G_2 MANIFOLDS OF COHOMOGENEITY TWO

BENJAMIN MCKAY

ABSTRACT. There are no G_2 holonomy manifolds of cohomogeneity two. The method of proof can be applied to many similar problems, and requires no assumption about the algebra of the symmetry group, or the geometry of the group action.

CONTENTS

1. INTRODUCTION

First, a word from our sponsor (particle physics). The mathematical reader should not expect to understand the physics, but (paraphrasing A. N. Varchenko) just let the words wash over you like music. Acharya & Witten [1] and Atiyah & Witten [2] have demonstrated that singularities in G_2 holonomy manifolds[1] can compactify M-theory[2] yielding effective quantum field theories in 3+1 dimensional Minkowski space with realistic gauge groups, chirality and supersymmetry. The mathematician's responsibility is to generate a zoo of such singularities, so that their physics can be probed, as possible candidates for a final physical theory. Enough physics.

This article is not the first, but certainly the broadest in scope in this hunt for new singularities. Our aim is to classify all G_2 holonomy manifolds of cohomogeneity two, i.e. with a symmetry group acting with orbits of codimension two. No symmetry group can act transitively, since G_2 holonomy manifolds are Ricci flat, and homogeneous Ricci flat manifolds are flat (a result of Alekseevski; see Besse [3]). Therefore the minimal cohomogeneity of a symmetry group must be 1; these

---

Date: November 20, 2003.

[1]See Joyce [8] for definitions of holonomy and G_2, and examples.

[2]See Polchinski [10] and Duff [6] for some idea as to what M-theory might be.

have probably very nearly been classified in a long literature. The most complete study of cohomogeneity one G_2 manifolds is Cleyton & Swann [5] (the authors need to assume that the symmetry group is compact, and complete the classification only for compact simple groups).

When studying cohomogeneity one, the orbits are hypersurfaces, and so the problem reduces to a system of ordinary differential equations describing the development of various fields in the direction perpendicular to those hypersurfaces. The relevant ordinary differential equations have turned out (for all symmetry group actions so far analysed) to be integrable explicitly. Cohomogeneity two is more interesting, since the relevant differential equations are now partial differential equations for functions of two variables (the two normal directions to the group orbits). We might hope for known integrable systems of partial differential equations to appear.

Commonly used methods for symmetry reduction do not apply in this context; they usually employ the hypothesis that the symmetry group, the manifold, and the action of the group on the manifold, are given. Then one forms the quotient of the manifold by the group action, and produces a reduction of the field equations by considering the correspondence of fields on the quotient with group invariant fields on the original manifold. However, in our setting neither the group nor the manifold is assumed known, let alone the action of the group on the manifold. Two experts in symmetry reduction of classical physical fields and spacetime models told the author that there is no method for classifying symmetric fields without *ad hoc* hypotheses. The importance of this article is that we make no such hypotheses, beyond controlling the cohomogeneity.

The study of exceptional holonomy groups is made more complicated by the problem of relating the holonomy group, a global invariant, to the local invariants of the geometry. Consider for example a flat torus. If we only slightly perturb the metric in an arbitrarily small region, the holonomy even at distant points jumps from a discrete group to a continuous group. The Ambrose–Singer theorem is usually used to ensure a lower bound on the holonomy: the Lie algebra generated by the values of the curvature tensor is contained in the Lie algebra of the holonomy group. To obtain an upper bound, we have used G_2 structures rather than Riemannian metrics. A torsion-free G_2 structure is guaranteed to have Riemannian metric with holonomy contained in G_2.

An enormous problem, to which I have paid no attention, is that ultimately physicists would like compactifications on compact G_2 holonomy manifolds with singularities. Atiyah & Witten were able to make use of noncompact singular G_2 holonomy manifolds, to understand certain features of M-theory on any compact manifold which has similar singularities, should such a compact manifold exist. Therefore noncompact G_2 holonomy manifolds are already helpful, and may provide a zoo of singularities which might ultimately be found on compact spaces. A compact smooth Ricci-flat manifold always has a finite symmetry group unless it is locally a product (see Besse [3]), and therefore symmetry group methods are not useful in constructing these. Allowing singularities is not known to alleviate this lack of symmetry.

Our final result:

Theorem 1. *There are no Riemannian manifolds of G_2 holonomy with a Lie algebra of infinitesimal symmetries whose largest orbits have cohomogeneity two.*

The result is not nearly as exciting as the method, which is quite generally applicable. We will explain our method in detail in the case of 9 dimensional groups, and thereafter make less detailed explanations.

2. G_2 structure equations

Let M be a seven dimensional manifold with a torsion free G_2 structure, where G_2 is the exceptional 14 dimensional simple Lie group in its compact form, i.e. the automorphism group of the octave numbers.[3] Such a manifold bears a canonical Ricci-flat Riemannian metric and a parallel closed and coclosed 3-form ϕ which is positive in the sense of Bryant [4] and Hitchin [7]. Because G_2 is simple and connected, M has a canonical orientation, and because G_2 is simply connected, M has a spin structure. Because Ricci-flat manifolds are real analytic (see Besse [3]), so is M and its Riemannian metric; the parallel nature of the 3-form ensures that it is real analytic as well. Moreover, the automorphism group of the G_2 structure must also be a real analytic group with a real analytic action.

