence of a chain joining x to y (if M is connected) is an elementary application of the continuity method. Therefore the Kobayashi pseudometric is defined, symmetric, and satisfies the triangle inequality. But it may fail to be positive between distinct points. Note that we could restrict to using only trees of disks instead of graphs of disks, i.e. ask that the metric space formed by our graph of disks be strictly convex. This would provide the same metric. Or we could be more generous, and allow any Riemann surface with nodes and two marked points whose universal cover is a graph of disks—a hyperbolic curve with at most nodal singularities.

We will say that a pseudocomplex manifold is *Kobayashi hyperbolic* if the Kobayashi pseudometric gives positive distance between distinct points.

As usual, we will say that a Riemann surface without boundary is hyperbolic if it admits a metric of constant negative curvature in its conformal class.

Proposition 24. *A pseudocomplex manifold which is Kobayashi hyperbolic is Brody hyperbolic.*

Proof. Suppose that our pseudocomplex manifold fails to be Brody hyperbolic, so that we have a basic parameterized E curve $\phi : \mathbb{C} \rightarrow E$. Take a sequence of concentric disks in \mathbb{C} of arbitrarily large radius. Impose the Poincaré metric on each, conformally matching the induced metric. Then any pair of points of M in the image of $p \circ \phi$ are joined by Kobayashi chains of arbitrarily small length. \square

Given a parameterized E disk $\phi : D \rightarrow E$ with D bearing the Poincaré metric, we define its *velocity* to be $(p \circ \phi)'(0) \cdot e \in TM$ where $0 \in D$ is any fixed point, and e is any fixed unit vector $e \in T_0D$. If we holomorphically map the disk to itself, by dilation, we can slow down the velocity, scaling it by any t with $0 \leq t \leq 1$. The question is whether we can speed it up.

Proposition 25. *A compact and proper pseudocomplex 4-manifold is Brody hyperbolic precisely when it is Kobayashi hyperbolic.*

Proof. The argument is identical to Brody's in [15]. We sketch it. Suppose that $E \subset \widetilde{\text{Gr}}_2(TM)$ is our proper pseudocomplex structure on a compact manifold M . Take any Riemannian metric on M . Given a sequence of parameterized E disks, $\phi_j : D \rightarrow E$, with arbitrarily large velocity vectors, Brody [15] presents an argument that shows that we can replace these ϕ_j with new parameterized E disks ψ_j so that

$$(p \circ \psi_j)'$$

is largest at the origin of the disk. and so that the norm of the velocity of ψ_j becomes arbitrarily large. We can then reparameterize to produce, just as Brody does, a sequence of parameterized E disks $\xi_j : D_j \rightarrow E$, using disks D_j in the complex plane of arbitrarily large radii, so that the velocities of these ξ_j disks are bounded. Since their first derivatives are largest at the origin, this ensures that their first derivatives are bounded, so that some subsequence converges uniformly. The uniformity ensures by Corollary 20 that they converge uniformly with all derivatives to an E line $\xi : \mathbb{C} \rightarrow E$. \square

Given an infinitesimal symmetry of a pseudocomplex structure $E \subset \widetilde{\text{Gr}}_2(TM)$ described by a vector field X on M , we can prolong X to produce a vector field \hat{X} on E , defined by the requirement that the flow of \hat{X} on E is given by differentiating the flow of X , and applying its derivative to a 2-plane of tangent vectors:

$$\exp\left(t\hat{X}\right)e = (\exp(tX))'(m) \cdot e$$

for $e \in E_m \subset \widetilde{\text{Gr}}_2(T_mM)$. This flow is defined through each point, for small values of t , not only for real t values but also for complex t values since \hat{X} is easily seen to be a J holomorphic vector field. The flow defines a foliation of E by J holomorphic curves, wherever $\hat{X} \neq 0$, in other words above points of M where $X \neq 0$. The lift \hat{X} is complete precisely when X is.

Theorem 12. *The symmetry group of a Brody hyperbolic proper pseudocomplex structure on a compact manifold is finite.*

Proof. Take an infinitesimal symmetry X . The flow of \hat{X} is complete, by compactness, for X any infinitesimal symmetry. Not every flow curve of \hat{X} is actually a parameterized E curve for M . However, if we pick a flow curve Σ of X , passing through a point where $X \neq 0$, it is easy to see that it will be an E curve, and that its lift $\hat{\Sigma}$ will be a flow curve of \hat{X} . The completeness ensures that its domain will be the entire complex line. Therefore X must vanish everywhere.

