COMPLETE PROJECTIVE CONNECTIONS

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Abstract. The first examples of complete projective connections are uncovered: on surfaces, normal projective connections whose geodesics are all closed and embedded are complete. On manifolds of any dimension, normal projective connections induced from complete affine connections with slowly decaying positive Ricci curvature are complete.

Contents

1. Introduction

This article is a step toward global analysis of Cartan geometries; a new avenue of research, where almost nothing is known. Completeness of projective connections is subtle, even on compact manifolds, and there seems to be no easy way to decide whether a projective connection is complete. In my recent work [21], I discovered that complete complex projective connections are flat, and I decided to look for

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complete real projective connections which are not flat. I was surprised to find that none were known; in this article you will find the first examples.[1]

Definitions are presented later, but for the moment recall that every Riemannian manifold has a distinguished projective connection, and that this imposes on every geodesic a natural choice of parameterization, well defined modulo projective transformations. This parameterization is *not* the arc length parameterization, in many examples, but it is unchanged if we change metric, as long as we keep the same geodesics. For example, in real projective spaces $\mathbb{P}^n$, the projective parameterization is the obvious parameterization: the geodesics are projective lines. But therefore in affine space, thought of as an affine chart of $\mathbb{P}^n$ (which gives it the usual straight lines as geodesics), following the projective parameterization can run us off to infinity in finite time. This is the bizarre incompleteness of affine space. Worse: since a flat torus is a quotient of affine space, it is also incomplete! We wrap around a geodesic infinitely often in finite time. The only complete examples known (before the results below) were the sphere and projective space (with standard metrics).

The concept of completeness of Cartan geometries is tricky to define, raised explicitly for the first time by Ehresmann [8] (also see Ehresmann [9, 10], Kobayashi [14], Kobayashi & Nagano [17], Clifton [7], Bates [1]), and plays a central role in Sharpe's book [24], but is also clearly visible beneath the surface in numerous works of Cartan. Roughly speaking, completeness concerns the ability to compare a geometry to some notion of flat geometry, by rolling along curves. There were no examples of complete projective connections except for the sphere and projective space (which are both flat) until now:

Theorem 1. *Every normal projective connection on a surface, all of whose geodesics are closed embedded curves, is complete.*

Theorem 2. *Every complete torsion-free affine connection with positive Ricci curvature decaying slower than quadratically induces a complete normal projective connection.*

The first theorem is more exciting, since it is purely global and depends directly on the projective connection.

Example 1. The projective connection on the product $S^n \times S^n$ of round spheres is projectively complete for $n > 1$. However, it is projectively incomplete for $n = 1$, since $S^1 \times S^1$ is the torus. But $S^1 \times S^1 \subset S^n \times S^n$ is a totally geodesic submanifold: a totally geodesic submanifold can have a different projective parameterization from the projective parameterization associated to the ambient manifold.

Analysis of projective connections is difficult because already on the simplest example, projective space, the automorphism group is not compact, a kind of inherent slipperiness. A sort of antithesis of completeness is known as *projective hyperbolicity*; see Kobayashi [15] and Wu [27] for examples of projective hyperbolicity.

[1]More recently I have discovered that Tanaka [25] p. 21 announced in a remark that he could prove that Einstein metrics of positive Ricci curvature on compact manifolds are projectively complete, although the proof did not appear. He actually says negative Ricci curvature, but must clearly mean positive Ricci curvature. It seems likely that the intended proof is the same as mine. I have also discovered that Blumenthal [2] proved that completeness of various Cartan geometries is preserved under submersions, but this did not generate examples other than the standard Hopf fibrations.

Élie Cartan [5] introduced the notion of projective connection; Kobayashi & Nagano [17], Gunning [12] and Borel [3] provide a contemporary review; we will use the definitions of Kobayashi & Nagano. This article may be difficult to follow without the article of Kobayashi & Nagano in hand.

2. The flat example: projective space

First, let us consider projective space $\mathbb{P}^n = \left(\mathbb{R}^{n+1} \backslash 0\right)/\mathbb{R}^{\times}$. Projective space is glued together out of affine charts, and the transition functions are affine transformations, so preserve straight lines, i.e. geodesics. The geodesic-preserving transformations of projective space are precisely the projective linear transformations, forming the group $\mathbb{P}\,\mathrm{GL}\,(n+1,\mathbb{R})$ (a well-known result in geometry due to David Hilbert [13]).

We will think of $\mathbb{P}^n$ as the space of tuples

$$\begin{pmatrix} x^0 \\ \vdots \\ x^n \end{pmatrix}$$

of numbers, not all zero, modulo rescaling. Write the corresponding point of $\mathbb{P}^n$ as

$$\begin{bmatrix} x^0 \\ \vdots \\ x^n \end{bmatrix}.$$

$\mathbb{P}^n$ is acted on transitively by the group $G = \mathbb{P}\,\mathrm{GL}\,(n+1,\mathbb{R})$ of projective linear transformations, i.e. linear transformations of the x variables modulo rescaling. We will write $[g]$ for the element of $\mathbb{P}\,\mathrm{GL}\,(n+1,\mathbb{R})$ determined by an element $g \in \mathrm{GL}\,(n+1,\mathbb{R})$. The stabilizer of the point

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the group G_0 consisting of $[g]$ where g is a matrix of the form

$$[g] = \begin{bmatrix} g^0_0 & g^0_j \\ 0 & g^i_j \end{bmatrix}$$

where $i, j = 1, \dots, n$. The Lie algebra of $\mathbb{P}\,\mathrm{GL}\,(n+1,\mathbb{R})$ is just $\mathfrak{sl}\,(n+1,\mathbb{R})$, so consists of the matrices of the form

$$\begin{pmatrix} A^0_0 & A^0_j \\ A^i_0 & A^i_j \end{pmatrix}$$

with $A^0_0 + A^i_i = 0$. We define the Maurer–Cartan 1-form $\Omega \in \Omega^1\left(\mathbb{P}\,\mathrm{GL}\,(n+1,\mathbb{R})\right) \otimes \mathfrak{sl}\,(n+1,\mathbb{R})$ by $\Omega = g^{-1}\,dg$. This form satisfies $d\Omega = -\Omega \wedge \Omega$. Splitting into components, we calculate

$$\begin{aligned} d\Omega^i_0 &= -\left(\Omega^i_j + \delta^i_j \Omega^k_k\right) \wedge \Omega^j_0 \\ d\Omega^i_j &= -\Omega^i_k \wedge \Omega^k_j + \Omega^0_j \wedge \Omega^i_0 \\ d\Omega^0_i &= \left(\Omega^j_i + \delta^j_i \Omega^k_k\right) \wedge \Omega^0_j \end{aligned}$$

Following Cartan [6] we let $\omega^i = \Omega^i_0, \gamma^i_j = \Omega^i_j + \delta^i_j \Omega^k_k$, and $\Omega_i = \Omega^0_i$ then we find

$$\begin{aligned} d\omega^i &= -\gamma^i_j \wedge \omega^j \\ d\gamma^i_j &= -\gamma^i_k \wedge \gamma^k_j + \left(\omega_j \delta^i_k + \omega_k \delta^i_j\right) \wedge \omega^k \\ d\omega_i &= \gamma^j_i \wedge \omega_j . \end{aligned}$$

The group G_0 is a semidirect product: each element factors into two elements of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} .$$

It will be helpful later to see how each of these factors acts on our differential forms. This is not difficult, since the form $\Omega = g^{-1}\, dg$ satisfies

$$r^*_{g_0} \Omega = \operatorname{Ad}^{-1}_{g_0} \Omega,$$

for $g_0 \in G_0$ We leave to the reader to calculate that if we write g for the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}$$

and λ for the matrix

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$$

then

$$\begin{aligned} r^*_g \omega^i &= \left(g^{-1}\right)^i_j \omega^j \\ r^*_g \gamma^i_j &= \left(g^{-1}\right)^i_k \gamma^k_l g^l_j \\ r^*_g \omega_i &= \omega_j g^j_i \\ r^*_\lambda \omega^i &= \omega^i \\ r^*_\lambda \gamma^i_j &= \gamma^i_j + \left(\lambda_j \delta^i_k + \lambda_k \delta^i_j\right) \omega^k \\ r^*_\lambda \omega_i &= \omega_i - \lambda_j \gamma^j_i - \lambda_i \lambda_j \omega^j . \end{aligned}$$

We can reconsider projective geometry in terms of bundles. For any manifold M of dimension n, let FM (called the *frame bundle* of M) be the set of all isomorphisms of tangent spaces of M with $\mathbb{R}^n$. The group $G = \mathbb{P}\operatorname{GL}(n+1, \mathbb{R})$ acts transitively on $\mathbb{P}^n$, and also on the frame bundle $F\mathbb{P}^n$. The stabilizer of a point of $\mathbb{P}^n$ is $G_0 \subset G$; the stabilizer of a frame at a point is the subgroup $G_1 \subset G_0$ consisting of matrices of the form

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} .$$

It is easy to show that $\mathbb{P}^n$ has tangent spaces $T_P \mathbb{P}^n = P^* \otimes \left(\mathbb{R}^{n+1}/P\right)$.

We can identify

$$\begin{array}{ccc} F\mathbb{P}^n & \longleftrightarrow & \mathbb{P}\operatorname{GL}(n+1, \mathbb{R})/G_1 \\ \downarrow & & \downarrow \\ \mathbb{P}^n & \longleftrightarrow & \mathbb{P}\operatorname{GL}(n+1)/G_0 . \end{array}$$

We have another bundle over $\mathbb{P}^n$, $\mathbb{P}\operatorname{GL}(n+1, \mathbb{R})$ itself, which we can put on the top at the right side. We will build a corresponding bundle on the left side.

Consider the geodesics of projective space. These are the projective lines. If we think of projective space as the space of lines through 0 in a vector space, its geodesics correspond to 2-planes in that vector space. Thus the space of geodesics is $\operatorname{Gr}(2, n+1) = \mathbb{P}\operatorname{GL}(n+1, \mathbb{R})/G_2$ where G_2 consists of the matrices of the form

$$[g] = \begin{bmatrix} g_0^0 & g_1^0 & g_J^0 \\ g_0^1 & g_1^1 & g_J^1 \\ 0 & 0 & g_J^I \end{bmatrix}$$

where $I, J = 2, \dots, n$. Above the space of geodesics is the space of pointed geodesics, which is the space of choices of a 2-plane in our vector space with a line in that 2-plane, so it is $\mathbb{P}\operatorname{GL}(n+1)/G_+$ where $G_+ \subset G_0$ consists of matrices of the form

$$[g] = \begin{bmatrix} g_0^0 & g_1^0 & g_J^0 \\ 0 & g_1^1 & g_J^1 \\ 0 & 0 & g_J^I \end{bmatrix}.$$

Write $\mathfrak{g}$ and $\mathfrak{g}_0$ for the Lie algebras of G and G_0, etc.

3. Structure equations of projective connections

Given a right principal G_0-bundle $E \to M$, write $r_g : E \to E$ for the right G_0-action of $g \in G_0$. We will refer to G_0 as the *structure group* of the principal bundle.

Definition 1. A *projective connection* on an n-manifold M is a choice of principal right G_0-bundle $E \to M$ together with a 1-form $\Omega \in \Omega^1(E) \otimes \mathfrak{g}$ so that

(1) at each point $e \in E$, $\Omega_e : T_e E \to \mathfrak{g}$ is a linear isomorphism
(2) $r_g^* \Omega = \operatorname{Ad}_g^{-1} \Omega$, and
(3) for any $A \in \mathfrak{g}$, writing $\vec{A}$ for the unique vector field satisfying $\vec{A} \lrcorner \Omega = A$, we require further that
$$e^{\vec{A}} = r_{e^A}$$
whenever $A \in \mathfrak{g}_0$ (the left hand side is the flow of a vector field).