Let $B_2 \to M$ be the bundle of all linear maps identifying tangent spaces of M with the seven dimensional irreducible representation $V = \mathbb{R}^7$ of G_2 so that they identify the 3-form with the standard 3-form

$$\phi_0 = dx^{567} + dx^{125} - dx^{345} + dx^{136} + dx^{246} + dx^{147} - dx^{237}$$

where $dx^1, \dots, dx^7$ is a basis of V^* and $dx^{ij} = dx^i \wedge dx^j$ etc. (The subscript $_2$ on B_2 is purely cosmetic.) We will always use this basis of V^*, and in so doing we are following McLean [9]. Following Bryant [4] we know that the group of linear transformations of V fixing the standard 3-form is G_2, and therefore $\pi : B_2 \to M$ (taking each such identification $u : T_m M \to V$ to the point $m \in M$) is a right G_2 principal bundle, under the action

$$r_g(u) = g^{-1}u.$$

Moreover we can define seven 1-forms on B_2 by the equation

$$\begin{pmatrix} \xi^0 \\ \vdots \\ \xi^3 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} = u\pi'.$$

We will also define quaternion valued 1-forms

$$\xi = \xi^0 + i\xi^1 + j\xi^2 + k\xi^3$$
$$\eta = i\eta^1 + j\eta^2 + k\eta^3.$$

Write $\mathbb{H}$ for the quaternions, L_q for left multiplication by q and R_q for right multiplication by q. For exterior forms α and β valued in the quaternions (or in

[3] See Joyce [8] for definitions of any of these terms.

any associative algebra), of degrees a and b respectively:

$$\begin{aligned} L_\alpha \wedge \beta &= \alpha \wedge \beta \\ R_\alpha \wedge \beta &= (-1)^{ab} \beta \wedge \alpha \\ L_\alpha \wedge L_\beta &= L_{\alpha \wedge \beta} \\ L_\alpha \wedge R_\beta &= (-1)^{ab} R_\beta \wedge L_\alpha \\ R_\alpha \wedge R_\beta &= (-1)^{ab} R_{\beta \wedge \alpha}. \end{aligned}$$

As McLean [9] explains there are unique imaginary quaternionic 1-forms λ and ρ and a unique 1-form β valued in $\operatorname{Im} \mathbb{H}^* \otimes \mathbb{H}$ (where $\operatorname{Im} \mathbb{H}^*$ means real linear 1-forms on $\operatorname{Im} \mathbb{H}$) so that

$$d \begin{pmatrix} \xi \\ \eta \end{pmatrix} = - \begin{pmatrix} L_\lambda - R_\rho & \beta \\ -{}^t\beta & L_\rho - R_\rho \end{pmatrix} \wedge \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

The meaning of ${}^t\beta$ is that we take β, valued in $\operatorname{Im} \mathbb{H}^* \otimes \mathbb{H}$ and use the invariant metric $q \mapsto q\bar{q}$ on $\mathbb{H}$ to identify $\operatorname{Im} \mathbb{H}^* \otimes \mathbb{H}$ with $\operatorname{Im} \mathbb{H} \otimes \mathbb{H}^*$. Moreover, McLean points out that we can assume that β satisfies

$$\beta_1 i + \beta_2 j + \beta_3 k = 0.^4$$

Define the connection 1-form

$$\gamma = \begin{pmatrix} L_\lambda - R_\rho & \beta \\ -{}^t\beta & L_\rho - R_\rho \end{pmatrix}$$

so that

$$d \begin{pmatrix} \xi \\ \eta \end{pmatrix} = -\gamma \wedge \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

To put it more concretely, the connection 1-form is

$$\gamma = \begin{pmatrix} 0 & -\lambda^1 + \rho^1 & -\lambda^2 + \rho^2 & -\lambda^3 + \rho^3 & \beta^0_1 & \beta^0_2 & \beta^0_3 \\ \lambda^1 - \rho^1 & 0 & -\lambda^3 - \rho^3 & \lambda^2 + \rho^2 & \beta^1_1 & \beta^1_2 & \beta^1_3 \\ \lambda^2 - \rho^2 & \lambda^3 + \rho^3 & 0 & -\lambda^1 - \rho^1 & \beta^2_1 & \beta^2_2 & \beta^2_3 \\ \lambda^3 - \rho^3 & -\lambda^2 - \rho^2 & \lambda^1 + \rho^1 & 0 & \beta^3_1 & \beta^3_2 & \beta^3_3 \\ -\beta^0_1 & -\beta^1_1 & -\beta^2_1 & -\beta^3_1 & 0 & -2\,\rho^3 & 2\,\rho^2 \\ -\beta^0_2 & -\beta^1_2 & -\beta^2_2 & -\beta^3_2 & 2\,\rho^3 & 0 & -2\,\rho^1 \\ -\beta^0_3 & -\beta^1_3 & -\beta^2_3 & -\beta^3_3 & -2\,\rho^2 & 2\,\rho^1 & 0 \end{pmatrix}$$

with

$$\begin{pmatrix} \beta^1_1 + \beta^2_2 + \beta^3_3 \\ \beta^0_1 + \beta^3_2 - \beta^2_3 \\ -\beta^3_1 + \beta^0_2 + \beta^1_3 \\ \beta^2_1 - \beta^1_2 + \beta^0_3 \end{pmatrix} = 0$$

and satisfies

$$\nabla \gamma = d\gamma + \gamma \wedge \gamma = \begin{pmatrix} L_{\nabla\lambda} - R_{\nabla\rho} & \nabla\beta \\ -\nabla^t\beta & L_{\nabla\rho} - R_{\nabla\rho} \end{pmatrix}.$$

[4] McLean actually says that $i\beta_1 + j\beta_2 + k\beta_3 = 0$, which turns out not to be true. It does not affect his results.