By ellipticity of the G structure on E , the group of symmetries of a proper pseudocomplex structure on a compact manifold is a finite-dimensional Lie group (see [43]). But without any nonzero infinitesimal symmetries, it must be zero dimensional, so a discrete group.

By the uniform convergence theorem for symmetries (Theorem 20), we know that a uniform limit of symmetries converges with all derivatives. Suppose that we have a sequence of symmetries. We can bound their derivatives by taking an E disk, and looking at how its velocity gets stretched under a symmetry. The stretching factor must be bounded, or we will be able to recursively apply the symmetry to generate E disks of arbitrarily large velocity. This bounds the derivative of the symmetry, since there is an E disk with any velocity close enough to 0, by continuity. Therefore any sequence of symmetries have a convergent subsequence, so the symmetry group is compact. Because it is also discrete, it must be finite. \square

Corollary 21. *Let M be a 4-manifold and $E \subset \widetilde{\text{Gr}}_2(TM)$ a proper pseudocomplex structure on M . The universal cover of M , with the induced proper pseudocomplex structure, is compact and Brody hyperbolic precisely if M is, which happens precisely when both are compact and Kobayashi hyperbolic.*

Proposition 26. *A compact and proper pseudocomplex 4-manifold, equipped with an arbitrary Finsler metric $v \mapsto |v|$, is Kobayashi hyperbolic precisely when the norms $|v|$ of velocities of E disks are bounded.*

Proof. The proof consists in an elementary limit argument. See [15]. \square

We now define the Royden pseudonorm on TM . Given any vector $v \in TM$, we can also ask whether it can be scaled to become the velocity vector of a parameterized E disk. The Royden pseudonorm $|v|$ of any vector $v \in TM$ is defined to be the largest scale factor $t \geq 0$ so that tv occurs as the velocity of a disk.

A simple application of the continuity method (for closed disks) shows that $|v| > 0$ for $v \neq 0$, so that the problem with this norm is only that it may be infinite: $|v| = \infty$, as occurs for any tangent vector in $M = \mathbb{C}^2$. Define lengths of paths $\gamma : [a, b] \subset \mathbb{R} \rightarrow M$ to be

$$\text{length } \gamma = \int_a^b |\dot{\gamma}(t)| dt$$

and define the Royden pseudodistance between points to be the infimum of lengths of paths joining them. Another application of continuity shows that this length coincides with the Kobayashi pseudodistance; see [37] for details.

11. BANGERT'S THEORY OF LINES IN A TORUS

Let $\Lambda \subset \mathbb{R}^{2n}$ be a lattice. We will say that a symplectic structure on the torus $T^{2n} = \mathbb{R}^{2n}/\Lambda$ is *standard* if it pulls back to the standard $dq^i \wedge dp^i$ structure on \mathbb{R}^{2n} . It is unknown if there are any nonstandard symplectic structures.

Theorem 13. *A proper pseudocomplex structure on T^4 tamed by a standard symplectic structure is not Brody hyperbolic.*

Proof. The proof is the same as in [3], using the quasiminimality of E curves, the pseudoconvexity of small balls (see Section 12), and Gromov compactness. \square

This result shows that while Brody hyperbolic structures are an open set, they are not always dense. In fact, it is not very clear if there are any Brody hyperbolic pseudocomplex structures on the torus.

It remains a mystery (first considered by Jürgen Moser) whether entire foliations of complex tori survive (perhaps in a KAM sense) perturbation of the complex structure to a pseudocomplex structure.

12. PLURISUBHARMONIC FUNCTIONS AND PSEUDOCONVEXITY OF HYPERSURFACES

12.1. Pseudoconvexity. To define pseudoconvexity of hypersurfaces $H \subset M$, we first need to lift H up to $E \rightarrow M$. The tangent space of H at any point $m \in H$ contains a discrete set of 2-planes belonging to $E_m \subset \widetilde{\text{Gr}}_2(T_m M)$. We construct $\tilde{H} \subset E$ to be the set of all of these 2-planes. It is easy to see that \tilde{H} is a submanifold of E of dimension 3, and $p : \tilde{H} \rightarrow H$ given by $p : E \rightarrow M$ is a local diffeomorphism. Moreover, \tilde{H} is totally real for the almost complex structure of E . If E is a proper pseudocomplex structure on M , then in fact $\tilde{H} \rightarrow H$ is a diffeomorphism.

