RATIONAL CURVES AND PARABOLIC GEOMETRIES

BENJAMIN MCKAY

Abstract. The twistor transform of a parabolic geometry has two steps: lift up to a geometry of higher dimension, and then drop to a geometry of lower dimension. The first step is a functor, but the second requires some compatibility conditions. Local necessary conditions were uncovered by Andreas Čap [14]. I prove necessary and sufficient global conditions for complex parabolic geometries: rationality of curves defined by certain ordinary differential equations. I harness Mori's bend–and-break to show that any parabolic geometry on any closed Kähler manifold containing a rational curve is inherited from a parabolic geometry on a lower dimensional closed Kähler manifold. These results yield global theorems on complex ordinary differential equations, holomorphic 2-plane fields on 5-folds, and other complex geometric structures on low dimensional complex manifolds.

Contents

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1. Introduction

All manifolds and maps henceforth are assumed complex analytic, and all Lie groups, algebras, etc. are complex.

1.1. The problem.

A few definitions to get us started:

Definition 1. Let $G_0 \subset G$ be a closed subgroup of a Lie group, with Lie algebras $\mathfrak{g}_0 \subset \mathfrak{g}$. A *Cartan geometry* modelled on G/G_0 on a manifold M is a choice of principal right G_0-bundle $E \to M$, and 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$ called the *Cartan connection*, which satisifies the following conditions:

(1) Denote the right action of $g \in G_0$ on $e \in E$ by $r_g e$. The Cartan connection transforms in the adjoint representation:
$$r_g^* \omega = \mathrm{Ad}_g^{-1} \omega.$$
(2) $\omega_e : T_e E \to \mathfrak{g}$ is a linear isomorphism at each point $e \in E$.
(3) For each $A \in \mathfrak{g}$, define a vector field $\vec{A}$ on E by the equation $\vec{A} \lrcorner \omega = A$. Then the vector fields $\vec{A}$ for $A \in \mathfrak{g}_0$ generate the right G_0 action:
$$\vec{A}(e) = \left. \frac{d}{dt} r_{e^{tA}} e \right|_{t=0}.$$

Example 1. The bundle $G \to G/G_0$ is a Cartan geometry, with Cartan connection $\omega = g^{-1}\, dg$ the left invariant Maurer–Cartan 1-form on G; this geometry is called the *model Cartan geometry*.

Definition 2. If $G_- \subset G_+ \subset G$ are two closed subgroups of a Lie group G, and $E \to M$ is a Cartan geometry modelled on G/G_+, then $E \to E/G_-$ is a Cartan geometry modelled on G/G_- (with the same Cartan connection as $E \to M$) called the *G/G_--lift* of $E \to M$ (or simply the *lift* of $E \to M$). Conversely, $E \to M$ is called the *G/G_+-drop* of $E \to E/G_-$ (or simply to *drop* of $E \to E/G_-$). We will say that a Cartan geometry *drops* if it is isomorphic to the lift of a Cartan geometry on a lower dimensional manifold.

Henceforth, every semisimple Lie algebra/group will be assumed to come equipped with a choice of Borel subalgebra/subgroup (i.e. maximal solvable subalgebra/subgroup containing a Cartan subalgebra/subgroup).

Definition 3. A Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ of a semisimple Lie algebra is called *parabolic* if it contains the Borel subalgebra. A connected Lie subgroup of a semisimple Lie group is called *parabolic* if its Lie algebra is parabolic. (Unlike some authors, we consider a connected semisimple Lie group G to be a parabolic subgroup of itself.) A homogeneous space G/P (with P parabolic and G connected) is called a *rational homogeneous variety*. See Landsberg [55] for more information about rational homogeneous varieties.

Definition 4. A *parabolic geometry* is a Cartan geometry modelled on a rational homogeneous variety.

One motivation for studying parabolic geometries on complex manifolds is the hope that varieties of minimal rational tangents on various Fano manifolds will determine parabolic geometries (see Hwang [38]). Another motivation comes from the programme of Biquard and Mazzeo [8] to study parabolic geometries as conformal infinities of Einstein manifolds.

The main problem we will address: which parabolic geometries drop?

1.2. **The solution.**

Remark 1. Applications to differential equations appear in section 14 on page 34. We now describe the main theorems.

Example 2. Let $G = \mathrm{SO}(n+2, \mathbb{C})$ (for some $n > 2$) and let P be the stabilizer in G of a null line in $\mathbb{C}^{n+2}$. Then G/P is a hyperquadric. The group G is the symmetry group of a unique conformal structure on the hyperquadric. It turns out that any holomorphic conformal structure on any manifold imposes a parabolic geometry on that manifold, modelled on G/P (see Tanaka [72]).

Take $P \subset G$ a parabolic subgroup of a semisimple Lie group. Each node of the Dynkin diagram of G corresponds to a simple root. If the root space of that root lies in the Lie algebra of P, draw that node as a cross ($\times$). Draw all other nodes as dots ($\bullet$). The resulting diagram is called the *Dynkin diagram* of G/P. Each parabolic subgroup of each semisimple Lie group is determined, up to conjugacy, by its Dynkin diagram.

Example 3. A hyperquadric of dimension $2n-1$ is a rational homogeneous variety $\mathbb{P}\mathrm{O}(2n+1, \mathbb{C})/P$, and has Dynkin diagram:

$$\times - \bullet - \cdots - \bullet - \bullet \Rightarrow \bullet$$

Any drawing of the vertices of a Dynkin diagram in crosses and dots picks out a choice of connected complex semisimple Lie group and parabolic subgroup, uniquely up to coverings. The quotient G/P is independent of the coverings. To each Dynkin diagram marked with crosses and dots we associate the category of parabolic geometries modelled on the associated rational homogeneous variety, with arrows the local diffeomorphisms matching parabolic geometries. This category contains a distinguished object G/P.

Lifting parabolic geometries adds crosses to the Dynkin diagram of the model. The lift is bigger in dimension, being a bundle over the original manifold. Lifting is a monomorphism of categories. The "lowest" category contains the entirely dotted Dynkin diagram, corresponding to the model G/G, a point. As we add crosses, this point grows into the various G/P. The highest category is the Dynkin diagram with all nodes crossed, corresponding to the model G/B where B is the Borel subgroup. A parabolic geometry drops if it is isomorphic to a lift of a parabolic geometry on a lower dimensional manifold.

Example 4. Since there is only one cross on the Dynkin diagram of the hyperquadric, a conformal geometry drops just when it drops to a parabolic geometry modelled on

$$\bullet - \bullet - \cdots - \bullet - \bullet \Rightarrow \bullet$$

i.e. G/G (a point). A connected geometry modelled on a point is just a point. In other words, a conformal geometry on a connected manifold drops just when it is lifted from a point, i.e. just when it is isomorphic to the model hyperquadric.