Plugging γ in here gives

$$\begin{aligned}\nabla\lambda &= d\lambda + \lambda\wedge\lambda - \left(\beta\wedge {}^t\beta\right)_+ \\ \nabla\rho &= d\rho + \rho\wedge\rho + \left(\beta\wedge {}^t\beta\right)_- \\ \nabla\beta &= d\beta + \left(L_\lambda - R_\rho\right)\wedge\beta + \beta\wedge\left(L_\rho - R_\rho\right)\end{aligned}$$

where, since $\beta\wedge {}^t\beta$ is a 2-form valued in $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$, we can split it into

$$\beta\wedge {}^t\beta = \left(\beta\wedge {}^t\beta\right)_+ + \left(\beta\wedge {}^t\beta\right)_- .$$

To force the curvature to live in 2-forms valued in $\mathfrak{g}_2$ we need to impose the equation

$$\nabla\beta_1 i + \nabla\beta_2 j + \nabla\beta_3 k = 0.$$

The first Bianchi identity says

$$\nabla\gamma\wedge\begin{pmatrix}\xi\\ \eta\end{pmatrix} = 0$$

which translates to

$$\begin{aligned}0 &= \nabla\lambda\wedge\xi - \xi\wedge\nabla\rho + \nabla\beta\wedge\eta \\ 0 &= -{}^t\nabla\beta\wedge\xi + \nabla\rho\wedge\eta - \eta\wedge\nabla\rho.\end{aligned}$$

3. Codimension two foliations

The group G_2 acts transitively on lines through the origin of $\mathbb{R}^7$ with stabilizer conjugate to $\mathrm{SU}(3)$, so that the 6-plane perpendicular to a fixed line is the $\mathbb{C}^3$ representation of $\mathrm{SU}(3)$; see Bryant [4] for proof. But $\mathrm{SU}(3)$ acts transitively on lines in $\mathbb{C}^3$, so therefore G_2 acts transitively on oriented 2-planes in $\mathbb{R}^7$. The stabilizer of the oriented 2-plane spanned by e_6, e_7 is $G_3 = \mathrm{Sp}(1)\times\mathrm{U}(1)/\pm 1$.

Therefore if we have an oriented codimension 2 submanifold $\Sigma^5 \subset M^7$ we can build over it the bundle $B_\Sigma \to \Sigma$ of choices of frame $u : T_m M \to \mathbb{R}^7$ so that u identifies $T_m\Sigma$ with $\mathbb{R}^5 \subset \mathbb{R}^7$. From the definition of the 1-forms ξ, η we see that $B_\Sigma \subset B_2$ satisfies the equations $\eta^2 = \eta^3 = 0$. The Lie algebra $\mathfrak{g}_3$ of the stabilizer of the 2-plane is given in terms of matrices by setting $\beta = 0$ and making ρ sit in the imaginary complex numbers inside the quaternions; the reader can see this directly from the connection 1-form matrix. The same breaking of structure group happens if we pick a foliation F of M by oriented codimension 2 submanifolds. Analogously define the bundle $B_3 \subset B_2$ as a principal right G_3 bundle, the bundle of linear maps $u : T_m M \to V$ from B_2 identifying the 2-plane perpendicular to the leaf of the foliation through m with the oriented 2-plane spanned by e_6, e_7. On B_3 we have the structure equations:

$$d\begin{pmatrix}\xi\\ \eta\end{pmatrix} = -\begin{pmatrix}L_\lambda - R_{i\rho_1} & 0\\ 0 & L_{i\rho_1} - R_{i\rho_1}\end{pmatrix}\wedge\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}$$

where the torsion is

$$\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix} = \begin{pmatrix}-\xi\wedge\left(\rho^2 j + \rho^3 k\right) - \beta\wedge\eta \\ {}^t\beta\wedge\xi - \left(\rho^2 j + \rho^3 k\right)\wedge\eta - \eta\wedge\left(\rho^2 j + \rho^3 k\right)\end{pmatrix}.$$

Here β, ρ^2 and ρ^3 have become multiples of the components of ξ and η, giving 70 components of the torsion. One could write out the torsion completely, divide it into representations of G_3, and do the same for the curvature.

To find the representations, use the following observations: first, that under the action of the group $\operatorname{Sp}(1) \times \operatorname{Sp}(1) / \pm 1 \subset G_2$, whose elements we can label as

$$g = \begin{pmatrix} L_\ell R_r^{-1} & 0 \\ 0 & L_r R_r^{-1} \end{pmatrix}$$

with ℓ and r unit quaternions, the right action of this subgroup of the structure group satisfies

$$\begin{aligned} r_g^* \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= g^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \\ &= \begin{pmatrix} \ell^{-1} \xi r \\ r^{-1} \eta r \end{pmatrix} \end{aligned}$$

while the action on the connection 1-form is

$$r_g^* \gamma = Ad_g^{-1} \gamma$$

which we compute out explicitly (using the identities for quaternionic differential forms) to give

$$\begin{aligned} r_g^* \lambda &= \ell^{-1} \lambda \ell \\ r_g^* \rho &= r^{-1} \rho r \end{aligned}$$

(naturally the product action).

If we have a foliation of M^7 with unoriented 5 dimensional leaves, we can make a 2-1 cover $\hat{M} \to M$ consisting of the choices of points of M together with local orientations of the leaves, and then $\hat{M}$ will be a G_2 manifold as well, with isomorphic Lie algebra of local isometries, so without loss of generality we take our foliations to be oriented.