Along H , the G structure B on E can be reduced to a principal G_1 bundle, where G_1 is the set of complex matrices

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ \frac{a\bar{b}iL}{|c|^2} & 0 & \frac{a}{c} \end{pmatrix}$$

where a is real, b, c are complex, and L is a relative invariant, which we will call the *Levi invariant*. Moreover,

$$\begin{aligned} \theta &= \bar{\theta} \\ \pi &= iL\bar{\omega}. \end{aligned}$$

This is easy to calculate (see [41] for details). If $L \neq 0$ at all points of the bundle over H , then we will say that H is strictly pseudoconvex.

Following Cartan's work on real hypersurfaces in complex surfaces [22, 23], we find the following structure equations on the reduced bundle of any pseudoconvex H :

$$\begin{aligned} d \begin{pmatrix} \theta \\ \omega \\ \beta \\ \gamma \\ \phi \end{pmatrix} &= \begin{pmatrix} -(\gamma + \bar{\gamma}) \wedge \theta + i\omega \wedge \bar{\omega} \\ -\beta \wedge \theta - \gamma \wedge \omega \\ -\phi \wedge \omega + \bar{\gamma} \wedge \beta + R\theta \wedge \bar{\omega} \\ -\phi \wedge \theta - 2i\bar{\beta} \wedge \omega - i\beta \wedge \bar{\omega} \\ i\beta \wedge \bar{\beta} + (\gamma + \bar{\gamma}) \wedge \phi + (S\omega + \bar{S}\bar{\omega}) \wedge \theta \end{pmatrix} \\ dR &= R(\gamma + 3\bar{\gamma}) - \bar{S}\omega - A\theta - B\bar{\omega} \\ dS &= S(3\gamma + 2\bar{\gamma}) - i\bar{R}\beta - E\omega - F\bar{\omega} - G\theta \end{aligned}$$

where $\pi = i\bar{\omega}$, and A, B, E, G, R, S are complex valued functions, and F is a real valued function (on the induced bundle over H). These are the same structure equations one usually encounters in the theory of CR geometries for real hypersurfaces in complex surfaces.

12.2. Pseudoconvex foliations. If we have a foliation F by pseudoconvex hypersurfaces, we can construct a submanifold $\tilde{F} \subset E$ consisting of the \tilde{H} submanifolds where H runs over all of the hypersurfaces of the foliation. This \tilde{F} is then a foliated 4-manifold. On each leaf, we can produce the subbundle of our G structure described above for hypersurfaces. Then this produces a subbundle of our G bundle over all of \tilde{F} , and on this subbundle, $\pi = i\bar{\omega}$. Moreover, on each leaf, $\theta = \bar{\theta}$.

A *defining function* for a foliation F by hypersurfaces is a function $f : M \rightarrow \mathbb{R}$ so that the level sets of f are the leaves of F . Locally, a defining function exists.

Proposition 27. *A defining function of a foliation by pseudoconvex hypersurfaces in a manifold M with pseudocomplex structure E must pull back to any E curve to have positive Laplacian, at all nonsingular points of the curve where the tangent plane to the projection of the curve to M is nearly tangent to a level set of the defining function.*

Proof. Nearly tangent means that the tangent plane of the E curve should be close to \tilde{F} , where F is the foliation. Without loss of generality, suppose that the E curve actually passes through a point of $\tilde{F} \subset E$. It is easy to calculate the Laplacian of f in terms of the subbundle structure equations:

$$-i\bar{\partial}\partial f = i\omega \wedge \bar{\omega} \quad \text{mod } \theta$$

so that if z is a holomorphic coordinate on an E curve, and $\omega = g dz$, then

$$\Delta f = \frac{1}{2}|g|^2$$

on the E curve. \square

Theorem 14. *Consider a nonconstant parameterized E curve $\phi : C \rightarrow E$. Suppose that the interior of C is mapped by $p \circ \phi : C \rightarrow M$ into the closure of a region $U \subset M$ with smooth strictly pseudoconvex boundary.*

$$\begin{array}{ccc} C & \xrightarrow{\phi} & E \\ & \searrow p \circ \phi & \downarrow p \\ & & U \subset M \end{array}$$

No interior point of C gets mapped to the boundary of U , and if a boundary point of C gets mapped to a boundary point of U :

$$z \in \partial C \mapsto p \circ \phi(z) = m \in \partial U,$$

then $p \circ \phi$ is an immersion near z , and

$$(p \circ \phi)'(z) \cdot T_z C \not\subset T_m \partial U$$

Proof. As in the complex case. \square

12.3. Plurisubharmonic functions. A *plurisubharmonic function* is a function $f : M \rightarrow \mathbb{R}$ whose restriction to any nonsingular E curve has positive Laplacian. The story of plurisubharmonic functions is radically different from the almost complex case.