Lifting first, and then dropping to some other category of parabolic geometries, is called a *twistor correspondence*. Picture a twistor correspondence as adding some crosses to the Dynkin diagram of the model, and then taking some crosses away. We will see that every Cartan geometry has a notion of curvature. The local conditions on curvature (computed by Čap) required to drop are called *twistor equations*; they reproduce many famous examples of twistor theory [37]. Characterizing parabolic geometries which drop tells us when a global twistor correspondence is possible in parabolic geometries.

Definition 5. A *Pfaffian system* is a vector subbundle of the cotangent bundle of a manifold. A local section of a Pfaffian system is therefore a locally defined 1-form. A vector v is *integral* for a Pfaffian system if $v \lrcorner \alpha = 0$ for every local section α of the Pfaffian system. An *integral manifold* of a Pfaffian system is an immersed submanifold whose tangent vectors are integral. A *Cauchy characteristic vector* of a Pfaffian system is a tangent vector v for which $0 = v \lrcorner \alpha = v \lrcorner d\alpha$ for every local section α of the Pfaffian system. A Cauchy characteristic submanifold is an immersed submanifold whose tangent vectors are Cauchy characteristic vectors. A Pfaffian system is *holonomic* if every integral vector is a Cauchy characteristic vector.

In this language, the Frobenius theorem says that a Pfaffian system is holonomic just when the manifold it lives on is foliated by integral submanifolds whose dimension is the corank of the Pfaffian system.

If $\phi : X \to Y$ is a smooth submersion of manifolds, and I is a Pfaffian system on Y, then $\phi^* I$ is also a Pfaffian system via the obvious inclusion $\phi^* I \subset \phi^* T^* Y \subset T^* X$. The fibers of $X \to Y$ are Cauchy characteristic submanifolds. The preimages in X of integral manifolds in Y are integral manifolds in X. It is well known (see Bryant et. al. [12] p. 33 corollary 2.3) that a Pfaffian system I on a manifold X is the pullback via a fiber bundle mapping $X \to Y$ with connected fibers just when the fibers of $X \to Y$ are Cauchy characteristic submanifolds.

Remark 2. We will henceforth follow tradition in writing each Pfaffian system as an expression $\eta = 0$ where η is a 1-form valued in a vector space. Such expressions are to be interpreted as meaning that, if η is a V-valued 1-form, with V a vector space, say $\eta = \eta^i e_i$ for some basis $e_1, \dots, e_p$ of V, then the Pfaffian system has $\eta^1, \dots, \eta^p$ as basis of sections. An expression like $\eta = 0$ is natural, since we are concerned only with the integral manifolds of the Pfaffian system, so we think of each Pfaffian system as if it were an equation whose solutions are integral manifolds.

Definition 6. Take G/P a rational homogeneous variety. Let $\mathfrak{p} \subset \mathfrak{g}$ be the Lie algebras of $P \subset G$. Take α a negative root of G. Let $\mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha$ be the Lie subalgebra of $\mathfrak{g}$ spanned by the root spaces of α and $-\alpha$. Let $\mathfrak{b}_\alpha \subset \mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha$ be the Borel subalgebra containing the root space of $-\alpha$, so $\mathfrak{b}_\alpha \subset \mathfrak{p} \cap \mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha$. Let $B_\alpha \subset \operatorname{SL}\left(2,\mathbb{C}\right)_\alpha \subset G$ be the Lie subgroups of G associated to the Lie algebras $\mathfrak{b}_\alpha \subset \mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha \subset \mathfrak{g}$. Let $E \to M$ be a parabolic geometry modelled on G/P, with Cartan connection 1-form ω. The Pfaffian system $\omega = 0 \mod \mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha$ on E is holonomic and has the fibers of $E \to E/B_\alpha$ as Cauchy characteristics, so descends to a Pfaffian system on E/B_α, which we also refer to as $\omega = 0 \mod \mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha$ (even though ω is not defined on E/B_α). An *α-circle* is a maximal connected integral curve of the foliation $\omega = 0 \mod \mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha$ on E/B_α. An α-circle will also be called a *circle associated to α*.

In [58] I argued that this use of the term *circle* is consistent with the Cartan geometry literature, and in particular with the literature in CR-geometry and conformal geometry.

Definition 7. A subset of a complex manifold M is called a *rational curve* if that subset is the image of a nonconstant holomorphic map $\mathbb{P}^1 \to M$.

Theorem 1. *Take $P_- \subset P_+$ parabolic subgroups of a semisimple group G, with Lie algebras $\mathfrak{p}_- \subset \mathfrak{p}_+$. Consider the set of roots of G whose root spaces lie in $\mathfrak{p}_+$ but not in $\mathfrak{p}_-$. A parabolic geometry modelled on G/P_- is lifted from a parabolic geometry modelled on G/P_+ just when the circles associated to those roots are rational curves.*

Corollary 1. *Consider two parabolic subgroups $P, Q \subset G$ of a semisimple group, containing the same Borel subgroup. Let $\langle P \cup Q \rangle \subset G$ be the parabolic subgroup generated by $P \cup Q$. A parabolic geometry modelled on G/P always lifts to one modelled on $G/\left(P \cap Q\right)$. The lift can only drop to a parabolic geometry modelled on G/Q if it drops to a parabolic geometry modelled on $G/\left\langle P \cup Q \right\rangle$.*

Remark 3. In a picture,

$$
\begin{array}{ccccc}
 & & G/\left(P \cap Q\right) & & \\
 & \swarrow & & \searrow & \\
G/P & & & & G/Q \\
 & \searrow & & \swarrow & \\
 & & G/\left\langle P \cup Q \right\rangle, & &
\end{array}
$$

you can always lift up a parabolic geometry modelled on the variety on the left side to one modelled on the variety on the top. You can subsequently drop down to one modelled on the variety on the right side if and only if your original geometry was lifted from one modelled on the variety on the bottom.

Remark 4. This result proves that twistor correspondences in parabolic geometries are trivial, reducing to the obvious observation that you can lift up a geometry modelled on the bottom variety either to one modelled on the variety on the left side or on the right side.

Remark 5. Lets say that a parabolic geometry is *fundamental* if it does not drop. Clearly only fundamental geometries are of any interest, since nonfundamental geometries are more naturally studied directly on the lower dimensional manifold. By this theorem, each parabolic geometry has a unique lowest drop, to a fundamental geometry with a model which we will call G/P^{nat}, (the "natural model") so that the parabolic geometry drops from being modelled on G/P to being modelled on G/Q just if $P \subset Q \subset P^{\text{nat}}$. Fundamental geometries bear no twistor correspondences.

1.3. **Obstruction theory.** The reader might wish for evidence that theorem 1 on the previous page sheds new light. The following theorems provide that evidence, starting with theorem 3.

Definition 8. A curve C in a complex manifold M is *free* if the sheaf $TM|_C$ on C is spanned by its global sections.

Theorem 2. *In a complex manifold with Cartan geometry, every rational curve is free.*

Definition 9. The *canonical bundle* of a complex manifold M is the line bundle $K_M = \det T^*M$, whose local sections are holomorphic volume forms.