4. Cohomogeneity two

Suppose that H is a Lie group acting on M preserving the G_2 structure, with generic orbit of codimension 2. Without loss of generality we can assume that all orbits have codimension 2 and that they are oriented. These orbits constitute a foliation F by oriented codimension 2 submanifolds and we construct the principal G_3 subbundle B_3. The group H acts on the bundle B_3 by the canonical prolongation of the action from M so that G_3 commutes with H acting on B_3. The components of the torsion and curvature on the bundle B_3 will be constant along the group orbits in B_3. The action of G_3 will permute the group orbits in B_3. Since the group orbits in B_3 project to codimension two submanifolds in M they must satisfy $\eta^2 = \eta^3 = 0$ and $\xi^0 \wedge \cdots \wedge \xi^3 \wedge \eta^1 \neq 0$. But there could be relations on each orbit in B_3 among the differentials $\lambda^1, \ldots, \lambda^3, \rho^1$ (which are elements of the connection 1-form) modulo semibasic 1-forms. So the orbits of H on B_3 must have dimension between 5 and 9. But since they bear an invariant coframing, they must be locally isomorphic to the group itself, i.e. locally diffeomorphic, with the coframing being a constant combination of the components of the Maurer–Cartan 1-form on the group. Only a discrete subgroup of H can fix a point of B_3. We summarize the relations among our 1-forms and functions as follows (where $\cong$ means vanishing modulo semibasic 1-forms, i.e. vanishing on the fiber of $B_3 \to M$):

Case: (0)
Structure group: $G_3 = \mathrm{Sp}(1) \times \mathrm{U}(1) / \pm 1$
Fiber:
$0 = \xi = \eta = \rho^2 = \rho^3 = \beta$
$0 \neq \lambda^1 \wedge \lambda^2 \wedge \lambda^3 \wedge \rho^1$
Symmetry group orbit in bundle:

$0 \neq \xi^0 \wedge \xi^1 \wedge \xi^2 \wedge \xi^3 \wedge \eta^1$
Bundle:
$0 \cong \rho^2 \cong \rho^3 \cong \beta$
$0 \neq \xi^0 \wedge \xi^1 \wedge \xi^2 \wedge \xi^3 \wedge \eta^1 \wedge \eta^2 \wedge \eta^3 \wedge \lambda^1 \wedge \lambda^2 \wedge \lambda^3 \wedge \rho^1$

We will construct an invariant principal subbundle of B_3 as follows: we look for all of the H orbits in B_3 on which the relations among the elements of the connection 1-form sit in a given normal form. These normal forms will have to be worked out below. Because the connection 1-form varies in the adjoint representation, we will find that this determines a principal subbundle.

Then we can apply the usual normalization of torsion and reduction of structure group applied in the equivalence method. Frequently we refer to "taking derivatives". By this apparently innocent expression, we mean that we write down all of the equations we have so far, take exterior derivative, and then plug in, for each scalar invariant say I, and basis of 1-forms ω^α, equations like $dI = I_\alpha \omega^\alpha$. When we talk about taking four derivatives, we mean that we repeat this process four times. We will present a detailed description of the method next when we study nine dimensional symmetry groups. After that, we will present abbreviated discussions of the remaining cases. It is crucial that we have assumed in each case vanishing of the Ricci curvature. This prevents obvious counterexamples, such as products $S^2 \times \mathbb{R}^5$, which have holonomy inside G_2 and are not flat, but are also not Ricci flat.

4.1. **Nine dimensional groups.** Since the bundle B_3 is 11 dimensional and bears an invariant coframing ξ, η, λ, ρ we find that each orbit in B_3 of the symmetry group H is nine dimensional precisely when the group H is. We can take $\xi, \eta, \lambda, \rho^1$ as a coframing on the orbits of H in B_3. Taking exterior derivatives of the structure equations and using the independence of the elements of this coframing, we obtain the following structure equations on H:

$$d \begin{pmatrix} \xi^0 \\ \xi^1 \\ \xi^2 \\ \xi^3 \\ \eta^1 \\ \lambda^1 \\ \lambda^2 \\ \lambda^3 \\ \rho^1 \end{pmatrix} = \begin{pmatrix} \left(\lambda^1 - \rho^1 + s\eta^1\right) \wedge \xi^1 + \lambda^2 \wedge \xi^2 + \lambda^3 \wedge \xi^3 \\ -\left(\lambda^1 - \rho^1 + s\eta^1\right) \wedge \xi^0 + \lambda^3 \wedge \xi^2 - \lambda^2 \wedge \xi^3 \\ -\lambda^2 \wedge \xi^0 - \lambda^3 \wedge \xi^1 + \left(\lambda^1 + \rho^1 - s\eta^1\right) \wedge \xi^3 \\ -\lambda^3 \wedge \xi^0 + \lambda^2 \wedge \xi^1 - \left(\lambda^1 + \rho^1 - s\eta^1\right) \wedge \xi^2 \\ -2s\left(\xi^0 \wedge \xi^1 - \xi^2 \wedge \xi^3\right) \\ -2\lambda^2 \wedge \lambda^3 + t\left(\xi^0 \wedge \xi^1 + \xi^2 \wedge \xi^3\right) \\ 2\lambda^1 \wedge \lambda^3 + t\left(\xi^0 \wedge \xi^2 - \xi^1 \wedge \xi^3\right) \\ -2\lambda^1 \wedge \lambda^2 + t\left(\xi^1 \wedge \xi^2 + \xi^0 \wedge \xi^3\right) \\ -\left(3t + 2s^2\right)\left(\xi^0 \wedge \xi^1 - \xi^2 \wedge \xi^3\right) \end{pmatrix}.$$

We are inclined to replace ρ^1 by $\epsilon = \rho^1 - s\eta^1$ for which

$$d\epsilon = -3t\left(\xi^0 \wedge \xi^1 - \xi^2 \wedge \xi^3\right).$$

In particular notice that $2s\epsilon - 3t\eta^1$ is closed. If $t \neq 0$ then we can replace η^1 by $\nu = \eta^1 - (2s/3t)\epsilon$ and then the s invariant no longer appears in the structure equations. Moreover, we can replace ξ by $\alpha = \xi/\sqrt{|t|}$ and rescale ν appropriately, to find that neither t nor s now appear in the structure equations. Therefore for t nonzero of a given sign, the group has a fixed isomorphism type. If $t = 0$ and $s \neq 0$ then we can replace η^1 by $\mu = \eta^1/s$ so that s no longer appears in the structure equations. Therefore there are at most 5 Lie algebras of dimension 9 which can act as symmetries of a torsion free G_2 manifold with generic orbit of codimension 2.