Theorem 15. *Let E be a proper pseudocomplex structure on a 4-manifold M . There is a plurisubharmonic function near any point of M precisely when M is almost complex.*

Proof. Take a function $f : M \rightarrow \mathbb{R}$, and pull it back to E . Now differentiate it:

$$df = f_1 \theta + f_{\bar{1}} \bar{\theta} + f_2 \omega + f_{\bar{2}} \bar{\omega}$$

and then take its second derivative:

$$d\bar{\partial}f = f_{2\bar{2}} \omega \wedge \bar{\omega} + \bar{f}_2 \bar{S}_2 \bar{\pi} \wedge \omega - f_2 S_2 \pi \wedge \bar{\omega}.$$

If $f_2 S_2(e) \neq 0$, then there is a 2-plane in $T_e E$ on which $\theta = \bar{\theta} = 0$ and on which

$$-id\bar{\partial}f < 0$$

and another on which

$$-id\bar{\partial}f > 0.$$

By local solvability of elliptic partial differential equations, we can then find E curves on which the Laplacian takes either sign. Therefore we will need $f_2 S_2 = 0$ at each point to have plurisubharmonicity of f . But it is easy to calculate that $f_2 = 0$ forces f to be constant. Therefore we need $S_2 = 0$. By Proposition 13 (assuming E proper), this occurs precisely for E almost complex.

Conversely, if E is almost complex, then in adapted coordinates the function

$$f(z, w) = |z|^2 + |w|^2$$

is plurisubharmonic near the origin. \square

13. BISHOP DISKS

Given a real surface $\Sigma \subset M$ immersed in a 4-manifold M with pseudocomplex structure E , we can define its *lift* to be $\tilde{\Sigma}$ to be the set of its tangent planes, thought of as a surface inside $\widetilde{\text{Gr}}_2(T_m M)$. Under the projection

$$\widetilde{\text{Gr}}_2(TM) \rightarrow M$$

this $\tilde{\Sigma}$ is taken diffeomorphically to Σ . We call a point of intersection of $\tilde{\Sigma}$ with E an *elliptic point* if it is a positive intersection, a *hyperbolic point* if it is a negative intersection, and otherwise we call it a *parabolic point*. See [42] for more information about the theory of elliptic and hyperbolic points in complex surfaces. A point of Σ is called *totally real* if the tangent space to Σ at that point does not belong to E . The surface Σ is called a *totally real surface* if all of its points are totally real. A point which is not totally real will be called an *E point* of Σ .

13.1. Totally real surfaces. Suppose that Σ is totally real. We define a submanifold $\hat{\Sigma} \subset E$ to be the set of all 2-planes $P \subset T_m M$ which belong to E and strike the tangent planes of Σ in a line. By our microlocal geometry results, this $\hat{\Sigma}$ is a smooth 3-manifold immersed in E . It is easy to see that $\hat{\Sigma}$ is totally real as a submanifold of the almost complex manifold E .

Proposition 28. *Let X be a totally real immersed submanifold of an almost complex manifold Z . Every point $x \in X$ has a neighborhood U in Z in which there are no pseudoholomorphic curves $\phi : C \rightarrow Z$ from a compact Riemann surface C with boundary, so that $\phi(\partial C) \subset X$ and $\phi(C) \subset U$.*

Proof. Take adapted complex coordinates

$$z^1, \dots, z^n$$

as in [40], so that the totally real submanifold becomes Lagrangian (indeed, we can ask that it become the set of real points of \mathbb{C}^n). Then integrate the Kähler form

$$\frac{i}{2} \sum_j dz^j \wedge dz^{\bar{j}}$$

which must be positive on a “small” pseudoholomorphic curve, because the curve is nearly holomorphic. But the boundary sits in a Lagrangian manifold, which (if we use a small enough neighborhood) contradicts the quasi-minimality of the pseudoholomorphic curve. \square

Corollary 22. *The same result is true for totally real surfaces in pseudocomplex 4-manifolds.*

Proof. Looking upstairs in E , an E curve with boundary in a totally real surface $\Sigma \subset M$ becomes a pseudoholomorphic curve in E with boundary in $\hat{\Sigma}$. \square

13.2. Elliptic points and Bishop disks. When Σ has an E point, $\hat{\Sigma}$ is not well-defined. We can define another notion of lifting Σ to E : let $\Sigma' \subset E$ simply be the pullback bundle of $E \rightarrow M$ under $\Sigma \rightarrow M$, i.e. $p : E \rightarrow M$ is the projection, and $\Sigma' = p^{-1}\Sigma$.