Corollary 2. *Suppose that M is complex manifold containing a rational curve, and that the canonical bundle of M admits a metric of nonnegative curvature. Then M bears no holomorphic Cartan geometry.*

Remark 6. A closed complex Kähler manifold is a *Calabi-Yau* manifold if $c_1(TM) = 0$ and the fundamental group is finite. The canonical bundle of a Calabi-Yau manifold becomes trivial (so admits a flat metric) after taking a finite covering space. Most of the known examples of Calabi–Yau manifolds of finite fundamental group contain rational curves (see Yau [75], articles by Katz, Kawamata and Wilson, for example), and therefore admit no holomorphic Cartan geometry. Dumitrescu [26] proved similar obstruction theorems for affine geometries on Calabi–Yau manifolds.

Corollary 3. *If a complex manifold blows down, then it admits no complex Cartan geometry.*

But instead of blowing down, lets drop:

Theorem 3. *If a closed Kähler manifold M contains a rational curve, then every parabolic geometry on M drops to some lower dimensional parabolic geometry (i.e. M is not fundamental).*

Remark 7. Andrei Mustaţă pointed out that this theorem is reminiscent of a speculation of Pandharipande [63] p. 1. Pandharipande's speculation is that rationally connected varieties with all rational curves free might be rational homogeneous varieties.

Example 5. A closed complex manifold is called a *Fano* manifold if the canonical bundle admits a negative curvature metric. Fano manifolds bear rational curves (see [54]). Therefore parabolic geometries on Fano manifolds drop.

Remark 8. This theorem generalizes the Main Theorem of Hwang & Mok [40] p.55. Hwang and Mok used the language of G-structures. Lets say that a rational homogeneous variety G/P is *cominiscule* if $\mathfrak{g}/\mathfrak{p}$ is an irreducible P-module. As explained by Čap [15] (and proven by Tanaka [72] and Čap and Schichl [16]), if

G is a Lie group with a faithful irreducible representation, then Cartan's method of equivalence associates to any G-structure modelled on that representation a parabolic geometry modelled on a cominiscule rational homogeneous variety. Not every parabolic geometry with such a model arises in this way; those which do are called *normal* and can be characterized by certain curvature equations, far too complicated to write down here (see Čap and Schichl [16] for complete details). Thereby one translates the Main Theorem of Hwang and Mok into the language of parabolic geometries. One also needs to recall that the irreducible Hermitian symmetric spaces (which Hwang and Mok refer to) are precisely the cominiscule rational homogeneous varieties. After this translation, Hwang and Mok's result is precisely our theorem 3, but Hwang and Mok require the additional hypotheses

(1) that M is a uniruled smooth projective variety and
(2) that the parabolic geometry is normal and
(3) that the parabolic geometry is modelled on a cominiscule rational homogeneous variety.

Corollary 4. *The only closed Kähler manifold which bears a holomorphic conformal geometry, and contains a rational curve, is the hyperquadric, and the only conformal geometry it bears is the usual flat one.*

Remark 9. This strengthens the work of Belgun [6], who proved that complex null geodesics cannot be rational except on conformally flat manifolds.

Definition 10. A complex manifold is *rationally connected* if any two points are contained in a rational curve.

Corollary 5. *The only rationally connected closed Kähler manifolds which admit parabolic geometries are the rational homogeneous varieties.*

Corollary 6. *Every parabolic geometry on a rational homogeneous variety is isomorphic to its model.*

Remark 10. Caution: there are biholomophic rational homogeneous varieties $G_0/P_0 = G_1/P_1$ for which G_0 and G_1 do *not* have isomorphic Lie algebras (see [70]). So there are rational homogeneous varieties with more than one parabolic geometry, but these parabolic geometries must be isomorphic to their models.

Example 6. Smooth projective hypersurfaces of degree 1 and 2 are rational homogeneous varieties, and thus can only have parabolic geometries isomorphic to their models. Smooth cubic hypersurfaces of dimensions 2, 3 or 4 contain rational curves, so they cannot bear fundamental geometries. For instance, every cubic surface is the projective plane blown up at 6 points. The exceptional divisors of those 6 points are not free. Therefore no cubic surface has any complex Cartan geometry.

Example 7. All closed curves admit parabolic geometries modelled on the projective line. These parabolic geometries are parameterized by the quadratic differentials; see Gunning [36] for the complete classification.

Example 8. Lets call a complex manifold a *ball quotient* (or a *torus quotient*) to mean that it admits an unramified covering map from a ball in complex Euclidean space (or a torus). The closed surfaces which admit projective connections are $\mathbb{P}^2$, ball quotients, torus quotients, Hopf surfaces, Inoue surfaces, primary Kodaira surfaces, or principal bundles of elliptic curves over a curve of genus at least 2 (see

[49]). They each admit flat projective connections. The model, $\mathbb{P}^2$, admits only the model projective connection, and torus quotients only admit translation invariant projective connections (completely classified below, see example 13). For any other surface in the list, no other projective connections are known, besides the standard flat projective connection in each example. (There is only one *normal* projective connection on any ball quotient; see corollary 16 on page 31).

Example 9. The only parabolic geometries on surfaces are projective connections and conformal connections; the surfaces are classified; see Kobayashi and Ochiai [49, 50, 52]. The connections are not. Klingler [44] made a start on the classification.

Example 10. The classification of closed complex 3-folds admitting parabolic geometries is still open, even among the smooth projective 3-folds; see theorem 24 on page 41.

Remark 11. Some theorems in this paper cannot be stated without a lengthy discussion of parabolic geometries, so will appear at the end of the paper. Many of these theorems at the end of the paper require a hypothesis on curvature ("regularity") which is satisfied in all of the examples of parabolic geometries that arise in geometric constructions. Regularity is unfortunately difficult to define precisely. Moreover, some theorems will require the use of Mori's bend-and-break method (which is explained by Kollár and Mori [54]).

2. Cartan geometries

Subsections 2.1 to 2.4 review known results about Cartan geometries. All results of this section hold equally well for real or complex manifolds and Lie groups.

2.1. Curvature.

Let ω be the Cartan connection of a Cartan geometry modelled on a homogeneous space G/G_0.

Definition 11. The *curvature* form of a Cartan geometry is

$$\nabla\omega = d\omega + \frac{1}{2}\left[\omega, \omega\right].$$

Let $\bar{\omega} = \omega + \mathfrak{g}_0 \in \Omega^1(E) \otimes (\mathfrak{g}/\mathfrak{g}_0)$. The curvature form is semibasic, so can be written

$$\nabla\omega = \kappa\bar{\omega} \wedge \bar{\omega},$$

and $\kappa : E \to \mathfrak{g} \otimes \Lambda^2(\mathfrak{g}/\mathfrak{g}_0)^*$ is called the *curvature*. A Cartan geometry is called *flat* if $\kappa = 0$.