Write

$$\Omega = \begin{pmatrix} \Omega_0^0 & \Omega_j^0 \\ \Omega_0^i & \Omega_j^i \end{pmatrix}$$

with $\Omega_0^0 + \Omega_i^i = 0$. We follow Cartan and define 1-forms $\omega^i, \gamma_j^i, \omega_i$ $(i, j, k, l = 1, \dots, n)$, linearly independent, by the equations

$$\begin{aligned} \omega^i &= \Omega_0^i \\ \gamma_j^i &= \Omega_j^i - \delta_j^i \Omega_0^0 \\ \omega_j &= \Omega_j^0. \end{aligned}$$

(This is just a change of basis from the $\Omega_\bullet^\bullet$ 1-forms.) Since Ω is a 1-form valued in $\mathfrak{g}$, we can think of $\omega^\bullet$ as $\Omega \mod \mathfrak{g}_0$.

Lemma 1. *There are uniquely determined functions $K_{jk}^i, K_{ijk}, K_{jkl}^i$ (called the* curvature functions*) so that the structure equations of Cartan in table 1 on the following page are satisfied.*

Proof. This requires elementary applications of Cartan's lemma; see Kobayashi & Nagano [17] for proof. □

$$
\begin{aligned}
\nabla \omega^i &= d\omega^i + \gamma^i_j \wedge \omega^j \\
&= \frac{1}{2} K^i_{kl} \omega^k \wedge \omega^l \\
\nabla \gamma^i_j &= d\gamma^i_j + \gamma^i_k \wedge \gamma^k_j - \left(\omega_j \delta^i_k + \omega_k \delta^i_j\right) \wedge \omega^k \\
&= \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l \\
\nabla \omega_i &= d\omega_i - \gamma^j_i \wedge \omega_j \\
&= \frac{1}{2} K_{ikl} \omega^k \wedge \omega^l \\
0 &= K^i_{jk} + K^i_{kj} \\
0 &= K^i_{jkl} + K^i_{jlk} \\
0 &= K_{ikl} + K_{ilk}.
\end{aligned}
$$

Table 1. The structure equations of a projective connection

4. Elementary global aspects of projective connections

Definition 2. A projective connection is called *flat* if the curvature functions vanish.

Example 2. The model of a projective connection is the one on $\mathbb{P}^n$ given by taking $E = G$, $\Omega = g^{-1}\,dg$ the left invariant Maurer–Cartan 1-form, and the map $g \in \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R}) \to g\,[e_0] \in \mathbb{P}^n$. The model is flat.

Lemma 2. *A projective connection is flat just when it is locally (i.e. on open subsets of M) isomorphic to the model.*

Proof. Clearly local isomorphism implies flatness. Start with a flat projective connection.Take the exterior differential system $\Omega - g^{-1}\,dg = 0$ on $E \times \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$. (See Bryant et al. [4] for more on exterior differential systems, Cauchy characteristics, and integral manifolds.) It satisfies the conditions of the Frobenius theorem just when the curvature functions vanish. The $\mathfrak{g}$ orbits are Cauchy characteristics, so maximal connected integal manifolds are unions of these orbits. The group G_0 has finitely many path components, and the union of finitely many integral manifolds is an integral manifold, so each connected integral manifold is contained in a unique G_0-invariant and $\mathfrak{g}$-invariant integral manifold. Integral manifolds are the graphs of local isomorphisms, which descend to maps $M \to \mathbb{P}^n$ by G_0-equivariance. □

Example 3. The sphere S^n has a 2-1 covering map $S^n \to \mathbb{P}^n$. Pulling back the bundle from the model, and the 1-form $g^{-1}\,dg$ from the model, we find a flat projective connection on S^n.

Definition 3 (Ehresmann [10]). A projective connection is *complete* when the vector fields $\vec{A}$ are all complete (i.e their flows are defined for all time).

Example 4. The vector fields $\vec{A}$ on the model generate the right action of $\mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ on itself, and therefore the model is complete.

Example 5. A covering space of a complete projective connection is complete, because the relevant vector fields are pullbacks under covering maps. Therefore S^n is complete.

Definition 4. An *isomorphism* of projective connections $E_j \to M_j$ $(j = 0, 1)$ is a G-equivariant diffeomorphism $\Phi : E_0 \to E_1$ preserving the projective connection forms: $\Phi^* \Omega_1 = \Omega_0$. An *infinitesimal symmetry* of a projective connection $E \to M$ is a vector field X on E commuting with the G_0-action, and satisfying $\mathcal{L}_X \Omega = 0$.

Lemma 3. *If a projective connection is complete then every infinitesimal symmetry is a complete vector field.*

Proof. (Essentially the same as Bates [1].) The vector fields $\vec{A}$ commute with X, so they permute the flow lines of X around in all directions. Therefore the time for which the flow of X is defined is locally constant. But then it cannot diminish as we move along a flow line. Therefore the flow of X is defined for all time. □

Lemma 4. *A projective connection $E \to M$ is flat just when the infinitesimal symmetries act locally transitively on E, complete and flat just when the automorphism group is transitive on E.*

Proof. This forces invariance of the curvature, which therefore must be constant. The curvature is equivariant under the G_0 action, so lives in a G_0-representation. One can easily see that there are no nonzero G_0-invariant vectors in that representation. □

Theorem 3. *Every flat projective connection $E \to M$ is obtained by taking the universal covering space $\tilde{M} \to M$, mapping $\tilde{M} \to \mathbb{P}^n$ by a local diffeomorphism, pulling back the model projective connection, and taking the quotient projective connection via some morphism $\pi_1(M) \to \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$.*

Remark 1. The map $\tilde{M} \to \mathbb{P}^n$ is called the *developing map.*

Proof. Without loss of generality, assume that M is connected. Put the exterior differential system $\Omega = g^{-1}\,dg$ on the manifold $E \times \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$. By the Frobenius theorem, the manifold is foliated by leaves (maximal connected integral manifolds). Because the system is invariant under left action of $\mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ on itself, this action permutes leaves. Define vector fields $\vec{A}$ on $E \times \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ by adding the one from E with the one (by the same name) from $\mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$. The flow of $\vec{A}$ on $\mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ is defined for all time, so the vector field $\vec{A}$ on $E \times \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ has flow through a point (e, g) defined for as long as the flow is defined down on E. These vector fields $\vec{A}$ are Cauchy characteristics, so the leaves are invariant under their flows.

The group G_0 has finitely many components, so the G_0 orbit of a leaf is a finite union of leaves. Let Λ_0 and Λ_1 be G_0-orbits of leaves, containing points $(e_i, g_i) \in \Lambda_i$. After replacing these points by other points obtained through G_0 action, we can draw a path from e_0 to e_1 in E, consisting of finitely many flows of $\vec{A}$ vector fields, so such a path lifts to our leaf. Therefore Λ_1 must contain a point (e_0, g_0'). Therefore there is a G_0-orbit Λ of a leaf, unique up to $\mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ action.

The inclusion $\Lambda \subset E \times \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ defines two local diffeomorphisms $\Lambda \to E$ and $\Lambda \to \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$, both $\vec{A}$ and G_0 equivariant. Consider the first of these. Let F be a fiber of $\Lambda \to E$ over some point $e \in E$. Define local coordinates on E

by inverting the map $A \in \mathfrak{g} \mapsto e^{\vec{A}} e \in E$ near $A = 0$. This map is only defined near $A = 0$, and is a diffeomorphism in some neighborhood, say U, of 0. Then map

$$U \times F \to \Lambda$$

by $(A, f) \mapsto e^{\vec{A}} f$, clearly a local diffeomorphism. Therefore $\Lambda \to E$ is a covering map, and G_0-equivariant, so descends to a covering map $\tilde{M} = \Lambda/G_0 \to M = E/G_0$. Thus $\Lambda \to \tilde{M}$ is the pullback bundle of $E \to M$.

The map $\Lambda \to \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ is G_0-equivariant, so descends to a map $\tilde{M} \to \mathbb{P}^n$. By definition, on Λ we have $\Omega = g^{-1}\,dg$, so this map is pullback of projective connections. □

Corollary 1. *A complete flat projective connection $E \to M$ with $\dim M \geq 2$ is isomorphic to a quotient S^n/Γ with $\Gamma \subset \mathrm{SO}\,(n+1)$ a finite group.*

Proof. Suppose that M has a complete flat projective connection. Then the universal covering space $\tilde{M} \to M$ inherits a complete flat projective connection, and so by theorem 3 on the previous page, $\tilde{M}$ inherits its projective connection from a local diffeomorphism $\tilde{M} \to \mathbb{P}^n$, say

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R}) \\ \downarrow & & \downarrow \\ \tilde{M} & \longrightarrow & \mathbb{P}^n. \end{array}$$

The vector fields $\vec{A}$ are preserved by the map $\tilde{E} \to \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$. These $\vec{A}$ vector fields are complete: on $\mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ they are the left invariant vector fields. The flows of the $\vec{A}$ vector fields are transitive on $\mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$, so $\tilde{E} \to \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ is surjective. Quotienting by G_0, $\tilde{M} \to \mathbb{P}^n$ is a surjective local diffeomorphism. Infinitesimal symmetries of the projective connection on $\mathbb{P}^n$ pull back to infinitesimal symmetries of the projective connection on $\tilde{M}$, and are complete vector fields by lemma 3 on the preceding page. These infinitesimal symmetries generate a Lie group action for which $\tilde{M} \to \mathbb{P}^n$ is equivariant, by Palais's theorem [23]. Therefore $\tilde{M} \to \mathbb{P}^n$ is a covering map, and so $\tilde{M} = S^n$. The quotient $\tilde{M} \to M$ is by a discrete group Γ with morphism $\Gamma \to \mathrm{SL}\,(n+1, \mathbb{R})$. Since S^n is compact, S^n can only be a covering space of compact spaces, so with a finite group Γ of deck transformations. Every finite subgroup of $\mathrm{SL}\,(n+1, \mathbb{R})$ preserves a positive definite inner product on $\mathbb{R}^{n+1}$, so Γ sits in a conjugate of $\mathrm{SO}\,(n+1)$. □

5. Classification of projective connections on curves

Lemma 5. *Every projective connection $E \to M$ on a curve M is flat.*

Proof. Curvature is a semibasic 2-form, but M has only one dimension. □

Consider $\mathbb{P}^1 = \mathbb{A} \cup \infty$, $\mathbb{A} = \mathbb{R}$ and let $\mathbb{A}^+$ be the positive real numbers. We draw the universal cover $\widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ as

$$\dots \underline{\quad 0 \quad\quad \infty \quad\quad 0 \quad\quad \infty \quad} \dots$$

Pull back the standard flat projective connection on $\mathbb{P}^1$ to a projective connection on $\widetilde{\mathbb{P}}^1$. The automorphism group $\mathrm{Aut}\,\widetilde{\mathbb{P}}^1$ of that projective connection on $\widetilde{\mathbb{P}}^1$ is the obvious central extension of $\mathbb{P}\,\mathrm{GL}\,(2, \mathbb{R})$ by $\mathbb{Z}$, where $\mathbb{Z}$ acts by translating zeros to zeros in this picture; write this action as $n \in \mathbb{Z} : x \mapsto x + \infty_n$. To be more concrete, we can split up each 2×2 matrix g into $g = qr$, the usual QR-factorization from linear algebra. For $g \in \mathrm{SL}\,(2, \mathbb{R})$, $g = qr$ with

$$q = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},\ r = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$$

with $a > 0$. Think of a, b, ϕ as local coordinates on $\mathrm{SL}\,(2, \mathbb{R})$. (In terms of these coordinates, group operations are unbearably complicated.) The a, b, ϕ are clearly global coordinates on the universal covering group of $\mathrm{SL}\,(2, \mathbb{R})$, which is $\operatorname{Aut} \widetilde{\mathbb{P}}^1$. Moreover, $r \in G_0$, so ϕ is a global coordinate function on $\widetilde{\mathbb{P}}^1 = \operatorname{Aut} \widetilde{\mathbb{P}}^1 / G_0$, quotienting out the right G_0 action.