Pulling back the G structure to Σ' , one finds that the structure equations give, near a point $e_0 \in \Sigma'$ which lies above an E point of Σ :

$$\theta = f\omega + g\bar{\omega}$$

where $f, g : B|_{\Sigma'} \rightarrow \mathbb{C}$ are smooth functions vanishing at e_0 . We say that the point e_0 is *nondegenerate* if

$$df \wedge d\bar{f} \wedge dg \wedge d\bar{g} \neq 0$$

at e_0 . Write

$$df \wedge d\bar{f} \wedge dg \wedge d\bar{g} = h\omega \wedge \bar{\omega} \wedge \pi \wedge \bar{\pi}.$$

Proposition 29. *An E point e_0 is elliptic precisely when $h > 0$ and hyperbolic precisely when $h < 0$.*

Proof. This requires only calculating examples of surfaces with arbitrary 2-jet, since it depends only on the 2-jet of any surface. For more details, see [41]. \square

At an elliptic point, we can further refine the structure equations (see [41]), obtaining at the point e_0 :

$$d \begin{pmatrix} \omega \\ \pi \\ f \\ g \\ C \end{pmatrix} = \begin{pmatrix} -\gamma \wedge \omega - S_2 \pi \wedge \bar{\omega} \\ -\bar{\gamma} \wedge \pi + (2\bar{S}_2 + \bar{C}T_3) \omega \wedge \bar{\pi} + (\bar{C}\bar{S}_2 + T_3) \bar{\pi} \wedge \pi + q(\pi - \bar{\omega}) \wedge \omega + r\pi \wedge \bar{\omega} \\ -\pi + \bar{\omega} \\ \omega + C\bar{\omega} \\ C(\bar{\gamma} - \gamma) + (r - Cq - T_2)\omega + (s - Cr)\bar{\omega} + 2S_2\pi - C(T_3 + CS_2)\bar{\pi} \end{pmatrix}$$

Theorem 16 (Bishop–Ye). *Let M be a 4-manifold with pseudocomplex structure E , and $\Sigma \subset M$ an immersed surface with elliptic point $m \in \Sigma$. There is a smooth embedding $\phi : X \rightarrow M$ of a half ball $X \subset \mathbb{R}^3$ so that $\phi(0) = m$, and ϕ restricts to each half sphere to be an embedded E curve with boundary in Σ .*

Proof. The argument is the same as in [52]. Also see [41] for more details. Essentially the idea is the same as in [14]: the linearization of the problem of constructing the half spheres has one dimensional kernel and zero dimensional cokernel. This allows perturbing everything away from the flat case: $M = \mathbb{C}^2$,

$$\Sigma = \left\{ w = |z|^2 + \frac{\lambda}{2} (z^2 + \bar{z}^2) \right\}$$

and

$$X = \left\{ w = t, |z|^2 + \frac{\lambda}{2} (z^2 + \bar{z}^2) < t \mid t \in \mathbb{R} \right\}.$$

We can approximate in adapted coordinates (as in Corollary 16) with this example (see [42]). What makes the a priori estimates work is that very small E disks near to the origin have small Sobolev norm of the (nonlinear) operator

$$\sigma[w] = w_{\bar{z}} - Q(z, w, w_z)$$

while they have Sobolev norm of roughly fixed magnitude for $\sigma'[w]$, so that we can move $w(z)$ just a little, and effectively push toward $\sigma = 0$. \square

Theorem 17 (Uniqueness of Bishop disks). *There are at most two possible embedded maximal half spheres of Bishop disks as defined in the previous theorem, one on each side of a real surface with an elliptic point.*

Proof. To obtain the uniqueness of Bishop disks, we need to understand the intersections of such disks. Suppose that we have two disks with boundary in a surface, so that both boundaries are smooth curves and lie close to an elliptic point. Suppose further that the disks are embedded, and transverse to the real surface along their boundaries. Neither disk can touch the elliptic point itself since that would require tangency with the real surface at that point. Suppose that our disks have an interior intersection. Then we know (by McDuff’s arguments as adapted above to our situation) that the intersection survives small perturbation. However, we will need to handle boundary intersections as well.