Example 11. The model is flat. The Frobenius theorem tells us that a Cartan geometry is locally isomorphic to its model just when it is flat.

Remark 12. A Cartan connection is not a connection, unless it is modelled on a point, because ω is valued in $\mathfrak{g}$, *not* in $\mathfrak{g}_0$.

Example 12. Since the curvature 2-form is semibasic, every Cartan geometry on a curve is flat.

Example 13. Pick $G_0 \subset G$ a closed subgroup of a Lie group, with Lie algebras $\mathfrak{g}_0 \subset \mathfrak{g}$. Take any linear subspace $\Pi \subset \mathfrak{g}$ transverse to $\mathfrak{g}_0$, so that $\Pi \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_0$ is a linear isomorphism. Let $\Gamma : \mathfrak{g}/\mathfrak{g}_0 \to \Pi$ be the inverse of that isomorphism. Let

$M = \mathfrak{g}/\mathfrak{g}_0, E = M \times G_0$. Writing elements of E as $(x, h) \in E$, with $x \in \mathfrak{g}/\mathfrak{p}$ and $h \in G_0$, let

$$\omega = h^{-1}\, dh + \mathrm{Ad}_h^{-1} \left(\Gamma \left(dx\right)\right),$$

a translation invariant Cartan geometry on $\mathfrak{g}/\mathfrak{g}_0$. Every translation invariant Cartan geometry on any affine space is isomorphic to this one for some choice of Π.

A translation invariant Cartan geometry is flat just when $\Pi \subset \mathfrak{g}$ is an abelian subgroup. In particular, unless $\mathfrak{g}$ is abelian or $\mathfrak{g}_0 = \mathfrak{g}$, we can always find a subspace $\Pi \subset \mathfrak{g}$ for which the induced Cartan geometry is *not* flat. By translation invariance, this Cartan geometry induces a curved Cartan geometry on any torus quotient.

2.2. **Kernel.**

Definition 12. The *kernel* K of a homogeneous space G/G_0 is the largest normal subgroup of G contained in G_0:

$$K = \bigcap_{g \in G} g G_0 g^{-1}.$$

As G-spaces, $G/G_0 = (G/K)/(G_0/K)$.

Lemma 1 (Sharpe [68])**.** *Suppose that K is the kernel of G/G_0, with Lie algebra $\mathfrak{k}$, and that $E \to M$ is a Cartan geometry modelled on G/G_0, with Cartan connection ω. Let $E' = E/K$ and $\omega' = \omega + \mathfrak{k} \in \Omega^1(E) \otimes (\mathfrak{g}/\mathfrak{k})$. Then ω' drops to a 1-form on E', and $E' \to M$ is a Cartan geometry, called the* reduction *of E.*

Definition 13. We will say that a Cartan geometry is *reduced* if $K = 1$. A Cartan geometry is reduced just if its model is.

Remark 13. A Cartan geometry may have flat reduction without itself being flat.

2.3. **Flat Cartan geometries.**

Definition 14. If $G_0 \subset G$ is a closed subgroup of a Lie group and $\Gamma \subset G$ is a discrete subgroup, call the (possibly singular) space $\Gamma \backslash G/G_0$ a *double coset space.* If $\Gamma \subset G$ acts on the left on G/G_0 freely and properly, call the smooth manifold $\Gamma \backslash G/G_0$ a *locally Klein geometry.*

Proposition 1 (Sharpe [68])**.** *A locally Klein geometry $\Gamma \backslash G/G_0$ has a Cartan geometry $\Gamma \backslash G \to \Gamma \backslash G/G_0$ with the left invariant Maurer–Cartan 1-form of G as Cartan connection.*

Definition 15. A Cartan geometry is *complete* if all of the vector fields $\vec{A}$ are complete.

Theorem 4 (Ehresmann [28])**.** *Take two simply connected manifolds equipped with complete analytic Cartan geometries. An isomorphism between connected and simply connected open subsets extends uniquely to a global isomorphism.*

Definition 16. We will say that two homogeneous spaces G/G_0 and G'/G_0 are similar if the Lie group G' has the same Lie algebra $\mathfrak{g}$ as G, and contains G_0 as a closed subgroup, with the same Lie algebra inclusion $\mathfrak{g}_0 \subset \mathfrak{g}$.

A Cartan geometry modelled on G/G_0 is also a Cartan geometry modelled on G'/G_0, for any similar homogeneous space. Nonetheless we will think of a Cartan geometry as becoming a different Cartan geometry when we change its model.

Corollary 7 (Sharpe [68])**.** *A Cartan geometry on a connected manifold is complete and flat just when, after perhaps a change of similar models, it becomes locally Klein.*

Remark 14. For example: the Hopf fibration $S^1 \to S^3 \to S^2$ is a complete flat Cartan geometry modelled on $\mathrm{SO}(3) / \mathrm{SO}(2) = S^2$. The Hopf fibration is locally Klein when we change model to $\mathrm{SU}(2) / \mathrm{SO}(2)$, but is not locally Klein for the model $\mathrm{SO}(3) / \mathrm{SO}(2)$.

Remark 15. The reduction of a complete Cartan geometry is complete, but the converse is not always true.

Definition 17. A group Γ *defies* a group G if every morphism $\Gamma \to G$ has finite image. For example, if Γ is finite, or G is finite, then Γ defies G.

Theorem 5 (McKay [58])**.** *A flat Cartan geometry, modelled on a homogeneous space G/G_0, on a closed connected manifold M with fundamental group defying G, is locally Klein (perhaps after change of similar models). If M and G/G_0 are both simply connected, then the locally Klein geometry is isomorphic to the model.*

2.4. **Local solution.**

Definition 18. Pick a Cartan geometry $E \to M$ modelled on G/G_0. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the Lie algebras of $G_0 \subset G$. For each $e \in E$, the *local apparent structure algebra* at e is the set $\mathfrak{g}_0^{\text{app}}(e)$ of $A \in \mathfrak{g}$ for which $\kappa(A, B) = 0$, for all $B \in \mathfrak{g}$. The *apparent structure algebra* is

$$\mathfrak{g}_0^{\text{app}} = \bigcap_{e \in E} \mathfrak{g}_0^{\text{app}}(e).$$

Lemma 2. *Let $E \to M$ be a Cartan geometry modelled on G/G_0. An element $A \in \mathfrak{g}$ belongs to $\mathfrak{g}_0^{\text{app}}(e)$ just when for any $B \in \mathfrak{g}$,*

$$\overrightarrow{[A, B]} = \left[\vec{A}, \vec{B}\right]$$

at the point $e \in E$.