In terms of the standard affine chart, identifying

$$x \in \mathbb{R} \to \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{P}^1,$$

$x = \cot(\phi)$ maps $\widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$. Then $x = 0$ lies at $\phi = \pi/2$, and $x = \infty$ lies at $\phi = 0$. The affine (left) group action on $\mathbb{P}^1$, $x \mapsto mx + b$, lifts to a unique action on $\widetilde{\mathbb{P}}^1$ fixing all ∞'s. This is just the use of elements of G_0 on the left instead of the right. We see this by direct calculation: $\phi = 0$ or $\phi = \pi$ just when q commutes with r. Thus qr represents the same point of $\widetilde{\mathbb{P}}^1$ as q.

Theorem 4 (Kuiper [19], Gorinov [11])**.** *The projective connections on a closed connected curve (modulo isomorphism) are:*

elliptic		$\widetilde{\mathbb{P}}^1 / (\phi \mapsto \phi + \theta)$
parabolic	*(1)*	$\mathbb{A}/(x \mapsto x + 1)$ *or*
	(2)	$\widetilde{\mathbb{P}}^1/(x \mapsto x + \infty_n + 1)$
hyperbolic	*(1)*	$\mathbb{A}^+/(x \mapsto rx)$ *(some* $r > 1$*) or*
	(2)	$\widetilde{\mathbb{P}}^1/(x \mapsto rx + \infty_n)$

where we can assume that n *is an arbitrary positive integer, the angle* θ *can be any nonzero real number, and* $n \neq 0$*. In particular, the numbers* r, θ, n *are invariants of the projective connection. The projective connections on an open connected curve (modulo isomorphism) are the pullbacks to the following open subsets of* $\widetilde{\mathbb{P}}^1$*:*

elliptic		$\widetilde{\mathbb{P}}^1$
parabolic	(1)	$^{-\infty}$____ . . . ____$^{\infty}$
	(2)	$^{-\infty}$____ . . . ____$^{\infty}$____ . . .
hyperbolic		^{0}____ . . . ____$^{\infty}$

Any number of copies of the affine line may be contained in these open parabolic and hyperbolic curves in the . . . in the middle, and infinitely many must appear in any . . . at each end. The hyperbolic curves are those with an invariant metric $(dz/z)^2$ *(and consequently an invariant affine connection: the Levi-Civita connection), defined except at* $z = \infty$ *and* $z = 0$*. The parabolic are those with no invariant metric but have an invariant affine connection, defined except at* $z = \infty$*, and the elliptic are those which have neither.*

Proof. We will only outline the proof. The technique is to use the developing map, i.e. identify the universal cover locally with $\mathbb{P}^1$, and thereby globally with an open interval of $\widetilde{\mathbb{P}}^1$, following theorem 3 on page 7. If this open interval has an endpoint, we can slide it along by automorphisms of $\widetilde{\mathbb{P}}^1$, and put it where we like. If it has two endpoints, we have to be more careful: we can put one of them where we like, say at some ∞, but then if the other one winds up landing at another ∞ in the process, it is impossible to move it without moving the first one. On the other hand, if the second end point does not land on an ∞, we can slide it along by affine transformations to land on a 0. For an open curve, this finishes the story. Consider a closed curve. With this normalization completed, a certain subgroup of automorphisms is still available fixing the (0, 1 or 2) endpoints. With this, we can normalize the monodromy around the closed curve, which is an element of $\mathbb{P}\,\mathrm{GL}\,(2,\mathbb{R})$. But the monodromy must act without fixed points in the interior of $\tilde{C}$, while fixing all of the endpoints. This allows us to classify the possible monodromy elements up to conjugation by automorphisms. ☐

Theorem 5. *A projective connection on a curve is complete just when its universal cover is identified with $\widetilde{\mathbb{P}}^1$ by the developing map, i.e. either elliptic or closed parabolic of type (2) or closed hyperbolic of type (2).*

Proof. Let $E \to C$ be a projective connection on a curve. Following theorem 3 on page 7, the universal cover $\tilde{C}$ is mapped locally diffeomorphically to $\widetilde{\mathbb{P}}^1$, and therefore is a connected open subset. The bundle $E \to C$ lifts to a bundle $\tilde{E} \to \tilde{C}$. Completeness is invariant under covering maps, so $E \to C$ is complete just when $\tilde{E} \to \tilde{C}$ is. Clearly $\tilde{E}$ is an open subset of the automorphism group of $\widetilde{\mathbb{P}}^1$, under

$$
\begin{array}{ccccccc}
E & \longleftarrow & \tilde{E} & \longrightarrow & \operatorname{Aut}\widetilde{\mathbb{P}}^1 & \longrightarrow & \mathbb{P}\,\mathrm{GL}\,(2,\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C & \longleftarrow & \tilde{C} & \longrightarrow & \widetilde{\mathbb{P}}^1 & \longrightarrow & \mathbb{P}^1.
\end{array}
$$

Completeness is just precisely invariance of that open subset under the flow of all left invariant vector fields, i.e. under left translation by the identity component, i.e. the open subset being a union of components. But $\mathbb{P}\,\widetilde{\mathrm{GL}\,(2,\mathbb{R})} = \operatorname{Aut}\widetilde{\mathbb{P}}^1$ has precisely two components, and they are interchanged by right G_0 action, and $\tilde{E}$ is G_0 invariant. Therefore completeness is just equality of $\tilde{E}$ and $\operatorname{Aut}\widetilde{\mathbb{P}}^1$. So completeness of C is just completeness of $\tilde{C}$ which is just isomorphism of $\tilde{C}$ with $\widetilde{\mathbb{P}}^1$. ☐

6. Geodesics

Given an immersed curve $\iota : C \to M$ on a manifold M with projective connection $E \to M$, define the pullback bundle $\iota^* E$

$$
\begin{array}{ccc}
\iota^* E & \longrightarrow & E \\
\downarrow & & \downarrow \\
C & \longrightarrow & M.
\end{array}
$$

Consider the vector-valued 1-form

$$
\sigma = \begin{pmatrix} \omega^1 \\ \omega^2 \\ \vdots \\ \omega^n \end{pmatrix}
$$

on E, known as the *soldering form*. This 1-form σ vanishes on the fibers of $E \to M$, because the fibers have tangent spaces spanned by the vectors $\vec{A}$, for $A \in G_0$. Pick a point $e \in E$, and let $m \in M$ be the projection of $e \in E$ via the bundle map $E \to M$. The soldering form descends to an isomorphism

$$\sigma_e : T_m M \to \mathbb{R}^n.$$

This isomorphism depends on the choice of the point $e \in E_m$. Indeed the isomorphism varies by the obvious G_0-action as we move e by G_0-action:

$$\sigma_{r_g e} = h^{-1} \sigma_e$$

for

$$g = \begin{bmatrix} 1 & \lambda \\ 0 & h \end{bmatrix} \in G_0.$$

Restricted from $T_m M$ to $T_m C$, σ_e has image in $\mathbb{R}^n$ a line. This line varies under the G_0-action. Let $E_C \subset \iota^* E$ be the set of points of $\iota^* E$ at which this line is the span of e_1 in $\mathbb{R}^n$. It is easy to see (see Gardner [?]) that $E_C \subset \iota^* E$ is a principal G_+-subbundle, where $G_+ \subset G_0$ is the subgroup of projective transformations preserving the projective line through $[e_0]$ and $[e_1]$ as well as fixing the point $[e_0]$. Moreover by construction, on E_C we have $\omega^I = 0$. Taking exterior derivative of the equations $\omega^I = 0$, we find $\gamma^I_1 = \kappa^I \omega^1$, for some functions $\kappa^I : E_C \to \mathbb{R}$, which descend to a section κ of $(\iota^* TM/TC) \otimes (T^* C)^2$, called the *geodesic curvature* of C. (See Cartan [6], Leçons sur la théorie des espaces à connexion projective, pp. 91-111.)

Remark 2. To prove that κ^I descend to a section of this bundle, or prove other similar statements, the procedure is always the same as our proof that $TM = E \times_{G_0} \mathbb{R}^n$ in lemma 1 on page 18.

Definition 5. We define a *geodesic* to be a curve of vanishing geodesic curvature.

Equivalently, geodesics are the curves on M which are the projections to M of the integral manifolds $E_C \subset E$ of the differential system $\omega^I = \gamma^I_1 = 0$.

Definition 6. The *geodesic flow* is the flow of the vector field dual to ω^1.

The flow lines of geodesic flow are contained in these integral manifolds; since these flow lines are permuted by the action of G_+, the manifolds E_C for C a geodesic are precisely the G_+-orbits of flow lines of geodesic flow.

Definition 7. Given a connected immersed curve $\iota : C \subset M$, and a chosen point $c_0 \in C$, we will *roll* an immersed curve C onto $\mathbb{P}^n$ to produce an immersion $C \to \mathbb{P}^n$ (called its *development*), as follows. Take the differential system $\Omega = g^{-1} \, dg$ on $E \times \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$, restrict it to $E_C \times \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$. The Frobenius theorem once again tells us that $E_C \times \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ is foliated by leaves (i.e. maximal connected integral manifolds of that differential system).

Take the G_+ orbit of any leaf, say $\Lambda \subset E_C \times \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$, containing a point $(e, 1)$ with $\pi(e) = c$, and map $\Lambda \to \mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R}) \to \mathbb{P}^n$ by the obvious maps. By G_+-equivariance, this determines a map $C \to \mathbb{P}^n$. By $\mathbb{P}\,\mathrm{GL}\,(n+1, \mathbb{R})$ invariance, changing the choice of Λ changes the map $C \to \mathbb{P}^n$ by a projective automorphism.

Definition 8. Take an immersed curve $\iota : C \to M$ and a projective connection $E \to M$. Let $N_+ \subset G_+$ be the subgroup of G_+ acting trivially on $\mathbb{P}^1 \subset \mathbb{P}^n$. Then $\bar{E}_C = E_C/N_+ \to C$ is a principal right $\bar{G}_0$-bundle, where $\bar{G}_0 = G_+/N_+$. Let

$\bar{G} = G/N_+$. The 1-form $\bar{\Omega} = \Omega \mod \mathfrak{n}_+ \in \Omega^1\left(\bar{E}_C\right) \otimes \bar{\mathfrak{g}}$ is a projective connection on $\bar{E}_C \to C$. Call this the *induced* projective connection on C.

Lemma 6. *C is a geodesic just when*
1. *the development $C \to \mathbb{P}^n$ maps $C \to \mathbb{P}^1 \subset \mathbb{P}^n$, and*
2. *the induced projective connection is the pullback via $C \to \mathbb{P}^1$.*

Proof. This is immediate from the structure equations. □

It is elementary to prove:

Theorem 6 (Kobayashi [14])**.** *Let M be a manifold with projective connection. Every immersed curve in projective space is the development of a curve in M just when M is complete.*

Definition 9. A curve C in a manifold M with projective connection is *complete* when the induced projective connection $\bar{E}_C$ is complete.

Lemma 7. *A projective connection is complete just when all of its geodesics are complete.*

Proof. The geodesic flow is tangent to all of the manifolds E_C for all geodesics, and projects under $E_C \to \bar{E}_C$ to the geodesic flow. Therefore its completeness is identical to the completeness of all geodesics. But $r_g^* \vec{A} = \overrightarrow{\operatorname{Ad}_g A}$ for $g \in G_0$, permuting the vector fields dual to all of the ω^i. The vector fields $\vec{A}$ for $A \in \mathfrak{g}_0$ are always complete, moving up the fibers. □

Remark 3. It is unknown whether there are manifolds of dimension greater than one with all geodesics parabolic and closed, or with all geodesics elliptic and open.

7. Affine connections, projective connections, projective structures and normal projective connections

Just to clarify a few minor points in the literature, we would like to explain the relations between affine connections, projective connections, projective structures and normal projective connections. Kobayashi & Nagano [17] explain how to relate projective structures, torsion-free affine connections, and normal projective connections, but they don't explain the relation between arbitrary projective connections and arbitrary affine connections.

Let $E \to M$ be a projective connection. Clearly the bundle $E/\mathbb{R}^{n*} \to M$ is a principal right $\operatorname{GL}(n, \mathbb{R})$-bundle.