The group is locally identified with the orbit, and the vector fields dual to the 1-forms $\xi^I, \eta^1, \lambda^a, \epsilon$ form the Lie algebra. Let $\mathfrak{a}$ be the subalgebra cut out by the equations $\lambda^2 = \lambda^3 = \xi = 0$. This is clearly Abelian since if we plug in the equations defining $\mathfrak{a}$ we find that all exterior derivatives of our coframe elements vanish. Consider the centralizer $Z(\mathfrak{a})$. This consists precisely in the vectors w so that $d\omega(v, w) = 0$ for all $v \in \mathfrak{a}$ and ω any 1-form from the coframing. One checks easily that $Z(\mathfrak{a}) = \mathfrak{a}$ so that $\mathfrak{a}$ is maximal Abelian.

It is clear from the structure equations that the equations $(\xi = \lambda = \epsilon = 0)$ define a one dimensional normal subgroup. We can quotient out by that subgroup, and find that on the quotient, call it H', the 1-forms ξ, λ, ϵ are still defined, and form a coframing, while η^1 is no longer defined on H'.

On $\mathfrak{h}'$ the Killing form

$$K(x, y) = -\frac{1}{8}\operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y)$$

turns out to have eigenvalues (with indicated multiplicity)

$$-\frac{3t}{2}\ (4), \frac{3}{2}\ (3), \frac{1}{2}\ (1).$$

Therefore this group is semisimple precisely if $t \neq 0$, and for $t < 0$ it is compact, while for $t > 0$ it is not compact. Let us consider the complexified group $H'_{\mathbb{C}}$ (or at least its Lie algebra $\mathfrak{h}'_{\mathbb{C}}$), and for the moment assume that $t \neq 0$, so that $\mathfrak{h}'_{\mathbb{C}}$ is a complex semisimple Lie group. Take

$$\begin{aligned} \omega^1 &= \xi^0 + \sqrt{-1}\xi^1 \\ \omega^2 &= \xi^2 + \sqrt{-1}\xi^3 \\ \omega^3 &= \lambda^2 + \sqrt{-1}\lambda^3. \end{aligned}$$

Then the structure equations become

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \lambda^1 \\ \epsilon \end{pmatrix} = \begin{pmatrix} -\sqrt{-1}\left(\lambda^1 - \epsilon\right) \wedge \omega^1 - \omega^{\bar{2}} \wedge \omega^3 \\ -\sqrt{-1}\left(\lambda^1 + \epsilon\right) \wedge \omega^2 + \omega^{\bar{1}} \wedge \omega^3 \\ -2\sqrt{-1}\lambda^1 \wedge \omega^3 + t\omega^1 \wedge \omega^2 \\ -\sqrt{-1}\omega^3 \wedge \omega^{\bar{3}} + \frac{t\sqrt{-1}}{2}\left(\omega^1 \wedge \omega^{\bar{1}} + \omega^2 \wedge \omega^{\bar{2}}\right) \\ -\frac{3t\sqrt{-1}}{2}\left(\omega^1 \wedge \omega^{\bar{1}} - \omega^2 \wedge \omega^{\bar{2}}\right) \end{pmatrix}.$$

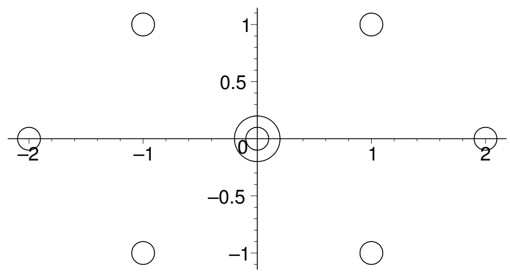

FIGURE 1. The root diagram of the nine dimensional symmetry group

To calculate roots, take the vector valued 1-form

$$\Omega = \begin{pmatrix} \omega^1 \\ \omega^{\bar{1}} \\ \omega^2 \\ \omega^{\bar{2}} \\ \omega^3 \\ \omega^{\bar{3}} \\ \lambda^1 \\ \epsilon \end{pmatrix}$$

(which is a coframing on the complexified Lie group) and look for vectors satisfying

$$-\lambda^1 \lrcorner\, d\Omega = \sqrt{-1}\Lambda_1 \Omega$$
$$-\epsilon \lrcorner\, d\Omega = \sqrt{-1}\Lambda_2 \epsilon$$

for whatever constants Λ_1, Λ_2. One finds that solutions exist for (Λ_1, Λ_2) among the points labelled in figure 1, with multiplicities as indicated. This is precisely the root diagram of $\mathfrak{sl}(3,\mathbb{C})$, so that $\mathfrak{h}'_{\mathbb{C}} = \mathfrak{sl}(3,\mathbb{C})$. The only real forms of $\mathfrak{sl}(3,\mathbb{C})$ are $\mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{su}(3)$ and $\mathfrak{su}(2,1)$. But $\mathfrak{su}(2,1)$ has rank one, while the others have rank two, and therefore $\mathfrak{h}'$ is one of $\mathfrak{sl}(3,\mathbb{R})$ or $\mathfrak{su}(3)$.

Now if $t \neq 0$ then we can subtract off $(2s/3t)\epsilon$ from η^1 so that the full group H has Lie algebra $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}$ if $t > 0$ and $\mathfrak{su}(3) \oplus \mathbb{R}$ if $t < 0$.