Proof. We evaluate $\left(\vec{A} \wedge \vec{B}\right) \lrcorner\, d\omega$ two ways. The first way:

$$\begin{aligned} \left(\vec{A} \wedge \vec{B}\right) \lrcorner\, d\omega &= \mathcal{L}_{\vec{A}} \left(\vec{B} \lrcorner\, \omega\right) - \mathcal{L}_{\vec{B}} \left(\vec{A} \lrcorner\, \omega\right) - \left[\vec{A}, \vec{B}\right] \lrcorner\, \omega \\ &= - \left[\vec{A}, \vec{B}\right] \lrcorner\, \omega. \end{aligned}$$

The second:

$$\begin{aligned} \left(\vec{A} \wedge \vec{B}\right) \lrcorner\, d\omega &= \left(\vec{A} \wedge \vec{B}\right) \lrcorner \left(-\frac{1}{2} [\omega, \omega] + \kappa \bar{\omega} \wedge \bar{\omega}\right) \\ &= - [A, B] + \kappa(A, B). \end{aligned}$$

Therefore

$$\overrightarrow{[A, B]} = \left[\vec{A}, \vec{B}\right] + \overrightarrow{\kappa(A, B)}.$$

∎

Lemma 3. *The apparent structure algebra is a Lie subalgebra of $\mathfrak{g}$ containing the structure algebra.*

Proof. The local apparent structure algebra is clearly a vector space, and contains the structure algebra $\mathfrak{g}_0$, so we have only to show that it is closed under Lie bracket. For any $A, B \in \mathfrak{g}_0^{\text{app}}(e)$, and any $C \in \mathfrak{g}$,

$$\begin{aligned}
\kappa\left([A, B], C\right) &= -\left[\overrightarrow{[A, B]}, \vec{C}\right] \lrcorner\, \omega - [[A, B], C] \\
&= -\left[\left[\vec{A}, \vec{B}\right], \vec{C}\right] \lrcorner\, \omega - [[A, B], C] \\
&= -\left(\left[\vec{A}, \left[\vec{B}, \vec{C}\right]\right] + \left[\left[\vec{A}, \vec{C}\right], \vec{B}\right]\right) \lrcorner\, \omega - [[A, B], C] \\
&= -\left(\overrightarrow{[A, [B, C]]} + \overrightarrow{[[A, C], B]}\right) \lrcorner\, \omega - [[A, B], C] \\
&= [A, [B, C]] + [[A, C], B] - [[A, B], C] \\
&= 0.
\end{aligned}$$

□

Theorem 6 (Čap [14]). *Let $G/G_- \to G/G_+$ be homogeneous spaces. A Cartan geometry $E \to M$ modelled on G/G_- is locally isomorphic near any point of M to the lift of a Cartan geometry modelled on G/G_+ just when $\mathfrak{g}_+$ lies in the apparent structure algebra.*

Proof. Locally quotient by the $\mathfrak{g}_+$-action; see Čap [14] for details. □

2.5. **The natural structure algebra.** We now begin to develop our new global tools for the study of Cartan geometries.

Definition 19. Let $E \to M$ be a Cartan geometry modelled on G/G_0, with apparent structure algebra $\mathfrak{g}_0^{\text{app}}$. Let $\mathfrak{g}_0^{\text{nat}} \subset \mathfrak{g}_0^{\text{app}}$ (the *natural structure algebra*) be the set of vectors $A \in \mathfrak{g}_0^{\text{app}}$ for which the vector field $\vec{A}$ on E is complete. (Recall $\vec{A}$ is defined by the equation $\vec{A} \lrcorner\, \omega = A$.)

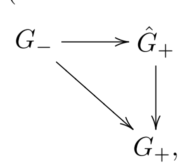
Lemma 4. $\mathfrak{g}_0^{\text{nat}} \subset \mathfrak{g}_0^{\text{app}}$ *is a Lie subalgebra.*

Proof. Let $\mathfrak{g}_0'$ be the Lie subalgebra of $\mathfrak{g}_0^{\text{app}}$ generated by $\mathfrak{g}_0^{\text{nat}}$. By Palais' theorem [62], since $\mathfrak{g}_0'$ is finite dimensional and generated by complete vector fields, $\mathfrak{g}_0'$ consists entirely of complete vector fields. Therefore $\mathfrak{g}_0' = \mathfrak{g}_0^{\text{nat}}$. □

2.6. **Global results on the twistor problem.**

Proposition 2. *Suppose that $G/G_- \to G/G_+$ are homogeneous spaces, and $G_- \subset G_+ \subset G$ are connected and have Lie algebras $\mathfrak{g}_- \subset \mathfrak{g}_+ \subset \mathfrak{g}$. Suppose that G_+/G_- is simply connected. Then every Cartan geometry $E \to M$ modelled on G/G_- which has natural structure algebra containing $\mathfrak{g}_+$ has a unique right G_+-action extending the G_--action and the $\mathfrak{g}_+$-action.*

Proof. On each orbit of $\mathfrak{g}_+$ in E, the $\mathfrak{g}_+$-action exponentiates to the action of a covering group of G_+. This action restricts to the given action of G_-. The covering groups $\hat{G}_+$ of G_+ are parameterized by the subgroups of $\pi_1(G_+)$. Those which contain G_- as a subgroup (i.e. with a commutative diagram



of Lie group morphisms, the top one being a monomorphism) are parameterized by the subgroups of $\pi_1(G_+)/\pi_1(G_-) = \pi_1(G_+/G_-)$. $\square$

It is not clear whether or not the action of G_+ turns $E \to E/G_+$ into a principal G_+-bundle; nonetheless we can see how the Cartan connection behaves.

Lemma 5. *Under the hypotheses of the previous proposition, conditions (1), (2), (3) of the definition of a Cartan connection (definition 1 on page 2) are satisfied (with $\mathfrak{g}_0, G_0$ replaced by $\mathfrak{g}_+, G_+$).*

Theorem 7. *Suppose that $G/G_+ \to G/G_-$ are homogeneous spaces, and $G_- \subset G_+ \subset G$ are connected and have Lie algebras $\mathfrak{g}_- \subset \mathfrak{g}_+ \subset \mathfrak{g}$. Suppose that G_+/G_- is closed and simply connected. Suppose that the only discrete subgroup $\Gamma \subset G_+$ acting freely and properly on G_+/G_- is $\Gamma = 1$. Then a Cartan geometry modelled on G/G_- drops to a Cartan geometry modelled on G/G_+ just when the natural structure algebra of that Cartan geometry contains $\mathfrak{g}_+$.*

Proof. Let the Cartan geometry be $E \to M$. Suppose that the natural structure algebra contains $\mathfrak{g}_+$. If $\mathfrak{g}_+ = \mathfrak{g}$, then the result follows immediately from corollary 7 on page 10. So lets assume that $\mathfrak{g}_+ \subset \mathfrak{g}$ is a proper Lie subalgebra.