Let $FM \to M$ be the bundle whose fiber over a point $m \in M$ consists in the linear isomorphisms $u : T_m M \to \mathbb{R}^n$. Make this into a principal right $\operatorname{GL}(n, \mathbb{R})$-bundle, by defining $r_g u = g^{-1} u$ for $g \in \operatorname{GL}(n, \mathbb{R})$. For each $e \in E$, $\Omega_e : T_e E \to \mathfrak{g}$ is onto, so if we write $\omega_e : T_e E \to \mathfrak{g}/\mathfrak{g}_0 = \mathbb{R}^n$ for the composition with the obvious projection, then $\omega \in \Omega^1(E) \otimes \mathbb{R}^n$. But $\omega = 0$ on vertical vectors on E, so for each point $e \in E$, ω_e determines a linear isomorphism $\underline{\omega}_e : T_m M \to \mathbb{R}^n$, where $e \in E_m$.

Lemma 8. *The map $e \in E \to \underline{\omega}_e \in FM$ descends to a $\operatorname{GL}(n, \mathbb{R})$-equivariant bundle isomorphism $E/\mathbb{R}^{n*} \to FM$. Let $\phi : FM \to M$ be the bundle map. Define 1-forms ω^i on FM by*

$$v \lrcorner \omega = u(\phi'(u)v)$$

for $v \in T_u FM$. Then $E \to FM$ pulls back ω^i to ω^i

Proof. The $\mathrm{GL}(n,\mathbb{R})$-equivariance is a calculation:

$$\underline{\omega}(r_g e) = g^{-1}\underline{\omega}(e),$$

from which the rest easily follows. □

Lemma 9. *If $E \to M$ is a projective connection, then $E \to E/\mathbb{R}^{n*} = FM$ is a trivial principal bundle right $\mathbb{R}^{n*}$-bundle. The $\mathrm{GL}(n,\mathbb{R})$-equivariant sections of this bundle determine affine connections on M with the given geodesics and given torsion K^i_{jk}. Changing the choice of section alters the parameterization of the geodesics.*

Proof. Existence of a global section s is elementary, for any principal $\mathbb{R}^{n*}$-bundle, using local convex combinations.

Let $s : FM \to E$ be a local $\mathrm{GL}(n,\mathbb{R})$-equivariant section. Then $s^*\omega^i = \omega^i$, clearly. Lets write γ^i_j still for the 1-forms $s^*\gamma^i_j$, and ω_i for $s^*\omega_i$.

The vector fields $\vec{A}$ on E, for $A \in \mathfrak{gl}(n,\mathbb{R})$, project to the corresponding vector fields $\vec{A}$ on FM given by the right action of $\mathrm{GL}(n,\mathbb{R})$. Therefore $\vec{A} \lrcorner \gamma = A$ on both E and FM. Therefore $\vec{A} \lrcorner \omega_i = 0$ on both E and FM.

The vectors dual to ω^i on E project to nonzero vectors on M, and therefore on FM, because ω^i are semibasic. Therefore ω^i, γ^i_j form a coframing on FM. So ω_i must be a combination of them, and since $\vec{A} \lrcorner \omega_i = 0$ for $A \in \mathfrak{gl}(n,\mathbb{R})$, we must have $\omega_i = a_{ij}\omega^j$ for some functions $a_{ij} : FM \to \mathbb{R}$.

Pick a geodesic $C \subset M$. Then the tangent spaces of E_C are cut out by the equations $\omega^I = \gamma^I_1 = 0$. These equations are expressed in semibasic 1-forms, so the integral manifolds will project to integral manifolds of the same system on FM. The projective parameterizations of a geodesic are those given by the geodesic flow through points of E_C, and therefore unless the section s stays inside a region where $\underline{\omega}^1$ is constant, the parameterization will not match a projective parameterization.

Because $\vec{A} \lrcorner \gamma = A$ for $A \in \mathfrak{gl}(n,\mathbb{R})$, γ determines a unique connection on FM, with horizontal space $\gamma = 0$. We leave the reader to show that the geodesics of the connection are the integral manifolds of the exterior differential system $\omega^I = \gamma^I_1 = 0$.

Recall that the torsion of an affine connection with connection 1-forms γ^i_j is given by equivariant functions T^i_{jk}, where

$$d\omega^i = -\gamma^i_j \wedge \omega^j + \frac{1}{2} T^i_{jk}\omega^j \wedge \omega^k,$$

so torsion is $T^i_{jk} = K^i_{jk}$, same as for the projective connection. □

The structure equations given by the choice of some section s are

$$\begin{aligned}
\nabla_s \omega^i &= d\omega^i + \gamma^i_j \wedge \omega^j \\
&= \frac{1}{2} K^i_{kl}\omega^k \wedge \omega^l \\
\nabla_s \gamma^i_j &= d\gamma^i_j + \gamma^i_k \wedge \gamma^k_j \\
&= \left(\frac{1}{2} K^i_{jkl} - a_{jl}\delta^i_k - a_{kl}\delta^i_j\right) \omega^k \wedge \omega^l,
\end{aligned}$$

relating the curvature of the affine connection to the curvature of the projective connection:

$$R^i_{jkl} = K^i_{jkl} - \left(a_{jl}\delta^i_k - a_{jk}\delta^i_l + a_{kl}\delta^i_j - a_{lk}\delta^i_j\right).$$

In particular, we can easily see that if the projective connection satisfies $K^i_{jil} = 0$, then we recover the curvature K of the projective connection from the curvature R of the affine connection via the equation

$$a_{ij} = \frac{nR^m_{imj} + R^m_{jmi}}{n^2 - 1},$$
$$K^i_{jkl} = R^i_{jkl} + \left(a_{jl}\delta^i_k - a_{jk}\delta^i_l + a_{kl}\delta^i_j - a_{lk}\delta^i_j\right).$$

This makes it easy to test for projective flatness of an affine connection, even with torsion.

Corollary 2. *The unparameterized geodesics of a projective connection are the unparameterized geodesics of some affine connection.*

Lemma 10. *Given a projective connection, there is a torsion-free projective connection (i.e. $K^i_{jk} = 0$) with the same geodesics, with the same projective parameterizations.*

Proof. Set $\tilde{\gamma}^i_j = \gamma^i_j + \frac{1}{2}K^i_{jk}$. □

Kobayashi [16] and Cartan [5] show that given a projective connection, there is a unique normal projective connection with the same unparameterized geodesics. By our result above, there are torsion-free connections with the same parameterized geodesics as this normal projective connection. Two affine connections are said to be *projectively equivalent* if they have the same unparameterized geodesics, and an equivalence class is called a *projective structure*. Kobayashi [16] and Kobayashi & Nagano [17] also show that given any torsion-free affine connection, there is a unique bundle $E \to FM$, whose sections are precisely the torsion-free affine connections, and that this bundle bears a unique normal projective connection with the given geodesics. Therefore a normal projective connection is essentially the same object as a projective structure.

Lemma 11 (Weyl)**.** *Two connections $\gamma, \tilde{\gamma}$ on FM have the same geodesics up to parameterization just when*

$$\tilde{\gamma}^i_j = \gamma^i_j + \left(\lambda_j\delta^i_k + \lambda_k\delta^i_j\right) + a^i_{jk}\omega^k$$

where $\lambda_j\omega^j$ is the pullback to FM of a 1-form λ on M, and $a^i_{jk}\omega^j \wedge \omega^k \otimes \frac{\partial}{\partial\omega^i}$ is the pullback to FM of a section of $\Lambda^2\left(T^*M\right) \otimes TM$.

Proof. Any two connection 1-forms have to agree on the vertical vectors, so can only differ by semibasic 1-forms:

$$\tilde{\gamma}^i_j = \gamma^i_j + p^i_{jk}\omega^k.$$

The equations of geodesics of $\tilde{\gamma}^i_j$ are $\omega^I = \tilde{\gamma}^I_1 = 0$, giving $\omega^I = \gamma^I_1 + p^I_{11}\omega^k$. For these to be the same Frobenius exterior differential systems, their leaves must have the same tangent spaces. The connections will share a geodesic C just when the submanifolds of FM:

$$\begin{array}{ccc} FM_C & \longrightarrow & FM \\ \downarrow & & \downarrow \\ C & \longrightarrow & M \end{array}$$

are the same. These are the leaves of the exterior differential system, which satisfies the conditions of the Frobenius theorem, i.e. the conormal bundle is spanned precisely by the 1-forms in the exterior differential system, so the systems must be identical. Therefore $p^I_{11} = 0$, for all $I > 1$. Since indices can be freely permuted in this argument, we have $p^i_{jj} = 0$ whenever $i \neq j$. By $\mathrm{GL}(n, \mathbb{R})$-equivariance of connection 1-forms, $p^i_{jj} = 0$ whenever $i \neq j$. Check that the expression

$$P \lrcorner \omega^i = p^i_{jk} \omega^j \wedge \omega^k$$

defines a section P of $T^*M \otimes T^*M \otimes TM$, by $\mathrm{GL}(n, \mathbb{R})$-equivariance. Split into $D = A + S$ with A antisymmetric and S symmetric. Then $S(v, v)$ must be a multiple of v for all vectors $v \in TM$, because $p^i_{jj} = 0$. Therefore $S(v, v) = \lambda(v)v$ for a unique 1-form λ. Moreover,

$$\tilde{\gamma}^i_j = \gamma^i_j + \left(\lambda_j \delta^i_k + \lambda_k \delta^i_j\right) \omega^k + a^i_{jk} \omega^k.$$

Equivariance under $\mathrm{GL}(n, \mathbb{R})$ ensures that a^i_{jk} determines a section of the appropriate bundle on M. □

Corollary 3 (Weyl)**.** *Two connections $\gamma, \tilde{\gamma}$ on FM have the same geodesics up to parameterization and the same torsion just when*

$$\tilde{\gamma}^i_j = \gamma^i_j + \left(\lambda_j \delta^i_k + \lambda_k \delta^i_j\right)$$

where $\lambda_j \omega^j$ is the pullback to FM of a 1-form λ on M.

Theorem 7. *Let ∇ be an affine connection on a manifold M, and let $\underline{\gamma}$ be the corresponding connection 1-form on FM. Define the 1-forms ω^i as above on FM. Define E to be the set of triples $(m, u, \tilde{\gamma})$ so that $m \in M$, $u : T_m M \to \mathbb{R}^n$ is a linear isomorphism, and $\tilde{\gamma}$ is the value at $u \in FM$ of a smooth connection 1-form with the same torsion and geodesics as $\underline{\gamma}$. Identify $\lambda \in \mathbb{R}^{n*}$ with the element of $\left(\lambda_j \delta^i_k + \lambda_k \delta^i_j\right) \in \mathbb{R}^{n*} \otimes \mathfrak{gl}(n, \mathbb{R})$. Make $\mathbb{R}^{n*}$ act on E on the right by $r_\lambda(m, u, \tilde{\gamma}) = (m, u, \tilde{\gamma} + \lambda \cdot \omega)$. Then E has the structure of a smooth manifold, and $\phi : (m, u, \tilde{\gamma}) \in E \to u \in FM$ is a principal right $\mathbb{R}^{n*}$-bundle, and $\Phi : (m, u, \tilde{\gamma}) \in E \to m \in M$ is a principal right G_0-bundle. Let the vector fields $\vec{\lambda}$ and $\vec{A}$ on E be the generators of the right action. Define 1-forms on E by pulling back ω^i, defining γ^i_j by the equation*

$$v \lrcorner \gamma^i_j = (\phi'(m, u, \tilde{\gamma})v) \lrcorner \tilde{\gamma}^i_j.$$

Then there are 1-forms ω_i on E so that

$$\vec{\lambda} \lrcorner \omega_i = \lambda_i \text{ and } \vec{A} \lrcorner \omega_i = 0$$

for $A \in \mathfrak{gl}(n, \mathbb{R})$. We can pick these 1-forms uniquely if we require that

$$d\gamma^i_j = -\gamma^i_k \wedge \gamma^k_j + d\gamma^i_j = -\gamma^i_k \wedge \gamma^k_j + \left(\omega_j \delta^i_k + \omega_k \delta^i_j\right) \wedge \omega^k + \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l$$

for some functions K^i_{jkl} and require further that $K^i_{jkl} + K^i_{jlk} = 0$ and $K^i_{jil} = 0$. Moreover, $\omega^i, \gamma^i_j, \omega_i$ constitute a projective connection $E \to M$ whose geodesics are the geodesics of ∇, and whose torsion is the torsion of ∇. This projective connection is normal just when ∇ is torsion-free.