The analysis of $t = 0$ is more subtle. Plugging in $t = 0$ we see that there is a morphism $\mathfrak{h}' \to \mathfrak{su}(2)_\lambda \oplus \mathbb{R}_\epsilon$. In the kernel of this morphism, we have $\mathfrak{h}'$ a semidirect product. But the Lie algebra $\mathfrak{h}$ might still be complicated. If $s = 0$ then $\mathfrak{h}$ is itself a semidirect product $\mathbb{R}_{\eta^1} \oplus (\mathfrak{su}(2)_\lambda \oplus \mathbb{R}_\epsilon) \ltimes \mathbb{R}^4_\xi$. The action of $\mathfrak{su}(2)$ on $\mathbb{R}^4_\xi = \mathbb{H}$ is just left multiplication by quaternions, and the action of ϵ is a rotation on the first two components of ξ and rotation backwards by the same angle on the last two.

Finally, one can compute the consequences of the relations uncovered so far on the invariants on the bundle B_3. It turns out that all invariants are forced to vanish (one sees this by take two derivatives of all of the 1-forms we have seen so far); consequently

Lemma 1. *The only torsion free G_2 structure on a connected and simply connected manifold acted on by a nine dimensional Lie group with generic orbit of codimension 2, which does not sit inside a larger such manifold, is the flat G_2 structure on $\mathbb{R}^7$. The group is unique up to conjugacy.*

4.2. **Eight dimensional groups.** Once again we imagine that we are given an eight dimensional group H acting as isomorphisms of a torsion free G_2 structure B_2 on a connected seven dimensional manifold M, with orbits of H in M having codimension 2, i.e. dimension 5. We construct the G_3 subbundle B_3. The orbits of H inside B_3 must satisfy some equations of the form

$$0 = \eta^2 = \eta^3 = Y_b \lambda^b - Z\rho^1 \pmod{\xi^J, \eta^1}$$

with not all of the Y_b and Z vanishing. By analyticity of torsion free G_2 structures (see Bryant [4]) either some Y_b is not zero on a dense open subset of B_3, or else they are all zero everywhere.

4.2.1. *Eight dimensional groups (I).* Suppose that one of the Y_b is not zero on a dense open subset of B_3. Note that the Y_b are constant on H orbits. Now get the structure group G_3 to permute the H orbits. It will thus permute their tangent spaces, and it will make the Y_b vary in the coadjoint representation, since these Y_b are dual to the λ^b. We can move up the fibers of B_1 until we enter an orbit of H on which $Y_1 > 0$ and $Y_2 = Y_3 = 0$. Rescaling the Y we find that on such an orbit

$$0 = \eta^2 = \eta^3 = \lambda^1 - Z\rho^1 \pmod{\xi^J, \eta^1}$$

for a well defined function Z. We may assume that this takes place on some H orbit over every point of M, by replacing M by a dense open subset. The subset B_4 of B_3 made up of these H orbits is a principal $G_4 = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathbb{Z}_2$ subbundle over M, because we have now restricted motion in the λ directions to fix λ^1. But H must preserve the subbundle B_4, and therefore H must have orbits in B_3 of at most 7 dimensions, and thus the dimension of H is at most 7.

4.2.2. *Eight dimensional groups (II).* Suppose that all Y_b vanish so that on every H orbit,

$$0 = \eta^2 = \eta^3 = \rho^1 \pmod{\xi^J, \eta^1}.$$

After taking account of the requirement that $d^2 = 0$, applied to the coframing $\xi, \eta, \lambda, \rho^1$, the structure equations on the bundle B_3 simplify to

$$d \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \rho_1 \end{pmatrix} = \begin{pmatrix} \lambda_2 \wedge \xi_2 + \lambda_1 \wedge \xi_1 - \rho_1 \wedge \xi_1 + \lambda_3 \wedge \xi_3 \\ -\lambda_1 \wedge \xi_0 - \lambda_2 \wedge \xi_3 + \rho_1 \wedge \xi_0 + \lambda_3 \wedge \xi_2 \\ -\lambda_3 \wedge \xi_1 - \lambda_2 \wedge \xi_0 + \lambda_1 \wedge \xi_3 + \rho_1 \wedge \xi_3 \\ -\lambda_3 \wedge \xi_0 + \lambda_2 \wedge \xi_1 - \rho_1 \wedge \xi_2 - \lambda_1 \wedge \xi_2 \\ 0 \\ 2\rho_1 \wedge \eta_3 \\ -2\rho_1 \wedge \eta_2 \\ 2\lambda_3 \wedge \lambda_2 \\ -2\lambda_3 \wedge \lambda_1 \\ 2\lambda_2 \wedge \lambda_1 \\ 0 \end{pmatrix}.$$

Again there are no curvature terms, and no terms representing the second fundamental form of the orbits, so that

Proposition 1. *The only torsion-free G_2 structure acted on by an eight dimensional group of isometries is the flat G_2 structure, and the group is unique up to conjugacy; it is the group generated by translations parallel to a given copy of $\mathbb{R}^5$, and rotations from a copy of* $\mathrm{SU}(2)$ *living inside G_2.*

4.3. **Seven dimensional groups.** Consider next a 7 dimensional group H acting on M with codimension 2 orbits, and on B_3 with 7 dimensional orbits. Each orbit then satisfies some pair of independent equations

$$0 = a_i \lambda^i + b\rho^1$$
$$0 = f_i \lambda^i + g\rho^1$$

modulo semibasic 1-forms. As we move around inside B_3 with its structure group G_3 we change these coefficients by the adjoint representation, so that we can easily arrange one of

$$\left.\begin{aligned} \lambda^2 &= 0 \\ \lambda^3 &= 0 \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} \rho^1 &= A_i \lambda^i \\ \lambda^1 &= 0 \end{aligned}\right\} \qquad (2)$$

modulo semibasic 1-forms ξ, η. In each case we can let B_4 be the subbundle of B_3 on which these equations hold (for some functions A_i as the case may be). This B_4 is then a principal right $G_4 = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathbb{Z}_2$ bundle. But in the case 2, the 1-form λ^1 must be semibasic on each orbit, while λ^2, λ^3 are semibasic on B_4, so the group H must have dimension no more than 6. Therefore we may restrict attention to case 1.