The G_- action on E is free and proper, and therefore the G_- action on each G_+-orbit $X = eG_+ \subset E$ is free and proper. So $X \to X/G_-$ is a smooth submersion. Hence a diagram of smooth G_+-equivariant submersions of G_+-homogeneous spaces:

$$\begin{array}{ccc} G_+ & \longrightarrow & X \\ \downarrow & & \downarrow \\ G_+/G_- & \longrightarrow & X/G_-. \end{array}$$

Clearly the stabilizer subgroup $\Gamma \subset G_+$ of a point $x \in X$ is a discrete subgroup, since the stabilizer Lie algebra is trivial. Therefore $G_+ \to X$ is a covering map, and so $G_+/G_- \to X/G_-$ is a covering map. But therefore $G_+/G_- \to X/G_-$ is a diffeomorphism, and therefore $G_+ \to X$ is a diffeomorphism. So G_+ acts freely on X and therefore freely on E.

The image down in M of each G_+-orbit in E is diffeomorphic to G_+/G_-, so is compact. Therefore each G_+-orbit in E is a closed set and a submanifold diffeomorphic to G_+. Take a convergent sequence of points $e_n \to e \in E$ and a convergent sequence of points $e_n g_n \to e' \in E$ with g_n a sequence in G_+. Write the map $E \to M$ as $\pi : E \to M$. The set of points $Z = \{e_1, e_2, \dots\} \cup \{e\}$ is compact, so the product $Z \times (G_+/G_-)$ is compact. The image of $Z \times (G_+/G_-)$ in M under $(z, gG_-) \mapsto \pi(zg)$ is compact. This image is the set

$$\pi(eG_+) \cup \bigcup_n \pi(e_n G_+),$$

which is therefore compact and contains $\pi(e_n)$ and $\pi(e)$ and $\pi(e_n g_n)$ and therefore contains $\pi(e')$. Because the G_+-orbits in E are submanifolds of lower dimension than E, we can arrange by small perturbations that all points e_n lie in different G_+-orbits, distinct from the G_+-orbit of e. So $\pi(e') \in \pi(eG_+)$, i.e. $e' \in eG_+$, say $e' = eg$.

Let

$$R = \{(e, eg) \mid e \in E \text{ and } g \in G_+\}.$$

We have proven that R is a closed subset in $E \times E$. Clearly R is an immersed submanifold of $E \times E$, since the map $(e, g) \mapsto (e, eg)$ is a local diffeomorphism to R. Moreover, this map is 1-1, since G_+ acts freely. Therefore $R \subset E \times E$ is a closed set and a submanifold. Therefore E/G_+ admits a unique smooth structure as a smooth manifold for which $E \to E/G_+$ is a smooth submersion (see Abraham and Marsden [1] p. 262).

Because $E \to E/G_+$ is a smooth submersion, we can construct a local smooth section near any point of E/G_+, say $s : U \to E$, for some open set $U \subset E/G_+$. Let $\rho : E \to E/G_+$ be the quotient map by the G_+-action. Define

$$\phi : U \times G_+ \to \rho^{-1}U,$$
$$\phi(u, g) = s(u)g.$$

This map ϕ is a local diffeomorphism, because ϕ is G_+-equivariant and $\rho \circ \phi(u, g) = u$ is a submersion. Moreover, ϕ is 1-1 because G_+ acts freely. Every element of $\rho^{-1}U$ lies in the image of our section s modulo G_+-action, so clearly ϕ is onto. Therefore $E \to E/G_+$ is locally trivial, so a principal right G_+-bundle. The Cartan connection of $E \to M$ clearly is a Cartan connection for $E \to E/G_+$. □

2.7. **Homogeneous vector bundles.** Let $E \to M$ be a Cartan geometry modelled on a homogeneous space G/G_0.

Definition 20. Given W any G_0-module, let $E \times_{G_0} W$ be the quotient of $E \times W$ by the right G_0-action

$$(e, w)g = \left(eg, g^{-1}w\right).$$

For example, $G \times_{G_0} W$ is called a *homogeneous vector bundle.* We will say that $E \times_{G_0} W$ is *modelled on* $G \times_{G_0} W$. Sections of $E \times_{G_0} W$ are identified with G_0-equivariant functions $E \to W$, by taking each G_0-equivariant function $f : E \to W$ to the section $s : M \to W$ defined by letting $s(m)$ be the orbit of $(e, f(e))$, where $e \in E$ projects to $m \in M$. We will say that W *solders* $E \times_{G_0} W$, and that the function $f : E \to W$ *quotients to* the associated section s.

Example 14. The curvature quotients to a section of the vector bundle $E \times_{G_0} W$ where $W = \mathfrak{g} \otimes \Lambda^2 \left(\mathfrak{g}/\mathfrak{g}_0\right)^*$; this section is also called the curvature.

Example 15. If the reduction of a Cartan geometry is flat, then so is the reduction of $(1 + f)\omega$ for f a section of $E \times_{G_0} W$ where $W = \mathfrak{k} \otimes (\mathfrak{g}/\mathfrak{g}_0)^*$, $\mathfrak{k}$ the Lie algebra of the kernel.

Lemma 6 (Sharpe [68] p. 188, theorem 3.15)**.** *Suppose that $E \to M$ is a Cartan geometry modelled on G/G_0. The tangent bundle is*

$$TM = E \times_{G_0} \left(\mathfrak{g}/\mathfrak{g}_0\right).$$

Remark 16 (Griffiths [34] p. 133). If W is a G-module (not just a G_0-module) then we can define global sections f_w of $G \times_{G_0} W$ by

$$f_w(g) = g^{-1}w,$$

for any fixed $w \in W$, trivializing the bundle $G \times_{G_0} W$. Equivalently, define a map

$$(g, w) \in G \times W \mapsto gw \in W.$$

This map is G_0-invariant, so trivializes the bundle $G \times_{G_0} W$. It is *not* necessarily G-invariant, but is rather G-equivariant.

Lemma 7. *Let G/G_0 be a homogeneous space. A G_0-module W extends to a G-module just when $G \times_{G_0} W$ is G-equivariantly trivial, i.e. G-equivariantly isomorphic to a trivial bundle $(G/G_0) \times W$.*

Proof. Suppose that $\Phi : G \times_{G_0} W \to (G/G_0) \times W_0$ is a G-equivariant trivialization, W_0 a G-module. Write $\Phi((g,w)G_0) = (gG_0, \phi(g)w)$. Write the representation of G_0 on W as $\rho : G_0 \to \mathrm{GL}(W)$. Thus $\phi : G \to W^* \otimes W_0$, and G_0-invariance says that $\phi(g)\rho(p) = \phi(gp)$. Assume without loss of generality that $W = W_0$ and $\phi(1) = 1$. Then clearly ϕ extends ρ from G_0 to G. □

2.8. **Development.** I could not find a discussion of development in the literature suitable for my purposes. The following applies to both real and complex manifolds.

Definition 21. Suppose that $E_0 \to M_0$ and $E_1 \to M_1$ are two Cartan geometries modelled on the same homogeneous space G/G_0, with Cartan connections ω_0 and ω_1, and X is any manifold, possibly with boundary. A smooth map $\phi_1 : X \to M_1$ is a *development* of a smooth map $\phi_0 : X \to M_0$ if there exists a smooth isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* E_1$ of principal G_0-bundles identifying the 1-forms ω_0 with ω_1.