For example, consider the real surface $\mathbb{R}\mathbb{P}^2 \subset \mathbb{C}\mathbb{P}^2$. Ignore for the moment the fact that it has no elliptic point, and think about two complex disks with boundary in this real surface. For example, take $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^2$ and slice it into two disks by cutting out the real points. Then we have two intersecting closed disks, which may cease to intersect if we perturb either one slightly. The delicate part is to make the disks intersect “on the same side.”

Let us follow essentially the argument of Ye [52]. Because the Fredholm theory tells us that the deformation theory of these disks is unobstructed, we can make them both slightly larger, extending them so that they pass across the real surface. Then at an intersection point, we can use adapted coordinates, holomorphic up to leading order terms, to see that the intersection of the two disks looks to leading order just like the complex case. But there is a further innovation we can introduce here: we can adapt our coordinates to the real surface near the intersection point. Since it is totally real near that point, there is no obstruction at any finite order to finding adapted coordinates for which the totally real surface is just the set of points at which the adapted coordinates take on real values. This is easy to check with the Cartan–Kähler theorem. However, we have to be very careful: there could be an obstruction beyond all orders. But we can pick an order at which to match up the real surface to the set of real points of our adapted coordinates, higher than the order of intersection of our disks at that point, and thereby pretend that the real surface is exactly the set of real points of our coordinates, with no loss of generality. Then we see that, with a little manipulation of the coordinates, one disk looks like

$$w = az^k + O(|z|^{k+1})$$

with $a \in \mathbb{R}$ while the other looks like

$$w = 0.$$

The integer k must be at least one. Under small perturbation through E disks we see that the intersection point can at worst break into k intersection points, distributed around a circle nearly evenly. As in complex geometry, it is clear then that the intersection persists under perturbation, and that moreover the intersection points are nearly complex conjugates of each other in these coordinates. In particular, intersections cannot disappear off the boundary of the disks, since they will actually “bounce off” the boundary.

Therefore if two E disks with boundary in a totally real surface intersect, then they continue to intersect after both of them receive a small perturbation, as long as the intersection was at finite order. However, as our example shows, they might fail to intersect after perturbation if they start off with infinite order intersection. By Aronszajn’s lemma, this can only occur if they can be extended to agree, so that we can glue them together as in the example.

If we had two half balls of Bishop disks, then either we could glue them together (if they lay on opposite sides of the real surface) or they would be identical (near the elliptic point—if we extend them to be maximal then they would agree) or one disk from one half sphere family has to have finite order intersection with one from the other half sphere family. \square

14. DIRECTIONS FOR FURTHER INVESTIGATIONS

The notion that arises from our work so far is that the moduli spaces of curves in a symplectic manifold should be thought of as having curves on them. While we have only obtained this result for the unobstructed part of four-dimensional moduli spaces of curves in a four-dimensional manifold, it is tempting to believe that this concept generalizes to arbitrary dimensions.

Recall that the theory of Gromov–Witten invariants centers on the map from moduli of curves in a symplectic manifold to moduli of abstract curves, forgetting the ambient manifold; (a little) more precisely, if $C \subset X$ is a holomorphic (or E) curve in a symplectic manifold X , then let us write X^C for the moduli space of deformations of the curve C as a holomorphic curve in X , compactified by adding stable curves. We can also map C to a point, say $*$, so that we obtain a map

$$X^C \rightarrow *^C$$

by forgetting how C sits in X . This $*^C$ is the Deligne–Mumford compactified moduli space. Gromov–Witten invariants arise from pulling back cohomology classes:

$$H^*(*^C) \rightarrow H^*(X^C).$$

If, as the author believes, these moduli spaces X^C carry their own E curves, then we may construct another moduli space: if C_1 is a curve in X^C , then we have a space

$$X^{C^{C_1}}$$

and maps

$$X^{C^{C_1}} \rightarrow *^{C^{C_1}}$$

which generate cohomology classes

$$H^*(*^{C^{C_1}}) \rightarrow H^*(X^{C^{C_1}}).$$

Presumably this generates a theory like that of Gromov–Witten invariants.

Another point of view on this story comes from the theory of partial differential equations. The duality between the E curves on 4-manifolds explained above is a (fairly elementary) example of a Bäcklund transformation. But the surprise is that it is “stable.” Consider the example of $X = \mathbb{C}\mathbb{P}^2$ with the standard complex structure, and Y the set of lines in X . We can make perturbations of the Cauchy–Riemann equations on X to any nonlinear elliptic equations as long as they are tamed by a symplectic structure. The space Y can be replaced by the space of rational E curves in X in the homology class of a complex line, and the Bäcklund transformation survives this perturbation. The author is unaware of any other example of this sort of stability for Bäcklund transformations, or of any evidence that any Bäcklund transformation is unstable in this sense.