Proof. It is obvious that $E \to FM$ is a principal right $\mathbb{R}^{n*}$-bundle. Define the right action of $g \in \mathrm{GL}(n, \mathbb{R})$ by

$$r_g \left(m, u, \tilde{\Gamma}\right) = \left(m, g^{-1} u, \mathrm{Ad}^{-1}_g \tilde{\Gamma} r^{-1}_{g*}\right).$$

Check that this fits together with the $\mathbb{R}^{n*}$-action into a G_0-action. For $g \in \mathrm{GL}(n, \mathbb{R})$, check that

$$r_g^* \omega^\bullet = g^{-1} \omega^\bullet$$
$$r_g^* \gamma = \mathrm{Ad}_g^{-1} \gamma.$$

Differentiate these equations to show that for $A \in \mathfrak{gl}(n, \mathbb{R})$,

$$\mathcal{L}_{\vec{A}} \omega^\bullet = -A \omega^\bullet$$
$$\mathcal{L}_{\vec{A}} \gamma = -\mathrm{Ad}_A \gamma.$$

Similarly, for $\lambda \in \mathbb{R}^{n*}$,

$$r_\lambda^* \omega^\bullet = \omega^\bullet$$
$$r_\lambda^* \gamma = \gamma + \lambda \omega,$$

so that

$$\mathcal{L}_{\vec{\lambda}} \omega^\bullet = 0$$
$$\mathcal{L}_{\vec{\lambda}} \gamma = \lambda \cdot \omega^\bullet.$$

Since the $\vec{A}$ and $\vec{\lambda}$ exhaust the vertical directions, Cartan's lemma ensures that:

$$0 = d\omega^i + \gamma^i_j \wedge \omega^j + \frac{1}{2} K^i_{kl} \omega^k \wedge \omega^l$$
$$0 = d\gamma^i_j + \gamma^i_k \wedge \gamma^k_j - \left(\omega_k \delta^i_j + \omega_j \delta^i_k\right) \wedge \omega^k + \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l,$$

for some functions K^i_{kl}, K^i_{jkl} antisymmetric in k, l, and some 1-forms ω_i with $\vec{\lambda} \lrcorner \omega_i = \lambda_i$.

Taking the initial $\underline{\gamma}$ 1-form to determine a section $\sigma(u) = \left(m, u, \underline{\gamma}\right)$ of $E \to FM$, calculate that

$$\sigma^* \omega^\bullet = \omega^\bullet$$
$$\sigma^* \gamma = \underline{\gamma},$$

which implies that

$$0 = \sigma^* \left(d\omega^i + \gamma^i_j \wedge \omega^j + \frac{1}{2} K^i_{kl} \omega^k \wedge \omega^l \right)$$
$$= d\omega^i + \underline{\gamma}^i_j \wedge \omega^j + \frac{1}{2} K^i_{kl} \omega^k \wedge \omega^l,$$

so that along that section of E, we have

$$K^i_{kl} = T^i_{kl}.$$

Check that

$$r_\lambda^* K^i_{kl} = K^i_{kl},$$

so that $K^i_{kl} = T^i_{kl}$ everywhere on E.

There is still some freedom to pick these ω_i. We can change them to

$$\tilde{\omega}_i = a_{ij} \omega^j$$

without changing the equations we have developed so far. This will alter the expression K^i_{jil}, changing it to

$$\tilde{K}^i_{jil} = K^i_{jil} + 2 a_{jl}.$$

Therefore there is a unique choice of ω_i 1-forms for which $K^i_{jil} = 0$. Taking exterior derivative of the equations so far, we find that the structure equations of a projective connection are satisfied. Moreover, the G_0-equivariance is assured because the condition which determined the ω_i was G_0-invariant. □

Remark 4. Notice the peculiar condition that $K^i_{jil} = 0$. This vanishing of what we might call *"unsymmetrized Ricci curvature"* is required to specify the choice of projective connection uniquely.

Corollary 4. *Any two projective connections on a manifold M, with the same torsion and unparameterized geodesics as a given affine connection, differ by a section of $T^*M \otimes T^*M$.*

Proof. If ω_i is changed to $\tilde{\omega}_i$ in the above proof, then $\tilde{\omega}_i - \omega_i = a_{ij}\omega^j$, and the structure equations tell us that

$$\begin{aligned} r_g^* a_{ij} &= a_{kl} g^i_k g^j_l \\ r_\lambda^* a_{ij} &= a_{ij}, \end{aligned}$$

which ensures that a_{ij} descends to a section of that bundle. □

8. Vector bundles and descent data

Definition 10. Let $E \to M$ be a projective connection, with connection 1-form $\Omega \in \Omega^E(\otimes)\,\mathfrak{g}$. We have a 1-form $\omega = \left(\omega^i\right) = \Omega \mod \mathfrak{g}_0 : T_eE \to \mathfrak{g}/\mathfrak{g}_0 = \mathbb{R}^n$. The linear map $\Omega : T_eE \to \mathfrak{g}$ is an isomorphism, identifying the vertical directions with $\mathfrak{g}_0$. Therefore the ω^i are semibasic, i.e. vanish on the vertical directions, being valued in $\mathbb{R}^n = \mathfrak{g}/\mathfrak{g}_0$. Moreover the 1-forms ω^i are a basis for the semibasic 1-forms for the map $E \to M$. At each point $e \in E$, ω^i is therefore the pullback via $\pi : E \to M$ of some 1-form from $\pi(e)$, say $\underline{\omega}^i : T_{\pi(e)}M \to \mathbb{R}$, so that

$$v \lrcorner \omega = \pi(e)' v \lrcorner \underline{\omega}^i$$

for all $v \in T_eE$. These $\underline{\omega}^i$ are *not* sections of T^*M, but of π^*T^*M.

Lemma 12. *If X is a vector field on M, define functions $X^i : E \to \mathbb{R}$ by*

$$X^i(e) = X \lrcorner \underline{\omega}^i(e).$$

Let

$$X^\bullet = \begin{pmatrix} X^1 \\ X^2 \\ \vdots \\ X^n \end{pmatrix} : E \to \mathbb{R}^n.$$

Then $r_g^* X^\bullet = g^{-1} X^\bullet$, for $g \in G_0$.

Proof. Pick any vector field Y on E so that $\pi'(e)Y(e) = X(\pi(e))$. We can do this by picking Y locally, and making affine combinations of local choices of Y. Therefore $r_{g*}Y - Y$ is vertical, for any $g \in G_0$. The G_0-equivariance of Ω says that

$$r_g^* \omega^\bullet = g^{-1} \omega^\bullet,$$

i.e.

$$r_{g*} v \lrcorner \omega^\bullet (r_g e) = g^{-1} \left(v \lrcorner \omega^\bullet(e)\right).$$

Therefore

$$\begin{aligned}
r_g^* X^\bullet(e) &= X^\bullet\left(r_g e\right) \\
&= X \mathbin{\lrcorner} \underline{\omega}^\bullet\left(r_g e\right) \\
&= Y \mathbin{\lrcorner} \omega^\bullet\left(r_g e\right) \\
&= g^{-1}\left(\left(r_{g*}^{-1} Y \mathbin{\lrcorner} \omega^\bullet(e)\right)\right) \\
&= g^{-1}\left(Y(e) \mathbin{\lrcorner} \omega^\bullet(e)\right) \\
&= g^{-1} X^\bullet(e).
\end{aligned}$$

□

Definition 11. Given a group G_0 and two spaces on which G_0 acts on the right, say X and Y, the diagonal right G_0-action is the one given by $(x, y)\, g_0 = (x g_0, y g_0)$. Let $X \times_{G_0} Y$ be the quotient by the diagonal G_0-action.

Proposition 1. *$TM = E \times_{G_0} \mathbb{R}^n$*

Proof. Given a vector field X on M, the functions $X^\bullet : E \to \mathbb{R}^n$ are G_0-equivariant, so form a section of $E \times_{G_0} \mathbb{R}^n \to M$. Clearly $X = 0$ just where that section vanishes, so this is an injection of vector bundles of equal rank, hence an isomorphism. Alternately, given G_0-equivariant function X^i, we define a section of $\pi^* TM \to E$ (where $\pi : E \to M$ is our projective connection bundle), by $X \mathbin{\lrcorner} \underline{\omega}^\bullet = X^\bullet$. But by G_0-equvariance of $X^\bullet$ and of $\omega^\bullet$, this X is G_0-invariant, so drops to a section of $TM \to M$. □

Remark 5. We will frequently state that various equivariant expressions on various principal bundles determine sections of various vector bundles, and in each case a proof along the lines of the above applies, so we will omit those proofs. For example:

Corollary 5. *$T^*M = E \times_{G_0} \mathbb{R}^{n*}$, etc.*

9. Positive Ricci curvature

Theorem 8. *Let M have a complete affine connection with Ricci curvature tensor $\operatorname{Ric}_{ij} = \frac{1}{2}\left(R^k_{ikj} + R^k_{jki}\right)$ positive definite, and bounded from below along any geodesic by*

$$\operatorname{Ric}_{ij}(t) \geq \frac{c}{4t^{2-\epsilon}} \delta_{ij},$$

for some constants $c, \epsilon > 0$ (these constants possibly dependent on the choice of geodesic), for all sufficiently large values of t, where t is the natural affine parameter along a geodesic determined by the affine connection. Then the induced projective connection on M is complete.

Proof. Pick coordinates x^i along the geodesic, which we can assume (by parallel transport of a frame) are constant on parallel transported vectors. There will be infinitely many zeroes to solutions of the equation

$$y'' + \operatorname{Ric}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} y = 0$$

where x^i are local coordinates along the geodesic, by comparison to the equation

$$y'' + \frac{c}{t^{2-\epsilon}} y = 0,$$

whose solutions are constant linear combinations of

$$\sqrt{t} J\left(\frac{1}{\epsilon}, \frac{\sqrt{c} t^{\epsilon/2}}{\epsilon}\right), \sqrt{t} Y\left(\frac{1}{\epsilon}, \frac{\sqrt{c} t^{\epsilon/2}}{\epsilon}\right),$$

where J and Y are Bessel functions. Kobayashi & Sasaki [18] proved that the projective parameterizations are precisely the ratios $u = (ay_1 + b)/(cy_0 + d)$, for $y = y_0, y = y_1$ any two linearly independent solutions of this equation, and a, b, c, d any constants with $ad - bc \neq 0$. Therefore the projective parameterization wraps around infinitely often, ensuring completeness of each geodesic by comparison to the classification of projective connections on curves. Completeness of every geodesic ensures completeness of the ambient projective connection by lemma 7 on page 12. □

Remark 6. Every example known of such an affine connection is found on a compact manifold. Kobayashi & Nagano [17] wonder whether projective completeness implies compactness, which would imply a strengthened Bonnet–Myers–Cheng theorem, i.e. that slower than quadratic Ricci curvature decay of a complete affine connection would force compactness. For Riemannian manifolds, there does not appear to be in the literature any proof that slower than quadratic Ricci curvature decay of the Levi-Civita connection would force compactness, but is not difficult to prove using results of David Wraith [26].

Remark 7. This theorem gives rise to many examples, but depends on Ricci curvature, which is not an invariant of a projective connection. We would like to find a criterion for completeness which can be checked in many examples, and which is projectively invariant. Even for the projective connections of Riemannian manifolds, it is unclear to what extent Ricci curvature bounds are really required. In fact they are not: we will provide examples of surfaces with projectively complete Riemannian metrics, whose curvature takes on both positive and negative values.

Example 6. The Killing form metric on a compact semisimple Lie group has positive Ricci curvature (Milnor [22]), and therefore is projectively complete.