Development is an equivalence relation. The map Φ is an integral manifold of the Pfaffian system $\omega_0 = \omega_1$ on $\phi_0^* E_0 \times E_1$, and so Φ is the solution of a system of (determined or overdetermined) differential equations, and conversely solutions to those equations determine developments. By lemma 6 on the previous page the developing map ϕ_1 has the same rank as ϕ_0 at each point.

Definition 22. Suppose that $E_0 \to M_0$ and $E_1 \to M_1$ are Cartan geometries with the same model G/G_0. We will say that M_1 *rolls freely* on curves in M_0 to mean that: for any points $e_0 \in E_0$ and $e_1 \in E_1$ in the fibers above any points $m_0 \in M_0$ and $m_1 \in M_1$, and C any simply connected curve, possibly with boundary, and $c \in C$ any point, any smooth map $\phi_0 : C \to M_0$ with $\phi_0(c) = m_0$ has a unique development $\phi_1 : C \to M_1$ with $\phi_1(c) = m_1$ with a unique isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* E_1$ so that $\Phi(e_0) = e_1$. Following Cartan's method of the moving frame, we will call the points e_0 and e_1 the *frames* of the development.

We obtain the same development if we replace the frames by $e_0 g_0$ and $e_1 g_0$, for any $g_0 \in G_0$. The curve mentioned could be a real curve, or (if the manifolds and Cartan geometries are complex analytic) a complex curve.

Example 16. A Riemannian geometry is a Cartan geometry modelled on Euclidean space. As Riemannian manifolds, the unit sphere rolls freely on curves in the plane. Rolling in this context has its obvious intuitive meaning (see Sharpe [68] pp. 375–390). The development of a geodesic on the plane is a geodesic on the sphere, so a portion of a great circle. The upper half of the unit sphere does *not* roll freely on the plane (by our definition), because if we draw any straight line segment in the plane of length more than π, we can't develop all of it to the upper half of the sphere.

Theorem 8. *Let M_1 be a manifold with a Cartan geometry. The following are equivalent:*

1. *M_1 is complete.*
2. *M_1 rolls freely on curves in its model.*
3. *M_1 rolls freely on curves in any Cartan geometry with the same model.*
4. *M_1 rolls freely on curves in some Cartan geometry with the same model.*

Remark 17. A complex parabolic geometry is complete just when it is isomorphic to its model (see McKay [58] or corollary 14 on page 22), greatly limiting the utility of development in this context.

Remark 18. Kobayashi [47] stated a weaker result without giving a proof. The result is often quoted (see Clifton [22]), but the proof has never appeared.

Proof. Suppose that $E_1 \to M_1$ is complete. The local existence and uniqueness of a development is clear by applying the Frobenius theorem to the Pfaffian system $\omega_1 = \omega_0$ on $\phi_0^* E_0 \times E_1$. The maximal connected integral manifolds project locally diffeomorphically to $\phi_0^* E_0$, because ω_0 is a coframing on them. The problem is reduced to proving that these maximal connected integral manifolds project onto $\phi_0^* E_0$.

If $C = \mathbb{CP}^1$, we can first try to prove existence and uniqueness of a development of $C \setminus \text{pt}$, for two different choices of $\text{pt} \in C$. If we can do that, then we can clearly prove existence and uniqueness of a development for all of C. Therefore lets assume that C is a real or complex curve which is not $\mathbb{CP}^1$.

After replacing C with a covering space, we can assume that $\phi_0^* E_0 \to C$ is a trivial bundle, with a global section s_0. If we can develop, then this global section is identified via the isomorphism Φ with a global section $s_1 : C \to \phi_1^* E_1$ so that

$$s_1^* \omega_1 = s_0^* \omega_0. \tag{1}$$

Conversely, if we can solve this equation, then there is a unique isomorphism Φ for which

$$\Phi\left(s_0 g_0\right) = s_1 g_0$$

for all $g_0 \in G_0$, by triviality of the bundles. So it suffices to solve equation 1.

It is elementary to solve equation 1 if $E_1 \to M_1$ is isomorphic to the model (i.e. it is elementary to develop to the model), since equation 1 is an ordinary differential equation of Lie type,

$$g^{-1}\, dg = s_0^* \omega_0,$$

so has a unique global solution with given initial condition $g = g_0$ at $t = t_0$. (See Bryant [11] for proof. Global existence and uniqueness of a development follow from writing the ordinary differential equations of Lie type as linear ordinary differential equations). Moreover, this suffices for our applications in this article, and suffices to prove Kobayashi's unproven theorem. On the other hand, if we can solve equation 1 when $E_0 \to M_0$ is isomorphic to the model, i.e. develop curves from the model, then we can first develop to the model and then develop from the model. So it suffices to assume that $E_0 \to M_0$ is the model.

Since local existence and local uniqueness is assured, global uniqueness is assured. To ensure global existence, we need only prove that we can extend the local solution along every map $[0,1] \to C$. So it suffices to prove the result for C a real curve. Take any smooth curve $g(t) \in G, 0 \le t \le 1$. Let

$$A(t) = g^{-1}(t) \frac{dg}{dt}.$$

We need to construct a curve $s_1 : [0,1] \to E_1$ so that

$$\frac{ds_1}{dt} \lrcorner\, \omega_1 = A(t).$$

For each $\varepsilon > 0$, construct a piecewise constant function $A^{\varepsilon} : [0,1] \to \mathfrak{g}$ so that $A^{\varepsilon} \to A$ uniformly as $\varepsilon \to 0$. The time-varying vector field $\vec{A}^{\varepsilon}$ on E_1 (and on G)

is clearly complete, because its flow is just the composition of the flows of various complete vector fields. The time-varying vector fields $\vec{A}^{\varepsilon}$ converge uniformly on compact sets of E_1 to the smooth time-varying vector field $\vec{A}$. Therefore $\vec{A}$ is complete. Its flow through $e_1 \in E_1$ is our required curve $s_1(t)$.

Conversely, suppose that M_1 rolls freely on curves in all Cartan geometries modelled on G/G_0. In particular we can take $M_0 = G/G_0$ the model. The one-parameter subgroups of G that don't lie in G_0 quotient to entire curves on G/G_0, which develop to entire curves in M_1, with isomorphism Φ. The infinitesimal generator $\vec{A}$ of the one-parameter subgroup is tangent to the one-parameter subgroup, and therefore to the bundles that the isomorphism is identifying, and thus the isomorphism identifies the flows of $\vec{A}$ on G and on E_1. Therefore $\vec{A}$ is complete on E_1.