The author suspects that if we have a dual pair of compact pseudocomplex four-manifolds, with both pseudocomplex structures being proper, then both four-manifolds are diffeomorphic to $\mathbb{C}\mathbb{P}^2$.

It would be nice to know which real analytic pseudocomplex structures admit a Schwarz reflection about any totally real hypersurface, roughly analogous to symmetric spaces.

REFERENCES

1. S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623–727.
2. ———, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math. **17** (1964), 35–92.
3. Victor Bangert, *Existence of a complex line in tame almost complex tori*, Duke Math. J. **94** (1998), no. 1, 29–40.
4. Eric Bedford and Bernard Gaveau, *Envelopes of holomorphy of certain 2-spheres in \mathbb{C}^2* , Amer. J. Math. **105** (1983), 975–1009.
5. Svetoslav Ī. Bilčev, *Systems of two first order partial differential equations (local theory)*, Izv. Vysš. Učebn. Zaved. Matematika **1970** (1970), no. 3 (94), 14–21.
6. ———, *Systems of two first order partial differential equations (the conservation laws)*, Izv. Vysš. Učebn. Zaved. Matematika **1970** (1970), no. 6 (97), 28–34.
7. ———, *The linearity of systems of two first order partial differential equations*, Izv. Vysš. Učebn. Zaved. Matematika **1** (1974), no. 140, 3–14.
8. ———, *Invariant foliations of the manifold of integral tangent elements for the asymmetric case of hyperbolic systems of first-order partial differential equations*, Godishnik Vissh. Uchebn. Zaved. Prilozhna Mat. **13** (1977), no. 4, 189–200 (1978).
9. ———, *Quasilinearity of a system of two first-order partial differential equations*, Godishnik Vissh. Uchebn. Zaved. Prilozhna Mat. **13** (1977), no. 4, 181–188 (1978).
10. ———, *A variational problem*, Godishnik Vissh. Uchebn. Zaved. Prilozhna Mat. **14** (1978), no. 4, 1–8 (1979).
11. Svetoslav Ī. Bilčev and Dočo T. Dočev, *Conservation laws for the asymmetric case of hyperbolic systems of two first-order partial differential equations*, Godishnik Vissh. Uchebn. Zaved. Prilozhna Mat. **13** (1977), no. 3, 167–174 (1978).
12. ———, *On the geometry of the asymmetric case of hyperbolic systems of two first-order partial differential equations with two unknown functions and two independent variables*, Godishnik Vissh. Uchebn. Zaved. Prilozhna Mat. **13** (1977), no. 3, 149–165 (1978).
13. Svetoslav Ī. Bilčev, Dočo T. Dočev, and Dimit'r K. Pantelev, *Systems of two first order partial differential equations (conservation laws)*, B''lgar. Akad. Nauk. Izv. Mat. Inst. **15** (1974), 21–26.
14. Errett A. Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1–22.
15. Robert Brody, *Compact manifolds in hyperbolicity*, Trans. Amer. Math. Soc. **235** (1978), 213–219.
16. Robert L. Bryant, *1994 Aisenstadt Lectures*, preprint.
17. ———, *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Diff. Geom. **17** (1982), 455–473.
18. ———, *Surfaces in conformal geometry*, The Mathematical Heritage of Hermann Weyl (Providence, R.I.) (R.O. Wells, ed.), Proceedings of Symposia in Pure Mathematics, vol. 48, Amer. Math. Soc., 1988, pp. 227–240.
19. Robert L. Bryant, Shiing-Shen Chern, Robert B. Gardner, Hubert L. Goldschmidt, and Phillip A. Griffiths, *Exterior differential systems*, Springer-Verlag, New York, 1991.
20. Robert L. Bryant, Phillip A. Griffiths, and Lucas Hsu, *Toward a geometry of differential equations*, pp. 1–76, in Yau [50], 1995.
21. Élie Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. France **52** (1924), 205–241, Also in [26], part. III, vol. 1, 825–861.
22. ———, *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, I*, Annali di Matematica Pura ed Applicata **11** (1932), 17–90, Also in [26], part. II, 1231–1304.