Remark 8. This theorem is similar to Tanaka [25] p. 21, but he uses the opposite sign convention for Ricci curvature (so that the sphere has negative Ricci curvature for him), and his result requires invariance of the Ricci curvature under parallel transport.

10. Left invariant projective connections on Lie groups

Theorem 9. *The isomorphism classes of left invariant projective connections on a Lie group H of dimension n with Lie algebra $\mathfrak{h}$ are invariantly identified with the linear maps $\mathfrak{h} \to \mathfrak{sl}(n+1, \mathbb{R})$ which are transverse to $\mathfrak{g}_0$, modulo the Lie algebra automorphisms of $\mathfrak{sl}(n+1, \mathbb{R})$ fixing $\mathfrak{g}_0$, and the Lie algebra automorphisms of $\mathfrak{h}$.*

Proof. Suppose that $E \to H$ is a left invariant projective connection, i.e. that the left action of H on itself lifts to a left action on E, preserving a projective connection. The actions of H and G_0 must commute. Pick a point $e \in E$ and map $(h, g) \in H \times G_0 \to r_{g_0} he \in E$. This map is clearly a diffeomorphism, so henceforth identify $E = H \times G_0$. Consider the projective connection Ω. Lets write the left invariant Maurer–Cartan 1-form on G_0 as $g_0^{-1}\, dg_0$, and similarly write the left invariant Maurer–Cartan 1-form on H as $h^{-1}\,dh$. Clearly $\Omega - g_0^{-1}\,dg_0$ vanishes on the fibers of $H \times G_0$, so $\Omega - g_0^{-1}\,dg_0$ is a multiple of $h^{-1}\,dh$. Lets write it as

$$\Omega = g_0^{-1}\,dg_0 + \operatorname{Ad}_{g_0}^{-1} C\left(h^{-1}\,dh\right) \tag{1}$$

where $C : H \times G_0 \to \mathfrak{h}^* \otimes \mathfrak{g}$ is a function. Check that C is constant under the left H action and under the right G_0 action, so that C is constant, an element of $\mathfrak{h}^* \otimes \mathfrak{g}$. Moreover, this element, thought of as a linear map $\mathfrak{h} \to \mathfrak{g}$, has image transverse to $\mathfrak{g}_0$. Conversely, suppose that we pick any element $C \in \mathfrak{h}^* \otimes \mathfrak{g}$, whose image is transverse to $\mathfrak{g}_0$. We can construct a left invariant projective connection by equation 1. It is easy to check that it is a projective connection. □

Corollary 6. *For any $A \in \mathfrak{g}$, write $\bar{A}$ for the corresponding element of $\mathfrak{g}/\mathfrak{g}_0$. A left invariant projective connection built from a linear map $C \in \mathfrak{h}^* \otimes \mathfrak{g}$ (with image of C transverse to $\mathfrak{g}_0$) is normal just when*

$$\overline{C([A,B])} = \overline{[C(A), C(B)]},$$

for any $A, B \in \mathfrak{g}$. It is flat just when C is a Lie algebra homomorphism.

Proof. This is an easy calculation, given Ω in equation 1: the curvature is

$$\begin{aligned} d\Omega + \Omega \wedge \Omega &= \kappa \bar{\Omega} \wedge \bar{\Omega} \\ &= \frac{1}{2} \operatorname{Ad}_{g_0}^{-1} \left(\left[C\left(h^{-1}\,dh\right), C\left(h^{-1}\,dh\right) \right] - C\left(\left[h^{-1}\,dh, h^{-1}\,dh \right] \right) \right). \end{aligned}$$

so that

$$\kappa\left(\bar{A}, \bar{B}\right) = [C(a), C(b)] - C\left([a,b]\right)$$

whenever $\bar{C}(a) = \bar{A}$ and $\bar{C}(b) = \bar{B}$, for any $a, b \in \mathfrak{h}$. □

Remark 9. The same approach will determine the isomorphism classes of left invariant Cartan geometries of any type, and their curvature.

We can always identify $\mathfrak{h} = \mathbb{R}^n$, and then we will have

$$C(A) = \begin{pmatrix} -\operatorname{tr} E(A) & F(A) \\ A & E(A) \end{pmatrix},$$

for some unique $E \in \mathfrak{h}^* \otimes \mathfrak{gl}(\mathfrak{h})$ and $F \in \mathfrak{h}^* \otimes \mathfrak{h}^*$. This is normal just when

$$\left(E(A) + \operatorname{tr} E(A)\right) B - \left(E(B) + \operatorname{tr} E(B)\right) A = [A, B].$$

Its curvature is given by

$$\kappa(A,B) = \begin{pmatrix} \kappa^0_0 & \kappa^0_\bullet \\ \kappa^\bullet_0 & \kappa^\bullet_\bullet \end{pmatrix}$$

where

$$\begin{aligned} \kappa^0_0 &= F(A)B - F(B)A + \operatorname{tr} E([A,B]) \\ \kappa^0_\bullet &= F(A)\left(E(B) + \operatorname{tr} E(B)\right) - F(b)\left(E(A) + \operatorname{tr} E(A)\right) \\ \kappa^\bullet_0 &= \left(E(A) + \operatorname{tr} E(A)\right) B - \left(E(B) + \operatorname{tr} E(B)\right) A - [A, B] \\ \kappa^\bullet_\bullet &= A\,F(B) - B\,F(A) + [E(A), E(B)] - E\left([A,B]\right). \end{aligned}$$

10.1. **Left invariant affine connections.**

Proposition 2. *The set of left invariant affine connections on a Lie group H of dimension n with Lie algebra $\mathfrak{h}$ is invariantly identified with $\mathfrak{h}^* \otimes \mathfrak{gl}(\mathfrak{h})$.*

Proof. Take any left invariant connection on the tangent bundle of H, and identify it as usual with a connection 1-form on FH. Let $\pi : FH \to H$ be the obvious projection map. Define the soldering 1-form $\omega \in \Omega^1(FH) \otimes \mathbb{R}^n$ by $v \lrcorner \omega_u = u(\pi'(u)v)$. It is easy to check that $r_g^* \omega = g^{-1}\omega$, for $g \in \mathrm{GL}(n, \mathbb{R})$. Then the connection 1-form $\gamma \in \Omega^1(FH) \otimes \mathfrak{gl}(n, \mathbb{R})$ transforms in the adjoint representation under right $\mathrm{GL}(n, \mathbb{R})$ action: $r_g^* \gamma = \mathrm{Ad}_g^{-1} \gamma$, and $d\omega + \gamma \wedge \omega = \frac{1}{2} T\omega \wedge \omega$ is the torsion of the connection.

Pick a point of FH, say u_0, above the point $1 \in H$; so $u_0 : \mathfrak{h} \to \mathbb{R}^n$. Fixing this identification, we can say that $\omega \in \Omega^1(FH) \otimes \mathfrak{h}$, and that $\gamma \in \Omega^1(FH) \otimes \mathfrak{gl}(\mathfrak{h})$. The map $(h, g) inH \times \mathrm{GL}(\mathfrak{h}) \mapsto h_* g^{-1} u_0 \in FH$ is a diffeomorphism. So identify FH with $H \times \mathrm{GL}(\mathfrak{h})$ by this diffeomorphism. This identifies $\omega_{(h,g)} = g^{-1} h^{-1}\, dh$, clearly. Moreover, $\gamma = g^{-1}\, dg + \mathrm{Ad}_g^{-1} \Gamma\left(h^{-1}\, dh\right)$ for a unique constant choice of $\Gamma \in \mathfrak{h}^* \otimes \mathfrak{gl}(\mathfrak{h})$. Moreover, reversing our steps ensures that all left invariant connections on the tangent bundle occur in this manner, for a unique choice of Γ, which can be selected arbitrarily. □

Example 7. For example, for any constant, we can take the choice $\Gamma = a\,\mathrm{ad}$, i.e. $\Gamma(A) = a\,\mathrm{ad}_A \in \mathfrak{gl}(\mathfrak{h})$, giving a canonical choice of connection to the tangent bundle of any Lie algebra. In particular, we can take $\Gamma = 0$. Taking $\Gamma = 0$ will give geodesic flow $g(t) = g(0), h(t) = h(0)\, e^{tg(0)A}$, for any constant $A \in \mathfrak{h}$, so a complete connection.

Example 8. If we pick any nondegenerate quadratic form on the Lie algebra $\mathfrak{h}$, then we can define ad^t by

$$\left\langle \mathrm{ad}_A^t B, C \right\rangle = \langle B, \mathrm{ad}_A C \rangle$$

for $A, B, C \in \mathfrak{h}$, and define ad' by

$$\mathrm{ad}'_A B = \mathrm{ad}_B^t A.$$

Then the Levi–Civita connection of the induced left invariant Riemannian metric on H is given by $\Gamma = \frac{1}{2}\left(\mathrm{ad} - \mathrm{ad}^t - \mathrm{ad}'\right)$.

If the metric is bi-invariant, then $\mathrm{ad}^t = -\,\mathrm{ad}$ and $\mathrm{ad}' = \mathrm{ad}$, so one has $\Gamma = \frac{1}{2}\,\mathrm{ad}$.

Corollary 7. *A left invariant connection on the tangent bundle of a Lie group is torsion-free just when $\Gamma = \frac{1}{2}\,\mathrm{ad} + S$, where $S \in \mathrm{Sym}^2(\mathfrak{h})^* \otimes \mathfrak{h}$.*

Proof. One easily calculates $d\omega + \gamma \wedge \omega$, and finds that Γ gives a torsion-free connection just when $\Gamma(A)B - \Gamma(B)A - [A, B]$ is symmetric in A and B. The torsion of the connection is

$$\frac{1}{2} T\omega \wedge \omega = g^{-1}\left(\Gamma\left(h^{-1}\, dh\right) \wedge h^{-1}\, dh - \frac{1}{2}\left[h^{-1}\, dh, h^{-1}\, dh\right]\right)$$

so that

$$T(A, B) = \Gamma(A)B - \Gamma(B)A - [A, B].$$

□

Corollary 8. *A torsion-free left invariant connection corresponds to a choice of $S \in \mathrm{Sym}^2(\mathfrak{h})^* \otimes \mathfrak{h}$ invariant under the adjoint action.*

Corollary 9. *A left invariant connection on the tangent bundle of a Lie group is flat just when Γ is a Lie algebra morphism.*

Proof. It is easy to compute that the curvature is

$$\begin{aligned} d\gamma + \frac{1}{2}[\gamma,\gamma] &= \frac{1}{2}\kappa\omega\wedge\omega \\ &= \frac{1}{2}\operatorname{Ad}_g^{-1}\left(\left[\Gamma\left(h^{-1}\,dh\right),\Gamma\left(h^{-1}\,dh\right)\right] - \Gamma\left(\left[h^{-1}\,dh, h^{-1}\,dh\right]\right)\right), \end{aligned}$$

so that

$$\kappa(A,B) = \left[\Gamma(A),\Gamma(B)\right] - \Gamma\left([A,B]\right).$$

□

Proposition 3. *If $\Gamma \in \mathfrak{h}^* \otimes \mathfrak{gl}(\mathfrak{h})$ determines a left invariant affine connection on a Lie group H of dimension n with Lie algebra $\mathfrak{h}$, then*

$$C(A) = \begin{pmatrix} -\frac{1}{n+1}\operatorname{tr}\Gamma(A) & \frac{1}{2(n+1)}\operatorname{tr}\Gamma\circ\operatorname{ad}_A \\ A & \Gamma(A) - \frac{1}{n+1}\operatorname{tr}\Gamma(A) \end{pmatrix}$$

determines the corresponding projective connection, which is normal just when Γ is torsion-free.

Proof. One easily calculates out the geodesic equation, to see that it agrees, and checks that the curvature is suitably trace-free. □

Lemma 13. *Let $\mathbb{B}$ be the Killing form of a Lie algebra $\mathfrak{h}$:*

$$\mathbb{B}(A,B) = \operatorname{tr}\operatorname{ad}_A\operatorname{ad}_B.$$

The symmetrized Ricci curvature of the natural torsion-free connection given by $\Gamma = \frac{1}{2}\operatorname{ad}$ (which is the Levi–Civita connection of the Killing form metric on any semisimple Lie group) is $r = -\frac{1}{4}\mathbb{B}$.