If we now focus attention on B_4 we find that all scalar invariants T must be constant along H orbits in B_4 and therefore their differentials can be expressed as $dT = T_2 \eta^2 + T_3 \eta^3$. Taking exterior derivatives of all of our equations we find the following:

$$d \begin{pmatrix} \xi^0 \\ \xi^1 \\ \xi^2 \\ \xi^3 \\ \eta^1 \\ \eta^2 \\ \eta^3 \\ \lambda^1 \\ \rho^1 \end{pmatrix} = \begin{pmatrix} \left(\lambda^1 - \rho^1\right) \wedge \xi^1 \\ -\left(\lambda^1 - \rho^1\right) \wedge \xi^0 \\ \left(\lambda^1 + \rho^1\right) \wedge \xi^3 \\ -\left(\lambda + \rho^1\right) \wedge \xi^2 \\ 0 \\ 2\rho^1 \wedge \eta^3 \\ -2\rho^1 \wedge \eta^2 \\ 0 \\ 0 \end{pmatrix}.$$

Again, we see no curvature or second fundamental form.

Proposition 2. *Every torsion-free G_2 structure which is acted on by a group of equivalences of dimension at least seven is flat, and the orbits are flat parallel affine subspaces, and the group is unique up to conjugacy.*

4.4. **Six dimensional groups.** Suppose that H is six dimensional. Then it has six dimensional orbits in B_3 and 5 dimensional orbits in M, and in B_3 its orbits satisfy $\eta^2 = \eta^3 = 0$. But B_3 has 11 dimensions, so there must be an additional 3 equations on the connection 1-forms satisfied on each orbit. Two cases emerge: (1) generically these equations allow one to solve for λ in terms of ρ^1, say

$$\lambda^a \cong A^a \rho^1$$

(modulo semibasic 1-forms; we will write $\cong$ to mean equality modulo semibasic 1-forms). Then we can use the adjoint action of the structure group G_3 to force the equations into the form

$$\rho^1 \cong Z\lambda^1, \lambda^2 \cong \lambda^3 \cong 0.$$

Alternatively: (2) we have $\lambda \cong 0$ on each orbit.

Consider case (1). We can acheive this form for our equations on a subbundle $B_4 \subset B_3$ with structure group $G_4 = \mathrm{U}(1) \times \mathrm{U}(1) / \pm 1$. Therefore on B_4 we find that $\rho^2, \rho^3, \beta, \lambda^2$ and λ^3 are semibasic and every scalar invariant T has differential

$$dT = T' \left(\rho^1 - Z\lambda^1\right) + \nabla T_2 \eta^2 + \nabla T_3 \eta^3.$$

and we can summarize the equations of case (1) in the table:

Case: (1)
Structure group: $\mathrm{U}(1) \times \mathrm{U}(1) / \pm 1$
Fiber:
$0 = \xi = \eta = \lambda^2 = \lambda^3 = \rho^2 = \rho^3 = \beta$
$0 \neq \lambda^1 \wedge \rho^1$
Symmetry group orbit in bundle:
$0 = \eta^2 = \eta^3, 0 \cong \rho^1 - Z\lambda^1 \cong \lambda^2 \cong \lambda^3 \cong \rho^2 \cong \rho^3 \cong \beta$
$0 \neq \xi^0 \wedge \xi^1 \wedge \xi^2 \wedge \xi^3 \wedge \eta^1 \wedge \lambda^1$
Bundle:
$0 \cong \lambda^2 \cong \lambda^3 \cong \rho^2 \cong \rho^3 \cong \beta$
$0 \neq \xi^0 \wedge \xi^1 \wedge \xi^2 \wedge \xi^3 \wedge \eta^1 \wedge \eta^2 \wedge \eta^3 \wedge \lambda^1 \wedge \rho^1$

In case (2), on the bundle B_3 we find ρ^2, ρ^3, β are semibasic and that every scalar invariant T has differential

$$dT = T_a \lambda^a + \nabla T_2 \eta^2 + \nabla T_3 \eta^3.$$

The equations look like

Case: (2)
Structure group: $G_3 = \mathrm{Sp}(1) \times \mathrm{U}(1) / \pm 1$
Fiber:
$0 = \xi = \eta = \rho^2 = \rho^3 = \beta$
$0 \neq \lambda^1 \wedge \lambda^2 \wedge \lambda^3 \wedge \rho^1$
Symmetry group orbit in bundle:
$0 = \eta^2 = \eta^3, 0 \cong \lambda \cong \rho^2 \cong \rho^3 \cong \beta$
$0 \neq \xi^0 \wedge \xi^1 \wedge \xi^2 \wedge \xi^3 \wedge \eta^1 \wedge \rho^1$
Bundle:
$0 \cong \rho^2 \cong \rho^3 \cong \beta$
$0 \neq \xi^0 \wedge \xi^1 \wedge \xi^2 \wedge \xi^3 \wedge \eta^1 \wedge \eta^2 \wedge \eta^3 \wedge \lambda^1 \wedge \lambda^2 \wedge \lambda^3 \wedge \rho^1$