23. ———, *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, II*, Annali Sc. Norm. Sup. Pisa **1** (1932), 333–354, Also in [26], part. III, vol. 2, 1217–1238.
24. ———, *Les espaces à connexion projective*, Mém. Sémin. Anal. vectorielle **4** (1937), 147–159, Also in [26], part. III, vol. 2 1397–1409.
25. ———, *Les espaces généralisés et l'intégration de certaines classes d'équations différentielles*, C. R. Acad. Sc. **206** (1938), 1689–1693, Also in [26], part. III, vol. 2, 1397–1409.
26. ———, *Œuvres Complètes*, Gauthier-Villars, Paris, 1952–1955.
27. Gaston Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, Gauthier-Villars, Paris, 1894.
28. Jesse Douglas, *The general geometry of paths*, Annals of Math. **29** (1928), 143–168.
29. J. J. Duistermaat, *On first order elliptic equations for sections of complex line bundles*, Compositio Math. **25** (1972), 237–243.
30. Michael Gage, *A note on skew-Hopf fibrations*, Proc. Amer. Math. Soc. **93** (1985), no. 1, 145–150.
31. Robert B. Gardner, *The method of equivalence and its applications*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989.
32. Herman R. Gluck and Frank W. Warner, *Great circle fibrations of the three sphere*, Duke Math. J. **50** (1983), 107–132.
33. Mikhael Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Inven. Math. **82** (1985), 307–347.
34. Helmut Hofer, Veronique Lizan, and Jean-Claude Sikorav, *On genericity for holomorphic curves in four-dimensional almost-complex manifolds*, J. Geom. Anal. **7** (1997), no. 1, 149–159.
35. Gary R. Jensen, *Higher order contact of submanifolds of homogeneous spaces*, Springer-Verlag, Berlin, 1977, Lecture Notes in Mathematics, Vol. 610.
36. A.N. Kolmogorov and A.P. Yushkevich (eds.), *Mathematics of the 19th Century : Geometry, Analytic Function Theory*, Birkhäuser, Basel, 1996, Translated from the Russian by Roger Cooke.
37. Boris S. Kruglikov and Marius Overholt, *The Kobayashi pseudodistance on almost complex manifolds*, dg-ga/9703005.
38. B. L. Laptev and B. A. Rosenfel'd, *Geometry*, pp. 1–117, in Kolmogorov and Yushkevich [36], 1996, Translated from the Russian by Roger Cooke.
39. Dusa McDuff, *Singularities of J-holomorphic curves in almost complex 4-manifolds*, J. Geom. Anal. **2** (1992), no. 3, 249–266.
40. ———, *Singularities and positivity of intersections of J-holomorphic curves*, Holomorphic curves in symplectic geometry, Birkhäuser, Basel, 1994, With an appendix by Gang Liu, pp. 191–215.
41. Benjamin McKay, *Duality and integrable systems of pseudoholomorphic curves*, Ph.D. thesis, Duke University, 1999.
42. Jürgen K. Moser and Sidney M. Webster, *Normal forms for real surfaces in \mathbb{C}^2 near complex tangents and hyperbolic surface transformations*, Acta Math. **150** (1983), 255–296.
43. Takushiro Ochiai, *On the automorphism group of a G-structure*, J. Math. Soc. Japan **18** (1966), 189–193.
44. Jean-Claude Sikorav, *Dual elliptic structures on $\mathbb{C}\mathbb{P}^2$* , lanl:math.SG/0008234.
45. Norman Steenrod, *The Topology of Fiber Bundles*, seventh ed., Princeton Landmarks in Mathematics and Physics, Princeton University Press, Princeton, New Jersey, 1999.
46. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, 1971.
47. Arthur Tresse, *Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre $y'' = \omega(x, y, y')$* , Ph.D. dissertation., Universtät Leipzig, Leipzig, 1896, Presented to the Prince Jablonowski Society of Leipzig. Unpublished.
48. I. N. Vekua, *Generalized Analytic Functions*, Pergamon Press, 1962, Translated into English from Russian; I. N. Sneddon, translation editor.
49. W. L. Wendland, *Elliptic Systems in the Plane*, Pitman, London, 1979.
50. Shing-Tung Yau (ed.), *Geometry, Topology and Physics for Raoul Bott*, Conference proceedings and lecture notes in geometry and topology, vol. 4, International Press Inc., Cambridge, Mass., 1995.

51. Rugang Ye, *Gromov's compactness theorem for pseudo holomorphic curves*, Trans. Amer. Math. Soc. **342** (1994), no. 2, 671–694.
52. ———, *Filling by holomorphic curves in symplectic 4-manifolds*, Trans. A.M.S. **350** (1998), no. 1, 213–250.

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