Proof. Pick a basis of $\mathfrak{h}$, say $e_1, e_2, \dots, e_n$. Suppose that the structure constants are c^i_{jk}, so that $[e_i,e_j] = c^k_{ij}e_k$. Then $\mathbb{B}_{ij} = c^k i\ell c^\ell_{jk}$. The curvature is

$$\begin{aligned} K^i_{jk\ell} &= \left[\frac{1}{2}\operatorname{ad}_{e_k}, \frac{1}{2}\operatorname{ad}_{e_\ell}\right] - \frac{1}{2}\operatorname{ad}_{[e_k,e_\ell]} \\ &= \frac{1}{4}c^i_{km}c^m_{\ell j} - \frac{1}{4}c^i_{\ell m}c^m_{kj} - \frac{1}{2}c^m_{k\ell}c^i_{mj}. \end{aligned}$$

Therefore the symmetrized Ricci tensor is

$$\begin{aligned} K_{j\ell} &= \frac{1}{2}\left(K^i_{ji\ell} + K^i_{\ell ij}\right) \\ &= \frac{1}{2}\left(\frac{1}{4}c^i_{im}c^m_{\ell j} - \frac{1}{4}c^m_{\ell i}c^i_{jm} + \frac{1}{4}c^i_{im}c^m_{j\ell} - \frac{1}{4}c^m_{ji}c^i_{\ell m}\right) \\ &= -\frac{1}{4}c^m_{ji}c^i_{\ell m} \\ &= -\frac{1}{4}\mathbb{B}_{j\ell}. \end{aligned}$$

□

Example 9. Consider this same natural torsion-free connection given by $\Gamma = \frac{1}{2}\,\mathrm{ad}$. Its geodesics are precisely the one-parameter subgroups (an easy calculation). Its symmetrized Ricci curvature is the Killing form, which is bi-invariant. Therefore the Ricci curvature in the direction of a given geodesic is constant. This ensures that this torsion-free connection is projectively complete just precisely when the Killing form is positive definite, i.e. precisely on the compact semisimple Lie groups.

Example 10. For example, $\mathrm{SL}\,(2,\mathbb{R})$ has projectively complete geodesics given precisely by the subgroups conjugate to $\mathrm{SO}\,(2)$, and has hyperbolic geodesics in the tangent directions of the 2-dimensional subgroups, and parabolic geodesics precisely in the directions of nilpotent elements, i.e. in directions conjugate to the subgroup generated by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

There is no other bi-invariant torsion-free connection on $\mathrm{SL}\,(2,\mathbb{R})$, because (by Clebsch–Gordan) there are no nonzero elements of $\mathrm{Sym}^2\left(\mathfrak{sl}\,(2,\mathbb{R})\right)^* \otimes \mathfrak{sl}\,(2,\mathbb{R})$ fixed under $\mathrm{SL}\,(2,\mathbb{R})$.

Example 11. Looking at the representations of $\mathfrak{sl}\,(2,\mathbb{R})$, we see that the biinvariant projective connections on $\mathrm{SL}\,(2,\mathbb{R})$ are precisely those of the form

$$C(A) = \begin{pmatrix} 0 & q\,A^* \\ A & p\,\mathrm{ad}_A \end{pmatrix}$$

where $A^*(B) = \mathbb{B}(A,B)$ is the dual covector in the Killing form, and p and q are arbitrary constants. The curvature is

$$\kappa(A,B) = \begin{pmatrix} 0 & 2pq\,\left[A,B\right]^* \\ (2p-1)\,\left[A,B\right] & q\left(A \otimes B^* - B \otimes A^*\right) + p(p-1)\,\mathrm{ad}_{[A,B]} \end{pmatrix}.$$

This projective connection is torsion-free just when $p = \frac{1}{2}$. Its symmetrized Ricci curvature is

$$r(A,B) = \left((n-1)q + p(p-1)\right)\mathbb{B}(A,B).$$

However, our theorem on Ricci curvature only applies when the projective connection is normal, and this happens only for the example we have already calculated. To see what is at issue more clearly, lets write the equations of the geodesic flow. First, for $g_0 \in G_0$, write

$$g_0 = \begin{pmatrix} a & b^* \\ 0 & c \end{pmatrix}.$$

The equations of geodesic flow, expressed in $\Omega = g_0^{-1}\,dg_0 + \mathrm{Ad}_{g_0}^{-1}\,C\left(h^{-1}\,dh\right)$, become

$$\begin{aligned} da &= \langle b, A\rangle \\ db &= -\frac{q}{\det c} c^t cA \\ dc &= -\frac{1}{\det c}\left((cA)\otimes b + p\,\mathrm{ad}_{cA}\,c\right) \\ h^{-1}\,dh &= \frac{1}{\det c} cA, \end{aligned}$$

where A is any fixed element of $\mathfrak{sl}\,(2,\mathbb{R})$. We can see that solving for b and c first should enable us to solve the whole system. Nonetheless, the author cannot see how to solve these ordinary differential equations, or how to estimate the time during which the solutions remain defined.

Remark 10. Blumenthal [2] proves that totally geodesic fiber bundles with projectively complete total space have projectively complete base space. For example, the Hopf fibration ensures projective completeness of complex projective space. But complex projective space has positive Ricci curvature, so we don't actually need Blumenthal's result to see projective completeness.

11. Jacobi vector fields

Consider a family of geodesics in a manifold M with projective connection. Let S be a surface, with a submersion $t : S \to \mathbb{R}$ whose fibers are curves $S_t \subset S$. Map $S \to M$ so that each fiber S_t maps into a geodesic. Let E_S be the union of the bundles E_{S_t}. On E_S,

$$\omega^I = a^I \, dt$$
$$\gamma^I_1 = a^I_1 \, dt.$$

Take the exterior derivative to find

$$\begin{aligned} \bar\nabla \begin{pmatrix} a^I \\ a^I_1 \end{pmatrix} &= d \begin{pmatrix} a^I \\ a^I_1 \end{pmatrix} + \begin{pmatrix} \gamma^I_J & 0 \\ -\delta^I_J \omega_1 & \gamma^I_J - \delta^I_J \gamma^1_1 \end{pmatrix} \\ &= \begin{pmatrix} K^I_{1J} & \delta^I_J \\ K^I_{11J} & 0 \end{pmatrix} \begin{pmatrix} a^J \\ a^J_1 \end{pmatrix} \omega^1 + \begin{pmatrix} \bar\nabla_t a^I \\ \bar\nabla_t a^I_1 \end{pmatrix} dt. \end{aligned}$$

where $\bar\nabla$ represents the covariant derivative in E_S calculated in the coframing $dt, \omega^1, \omega_1, \dots$. Therefore each motion S through geodesics determines, above each geodesic S_0 some functions $\left(a^I, a^I_1\right) : E_{S_0} \to \mathbb{R}^{2(n-1)}$, satisfying the equations

$$d \begin{pmatrix} a^I \\ a^I_1 \end{pmatrix} + \begin{pmatrix} \gamma^I_J & 0 \\ -\delta^I_J \omega_1 & \gamma^I_J - \delta^I_J \gamma^1_1 \end{pmatrix} \begin{pmatrix} a^J \\ a^J_1 \end{pmatrix} = \begin{pmatrix} K^I_{1J} & \delta^I_J \\ K^I_{11J} & 0 \end{pmatrix} \begin{pmatrix} a^J \\ a^J_1 \end{pmatrix} \omega^1. \tag{2}$$

Moreover, under the action of G_+,

$$r_g^* \begin{pmatrix} a^I \\ a^I_1 \end{pmatrix} = \begin{pmatrix} \left(g^{-1}\right)^I_J & 0 \\ \left(g^{-1}\right)^I_J g^0_1 & \left(g^{-1}\right)^I_J g^1_1 \end{pmatrix} \begin{pmatrix} a^J \\ a^J_1 \end{pmatrix}. \tag{3}$$

Definition 12. If $C \subset M$ is a geodesic, and $\left(a^I, a^I_1\right) : E_C \to \mathbb{R}^{2(n-1)}$ is a solution to equations (2), (3), call $\left(a^I, a^I_1\right)$ a *Jacobi vector field* (even if it doesn't arise from any actual variation through geodesics).

Lemma 14. *Suppose that a connected manifold M has a projective connection $E \to M$. Any infinitesimal symmetry of a projective connection $E \to M$ determines and is determined by a unique vector field on M.*

Proof. The equations of an infinitesimal symmetry are $\mathcal{L}_X \Omega = 0$ and $r_{g*} X = X$, X a vector field on E. But then if we let $X^\bullet = X \lrcorner \omega^\bullet$, we find immediately that $X^\bullet$ represents a vector field on M. Moreover, if $X^\bullet = 0$, then $X \lrcorner \Omega : E \to \mathfrak{g}$ is a G_0-equivariant function:

$$r_g^* \left(X \lrcorner \Omega\right) = \operatorname{Ad}_g^{-1} \left(X \lrcorner \Omega\right),$$

but is also invariant under the flow of X. Moreover, for any vector $\vec{A} \in \mathfrak{g}$,

$$\mathcal{L}_{\vec{A}} X \lrcorner \Omega = -\operatorname{Ad}_A X \lrcorner \Omega.$$

Pick any point $e \in E$, and let $A = X(e) \lrcorner \Omega(e)$. Consider the vector field $\vec{A}$. It agrees with X at e, and also satisfies $r_g^* \left(\vec{A} \lrcorner \Omega\right) = \operatorname{Ad}_g^{-1} \left(\vec{A} \lrcorner \Omega\right)$, so $\vec{A}$ agrees with

X up the fiber of e. Moreover, $\vec{A}$ and X have the same brackets with $\vec{B}$ for any $B \in \mathfrak{sl}(n+1)$. Therefore they agree above the path component of M containing $\pi(e)$, and since M is connected we find $X = \vec{A}$. But X is G_0-invariant, so A must be G_0-invariant, i.e $A \in \mathfrak{g}_0$ belongs to the center of $\mathfrak{g}_0$. Check that the center is 0. □

Along a curve S_0, we can construct the normal bundle $\nu S_0 = TM|_{S_0} / TS_0$. Given any section a of the normal bundle, define functions a^I by

$$a^I = a \lrcorner \underline{\omega}^I,$$

which is well defined because $\underline{\omega}^I$ vanishes on TS_0.

Lemma 15. *Given an immersed curve $S_0 \subset M$, and a section a of its normal bundle, the functions a^I satisfy*

$$r_g^* a^I = \left(g^{-1}\right)^I_J a^J$$

for $g \in G_+$. Conversely, given functions satisfying these equations, they determine a section of the normal bundle.

Corollary 10. *A Jacobi vector field determines and is determined by a section of the normal bundle. For a family of geodesics $x : S^1 \times \mathbb{R} \to M$, this section is*

$$a(x) = \left.\frac{\partial x}{\partial t}\right|_{t=0} \mod x_* TS^1,$$

the normal component of the velocity.

Proof. At each point $e \in E_S$, we see that

$$\begin{aligned} \frac{\partial}{\partial t} \lrcorner \underline{\omega}^I &= \frac{\partial}{\partial t} \lrcorner a^I \, dt \\ &= a^I. \end{aligned}$$

□

Let $N \subset G$ be the subgroup acting trivially on the projective line $\mathbb{P}^1$ containing $[e_0]$ and $[e_1]$, and $N_+ = N \cap G_+$. Let $\bar{G} = G/N$ and $\bar{G}_+ = G_+/N_+$.

Given an immersed curve C, the quotient $\bar{E}_C = E_C/N_+ \to C$ is a principal right $\bar{G}_+$-bundle, and when equipped with the 1-form

$$\bar{\Omega} = \Omega \mod \mathfrak{n}_+ = \begin{pmatrix} -\frac{1}{2}\gamma^1_1 & \omega_1 \\ \omega^1 & \frac{1}{2}\gamma^1_1 \end{pmatrix}$$

is a flat projective connection on C.