4.4.1. *Six dimensional groups I.* Consider case (1). On each orbit in the bundle we have

$$0 = \eta^2 = \eta^3 = \rho^1 - Z\lambda^1 - W_J \xi^J - R\eta^1.$$

But under the action of the structure group, which is $\mathrm{U}(1) \times \mathrm{U}(1) / \pm 1$, the W_J coefficients vary in a nontrivial representation of the first $\mathrm{U}(1)$ factor. If these W coefficients are not zero, then we can use this representation to find a subbundle on which they are in a normal form. But then on that subbundle, λ^1 will be semibasic, and so that subbundle is not invariant under the H action. This contradicts the assumption that the geometry is invariant under the H action. Therefore $W = 0$ everywhere, and on orbits of H in our bundle:

$$0 = \eta^2 = \eta^3 = \rho^1 - Z\lambda^1 - R\eta^1.$$

Now suppose that $Z \neq 0$. Then every H invariant function I on the bundle satisfies

$$dI = I_1 \left(\rho^1 - Z\lambda^1 - R\eta^1\right) + I_2\eta^2 + I_3\eta^3.$$

But now if $I_1 \neq 0$ we find that I varies in the λ^1 direction and therefore I is not constant on the group orbits. This is a contradiction, and so $I_1 = 0$ everywhere, i.e. every H invariant function I on the bundle must have differential $dI = I_2\eta^2 + I_3\eta^3$, so is pulled back from the surface M^7/H^5. However, if I is H invariant, then so is dI and therefore also I_2 and I_3, since η^2 and η^3 are H invariant. Calculating the exterior derivative of the equation $dI = I_2\eta^2 + I_3\eta^3$ we find that

$$d\begin{pmatrix} I_2 \\ I_3 \end{pmatrix} \cong \begin{pmatrix} 0 & -2\rho^1 \\ 2\rho^1 & 0 \end{pmatrix} \begin{pmatrix} I_2 \\ I_3 \end{pmatrix}$$

so that these dI_2 and dI_3 do not have the form $dJ = J_2\eta^2 + J_3\eta^3$ which we showed above they must have, unless $I_2 = I_3 = 0$. So finally we have shown that H invariant functions on the bundle must be constant unlesss $Z = 0$. But any function on M/H pulls back to an H invariant function, so we must have $Z = 0$ everywhere on our bundle.

At this point, we carry out a long calculation. We will not give the details, since it consists merely in differentiating the above equations four times, and long but elementary algebra. The result: all invariants vanish (including the curvature, and the second fundamental form of each leaf), and the structure equations become

$$\begin{aligned} d\xi &= -\lambda^1 i \wedge \xi - \xi \wedge \rho^1 i \\ d\eta &= 2\rho^1 i \wedge \eta \\ d\lambda^1 &= 0 \\ d\rho^1 &= 0. \end{aligned}$$

4.4.2. *Six dimensional groups II.* Consider case (2). After some calculation, one finds the structure equations

$$\begin{aligned} d\begin{pmatrix} \xi \\ \eta \end{pmatrix} &= -\begin{pmatrix} L_\lambda - R_\rho & 0 \\ 0 & L_\rho - R_\rho \end{pmatrix} \\ \rho &= \rho^1 i \\ d\lambda &= -\lambda \wedge \lambda \\ d\rho^1 &= 0. \end{aligned}$$

Once again the G_2 manifold is flat, and the group acting is unique up to conjugacy inside the isometry group of the flat G_2 manifold. We can pick any constants Y_p, $p = 1 \dots 3$ so that on the H orbits in B_3 we have $\lambda = Y\eta^1$. The group orbits in the G_2 manifold M are flat parallel affine subspaces.

Proposition 3. *Every torsion-free G_2 structure which is acted on by a group of equivalences of dimension at least six is flat, and the orbits are flat parallel affine subspaces, and the group is unique up to conjugacy, except if it is six dimensional.*

4.5. **Five dimensional groups.** Suppose that H is five dimensional. Then the manifold looks locally like a fiber bundle. If it is globally a fiber bundle, the base surface is equipped with various fields, including a gauge field, which are tied up by an elaborate system of coupled differential equations and inequalities.

On each H orbit we must have λ and ρ^1 semibasic, and $\eta^2 = \eta^3 = 0$. Therefore any scalar invariant T on B_3 must satisfy

$$dT = T_0\rho^1 + T_a\lambda^a + \nabla T_2\eta^2 + \nabla T_3\eta^3.$$

We still have β, ρ^2 and ρ^3 being semibasic. After an arduous calculation, taking these equations and differentiating them four times, we obtain vanishing of all of the invariants, and flatness.

References

1. Bobby Acharya and Edward Witten, *Chiral fermions from manifolds of G_2 holonomy*, hep-th/0109152. 1
2. Michael F. Atiyah and Edward Witten, *M-theory dynamics on a manifold of G_2 holonomy*, hep-th/0107177. 1
3. Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR **88f:**53087 1, 2, 3
4. Robert L. Bryant, *Metrics with exceptional holonomy*, Annals of Mathematics **126** (1987), 525–576. 3, 5, 10
5. Richard Cleyton and Andrew Swann, *Cohomogeneity-one G_2-structures*, math.DG/0111056, 2001. 2
6. M. J. Duff (ed.), *The world in eleven dimensions: supergravity, supermembranes and M-theory*, Studies in High Energy Physics Cosmology and Gravitation, IOP Publishing Ltd., Bristol, 1999. MR **2001b:**81002 1
7. Nigel Hitchin, *Stable forms and special metrics*, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), Amer. Math. Soc., Providence, RI, 2001, pp. 70–89. MR 1 871 001 3
8. Dominic D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. MR **2001k:**53093 1, 3
9. Robert C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), no. 4, 705–747. MR **99j:**53083 3, 4
10. Joseph Polchinski, *String theory. Vol. II*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1998, Superstring theory and beyond. MR **99h:**81184 1

University of South Florida, Saint Petersburg, Florida
E-mail address: bwmckay@stpt.usf.edu