The sheaf of infinitesimal symmetries on $\tilde{C}$ is the sheaf of solutions of a system of linear ordinary differential equations, so by Picard's theorem the local solutions extend globally, never becoming multivalued because $\tilde{C}$ is simply connected.

Lemma 16. *C is complete just when every infinitesimal symmetry of the projective connection on $\tilde{C}$ is complete.*

Proof. Follows from the classification of projective connections on curves. □

Lemma 17. *If C is complete then the map $\tilde{C} \to \mathbb{P}^1$ preserves and reflects infinitesimal symmetries.*

Proof. This is clear from the classification. Alternatively: the infinitesimal symmetries pullback to infinitesimal symmetries, because $\tilde{C} \to \mathbb{P}^1$ is a covering map. Infinitesimal symmetries comprise a 3 dimensional vector space, so they must match precisely. □

The equations of an infinitesimal symmetry

$$\mathcal{L}_X \begin{pmatrix} \omega^1 \\ \gamma^1_1 \\ \omega_1 \end{pmatrix},$$

if we set

$$\begin{pmatrix} X^1 \\ X^1_1 \\ X_1 \end{pmatrix} = X \lrcorner \begin{pmatrix} \omega^1 \\ \gamma^1_1 \\ \omega_1 \end{pmatrix}$$

give us

$$d \begin{pmatrix} X^1 \\ X^1_1 \\ X_1 \end{pmatrix} = \begin{pmatrix} X^1_1 \omega^1 - X^1 \gamma^1_1 \\ 2X^1 \omega_1 - 2X_1 \omega^1 \\ X_1 \gamma^1_1 - X^1_1 \omega_1 \end{pmatrix}. \tag{4}$$

Lemma 18. *If X is an infinitesimal symmetry, let*

$$\mathbb{B}(X) = 2X_1 X^1 + \frac{1}{2} \left(X^1_1\right)^2.$$

Then $\mathbb{B}(X)$ is a constant.

Proof. This is just the Killing form applied to $X \lrcorner \Omega'$. It is also easy to check by calculating the exterior derivative of $\mathbb{B}(X)$. □

Definition 13. An infinitesimal symmetry X is called *elliptic* if $\mathbb{B}(X) < 0$, *parabolic* if $\mathbb{B}(X) = 0$ and X is not everywhere 0, and *hyperbolic* if $\mathbb{B}(X) > 0$.

Lemma 19. *Every elliptic infinitesimal symmetry has no zeros. Zeros of a hyperbolic infinitesimal symmetry (if there are any zeros) are of order 1. Zeros of a parabolic infinitesimal symmetry (if there are any zeros) are of order precisely 2.*

Proof. Zeros here mean on $\tilde{C}$, so equate upstairs on $\bar{E}_{\tilde{C}}$ to zeros of X^1.

Either use the fact that this lemma holds true on $\mathbb{P}^1$, and local isomorphism of sheaves of infinitesimal symmetries, or more simply note that at a zero:

$$\mathbb{B}(X) = \frac{1}{2} \left(X^1_1\right)^2,$$

which can't be negative, ruling out ellipticity. Moreover, it is positive (hyperbolicity) just when $X^1_1 \neq 0$, i.e. just when $dX^1 \neq 0$, a zero of order 1. Any zero of order higher than 2 would ensure that the differential equations (4) for infinitesimal symmetries have the same initial conditions as the 0 infinitesimal symmetry. □

Lemma 20. *C is complete just when all parabolic infinitesimal symmetries of $\tilde{C}$ are complete.*

Proof. The infinitesimal symmetries form a Lie algebra spanned by the parabolic ones. By Palais' theorem [23], a finite dimensional Lie algebra generated by complete vector fields consists entirely of complete vector fields.

Alternatively, just look again at the classification of projective connections on curves. □

12. Normal projective connections on surfaces

Definition 14. A projective connection is called *normal* if

$$0 = K^i_{jk}$$
$$0 = K^i_{jil}$$

and

$$0 = K^i_{jkl} + K^i_{ljk} + K^i_{klj}.$$

Lemma 21 (Cartan [6])**.** *Given a projective connection, there is a unique normal projective connection with the same unparameterized geodesics.*

Lemma 22. *Let M be a surface with normal projective connection $E \to M$, and C a periodic geodesic. If C has a nonzero Jacobi vector field, which vanishes at some point of C, then C is a complete geodesic.*

Proof. The equations of a Jacobi vector field are identical to the flat case, since the relevant curvature vanishes. Therefore under development the Jacobi vector fields are identified locally with Jacobi vector fields on the model. Any parabolic infinitesimal symmetry on $\widetilde{\mathbb{P}}^1$ has a zero between any two zeros of a Jacobi vector field. But $\tilde{C} \subset \widetilde{\mathbb{P}}^1$ is just an open interval, with the same differential system for Jacobi vector fields, so the same is true on $\tilde{C}$. But this forces every parabolic infinitesimal symmetry to have zeros arbitrarily far along $\tilde{C}$ in both directions, since the Jacobi vector fields have zeros in C, so periodically placed zeros in $\tilde{C}$ arbitrarily far along. This forces the parabolic infinitesimal symmetries to be complete, since the flow of a parabolic infinitesimal symmetry will drive us toward its next zero. □

13. Tameness

We follow LeBrun & Mason [20] closely here; keep in mind that their paper treats projective structures, rather than the more general concept of projective connection, so one has to check that the results quoted below hold, with the same proofs, for arbitrary projective connections. Our aim is to show that every normal projective connection on a surface whose geodesics are all closed has a nonzero Jacobi vector field on each geodesic, i.e. lots of motions through closed geodesics.

Given a projective connection $\pi : E \to M$, define a map $\Pi : E \to \mathbb{P}TM$ by requiring that for any geodesic C, $e \in E_C \mapsto \Pi(e) = T_{\pi(e)}C \in \mathbb{P}TM$.

Lemma 23. *This map Π is well-defined and smooth and identifies $\mathbb{P}TM = E/G_+$. The foliation of E by the flow lines of the geodesic flow descends to a foliation of $\mathbb{P}TM$, whose leaves project to the geodesics in M.*

Proof. To show that Π is well-defined, we have only to show that for each point $e \in E$, there is a geodesic $C \subset M$ with $e \in E_C$. But we can just take the integral manifold of the geodesic exterior differential system in E passing through e, and it will project to an appropriate geodesic C. The geodesic flow line through e projects to an appropriate geodesic. This makes clear the smoothness of Π.

Suppose that we have two points $e_0, e_1 \in E$ with $\Pi(e_0) = \Pi(e_1) = \ell \in \mathbb{P}TM$. Then the integral manifolds E_{C_0} and E_{C_1} of the exterior differential system for geodesics with $e_j \in E_{C_j}$ must project to tangent geodesics: $T_{c_0}C_0 = T_{c_1}C_1$ for some c_0 and c_1. The 1-forms $\underline{\omega}^I(e_j)$ must therefore be linear multiples of one another, which forces e_0 and e_1 to be in the same G_+-orbits, by looking at how the ω^i transform under right G_+-action. Therefore $E_{C_0} = E_{C_1}$. So $\Pi : E/G_+ \to \mathbb{P}TM$ is 1-1. Under right G_0-action, $r_{g_0}\vec{A} = \overrightarrow{\mathrm{Ad}_{g_0}^{-1} A}$ ensuring that Π is onto, since this action acts transitively on semibasic directions. To ensure that the inverse map $\mathbb{P}TM \to E/G_+$ is smooth, take any local section of $E \to E/G_+$, and attach to each point $\ell \in \mathbb{P}TM$ the associated point of E; this is the point satisfying the equations $\bar{\omega}^I(e) = 0$ on ℓ. Once again, examining the right action of G_0, it is easy to check that this point e is uniquely determined and smoothly so. □

Definition 15. A projective connection $E \to M$ is called *tame* if the foliation of $\mathbb{P}TM$ is locally trivial. It is called *Zoll* if all geodesics are embedded closed curves.

Lemma 24 (LeBrun & Mason [20])**.** *Any Zoll projective connection on a compact surface is tame. The only surfaces which admit Zoll projective connections are the sphere and the real projective plane.*

Lemma 25 (LeBrun & Mason [20])**.** *Let M be a surface bearing a Zoll projective connection. The space of unoriented connected geodesics Λ (i.e. the space of leaves of the foliation of $\mathbb{P}TM$) is diffeomorphic to $\mathbb{P}^2$. The map $\mathbb{P}TM \to \Lambda$ is a smooth fiber bundle, taking $T_cC \mapsto C$ (for C any geodesic and $c \in C$).*

Lemma 26. *Let Λ be the space of geodesics of a tame Zoll projective connection M. The map $\mathbb{P}TM \to \Lambda$ linearly identifies Jacobi vector fields with tangent vectors to Λ.*

Proof. Every vector in $T\Lambda$ gives rise to an infinitesimal motion through geodesics, with nonvanishing normal velocity, and therefore by dimension count must account for all of the Jacobi vector fields, since the ordinary differential equation for Jacobi vector fields has well defined initial value problem. □

Corollary 11. *Zoll normal projective connections on surfaces are complete.*

Example 12. Zoll [28] provides the following examples of Zoll metrics: for any odd function $f : [-1, 1] \to (-1, 1)$, with $f(z) = -f(-z)$ and $f(-1) = f(1) = 0$,

$$ds^2 = \frac{(1 + f(z))^2}{1 - z^2}\, dz^2 + \left(1 - z^2\right) d\theta^2$$

is a Zoll metric on the 2-sphere (i.e. all geodesics are embedded and periodic), for $(z, \theta) \in [-1, 1] \times [0, 2\pi]$ longitude and latitude coordinates. The curvature is

$$\kappa = \frac{f(z) + 1 - zf'(z)}{(f(z) + 1)^3}.$$

For instance, if $f(z) \equiv 0$, then $\kappa \equiv 1$ giving the standard metric on the sphere. On the other hand, if $f(z)$ has large first derivative and small value at some point z_0 away from 0 (and, being odd, has the same behaviour at $-z_0$), then we will find negative curvature near $z = z_0$. For example

$$f(z) = \cos\left(\frac{\pi}{2} z\right) e^{-\alpha z^2} \sin\left(2 \arctan\left(\beta\left(z - z_0\right)\right) + 2 \arctan\left(\beta\left(z + z_0\right)\right)\right),$$

with large constants α and β, so that $\beta e^{-\alpha z_0^2}$ is large, and any $z_0 \neq 0$ with $-1 < z_0 < 1$, gives curvature close to 1 near $z = \pm 1$, and negative near $z = \pm z_0$, on a rotationally symmetric surface which is as close as we like to the usual metric on the sphere near the poles. For example, $\alpha = 1, \beta = 1/4, z_0 = 1/2$ already provides two bands of negative curvature. Similarly, we can construct any even number of negative curvature bands, at prescribed locations.

Remark 11. LeBrun & Mason [20] provide explicit examples of Zoll affine connections on the sphere, which must also determine complete projective connections by our results.

Remark 12. It is not clear if there is any simple relationship between projective and affine completeness. It is not known whether torsion-free Zoll affine connections are complete.

14. Conclusions

There is no clear common thread between our examples. Perhaps there is always a positive Ricci curvature affine connection for any Zoll normal projective connection. Perhaps Zoll normal projective connections are always complete. It seems a reasonable conjecture that any projective connection can be perturbed to a projective connection all of whose geodesics are hyperbolic, without altering its unparameterized geodesics. It appears unlikely that a projective connection can be made complete by such a perturbation.

Comment 1. Compare with Burstall and Rawnsley. Their almost complex structure should correspond to something here.

Comment 2. The crucial cutoff is $y'' + qy = 0$ with $q < a/x^2$ for some $a \leq 1/4$ implying finitely many zeroes, while $q > a/x^2$ for some $a > 1/4$ implies infinitely many zeroes. I should somehow be able to make use of this idea.

Comment 3. Need to understand the ρ tensor to explain the relation to Ricci curvature.

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