Finally, suppose that a Cartan geometry $\pi_1 : E_1 \to M_1$ rolls freely on curves in some Cartan geometry $\pi_0 : E_0 \to M_0$. So for every choice of points $e_0 \in E_0, m_0 \in M_0, e_1 \in E_1$ and $m_1 \in M_1$ with e_j in the fiber of E_j above m_j $(j = 0, 1)$, and for every curve $\phi_0 : C \to M_0$ with $\phi_0(c) = m_0$, some $c \in C$, there is a curve $\phi_1 : C \to M_1$ and isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* E_1$ so that $\omega_0 = \omega_1$ on the graph of Φ. Pick any $A \in \mathfrak{g}$. Consider a flow line $\Psi_0(t)$ of $\vec{A}$, so $\Psi_0(0) = e_0$ and $\Psi_0'(t) \lrcorner\, \omega_0 = A$. Let $\phi_0(t) = \pi_0(\Psi_0(t))$. Construct the curve ϕ_1 and isomorphism Φ. Then let $\Psi_1(t) = \Phi(\Psi_0(t))$. Clearly $\Psi_1(0) = e_1$ and $\Psi_1'(t) \lrcorner\, \omega_1 = A$, so Ψ_1 is a flow line of $\vec{A}$ on E_1. Therefore every flow line of $\vec{A}$ on E_1 is defined for as long a time as any flow line of $\vec{A}$ on E_0. If a flow line on E_1 is defined for a finite time only, then after we flow for almost all of that time, we must reach a point where the flow is defined only for a very short time. Therefore all flow lines of all elements $A \in \mathfrak{g}$ are defined for all time on E_1, i.e. E_1 is complete. □

Corollary 8. *A complex manifold which admits a holomorphic affine connection contains no rational curves.*

Proof. Such curves would develop to rational curves in the model, which is affine space. □

Remark 19. Development identifies the principal bundles $\phi_0^* E$ and $\phi_1^* E_1$, and thereby identifies the associated vector bundles $\phi_0^* E \times_{G_0} W$ and $\phi_1^* E_1 \times_{G_0} W$, for any G_0-module W.

3. Rational homogeneous varieties

3.1. **Langlands decomposition.** Knapp [46] proves all of the results we will use on Lie algebras; we give some definitions only to fix notation. Each parabolic subgroup $P \subset G$ of a semisimple Lie group has a *Langlands decomposition* (see Knapp [46]) $P = LAN$, where L is semisimple, A is abelian, and N is unipotent and normal. A root α of G is called a root of P if the associated root space $\mathfrak{g}^{\alpha}$ is a subspace of $\mathfrak{p}$. The roots of P all lie either on or on one side of some hyperplane in $\mathfrak{h}^*$. The roots on that hyperplane span the Lie algebras of L and A, while those on the chosen side of the hyperplane span the Lie algebra of N. Conversely, every hyperplane (with a chosen side) determines a unique parabolic subgroup P whose roots lie on the hyperplane or on the chosen side (up to Weyl group action). The Cartan [Borel] subalgebra for $\mathfrak{l}$ is just $\mathfrak{h} \cap \mathfrak{l}$ [$\mathfrak{b} \cap \mathfrak{l}$], and therefore $\mathfrak{l}$ intersects parabolic subalgebras of $\mathfrak{g}$ in parabolic subalgebras of $\mathfrak{l}$. Clearly the roots of $\mathfrak{l}$ are just those roots of $\mathfrak{g}$ lying in $\mathfrak{l}$. Thus the Dynkin diagram of L is given by cutting out the crosses from the Dynkin diagram of P.

For example, in figure 3(a) on page 20, we see the roots of the parabolic subgroup P of $\mathbb{P}\,\mathrm{SL}\,(3,\mathbb{C})$ which preserve a point of $\mathbb{P}^2$. The space $\mathfrak{l}$ is generated by the root spaces on the vertical axis, hence $\mathfrak{l} = \mathfrak{sl}\,(2,\mathbb{C})$. The space $\mathfrak{a}$ is the 1-dimensional subspace corresponding to the other root at the origin, and $\mathfrak{n}$ has roots drawn in figure 3(c).

Lemma 8. *Let $P_- \subset P_+ \subset G$ be parabolic subgroups. Every element of P_+ lies in a P_+-conjugacy class of an element of P_-.*

Proof. Write Langlands decompositions $P_- = L_- A_- N_-$, and $P_+ = L_+ A_+ N_+$. Every element of L_+ is conjugate to an element of the Cartan subgroup of L_+, which is the Cartan subgroup of $L_- A_-$; clearly $N_+ \subset N_-$ and $A_+ \subset A_-$. □

Corollary 9. *Every element of P_+ acts on P_+/P_- with a fixed point.*

Lemma 9 (Tits [73], Baston & Eastwood [5])**.** *Suppose that $P_- \subset P_+ \subset G$ are parabolic subgroups of a semisimple Lie group G. Under Langlands decompositions $P_- = L_- A_- N_-$, and $P_+ = L_+ A_+ N_+$, the kernel K of P_+/P_- contains $A_+ N_+$. Therefore $P_+/P_- = (P_+/K)\,/\,(P_-/K)$ is a rational homogeneous variety. The map $L_+ \to P_+/K$ is surjective. The group $P_+/K = L_+/L_-$ is semisimple, and $P_-/K \subset P_+/K$ is a parabolic subgroup.*

Lemma 10. *Take a rational homogeneous variety G/P. The subgroup P cannot fix any line in $\mathfrak{g}/\mathfrak{p}$.*

Proof. If a line is fixed in $\mathfrak{g}/\mathfrak{p}$ by P, then in $\mathfrak{g}$ the preimage of that fixed line in $\mathfrak{g}/\mathfrak{p}$ is a subspace, say $\mathfrak{p}_+ \subset \mathfrak{g}$, so that $[\mathfrak{p}, \mathfrak{p}_+] \subset \mathfrak{p}$. Clearly $\mathfrak{p}_+$ is spanned by $\mathfrak{p}$ and one other vector, and so is a parabolic Lie subalgebra. The associated parabolic subgroup P_+ has normal subgroup $P \subset P_+$. Therefore the quotient P_+/P is a one-dimensional Lie group. But such a quotient is a rational homogeneous variety, so must be a rational curve, and thus not a Lie group. □

3.2. **Homogeneous vector bundles on a line.** Let $B \subset \mathrm{SL}\,(2,\mathbb{C})$ be the Borel subgroup of upper triangular matrices. A homogeneous vector bundle V on $\mathbb{P}^1$ is then obtained from each B-module W by $V = \mathrm{SL}\,(2,\mathbb{C}) \times_B W$. This homogeneous vector bundle is $\mathrm{SL}\,(2,\mathbb{C})$-equivariantly trivial just when W is an $\mathrm{SL}\,(2,\mathbb{C})$-module. In particular, the bundle $\mathcal{O}\,(-1) \to \mathbb{P}^1$ is given by taking the B-module

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} w = aw,$$

$w \in \mathbb{C}$. More generally, the line bundle $\mathcal{O}\,(d) \to \mathbb{P}^1$ is given by the B-module

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} w = a^{-d} w.$$

We will also call this B-module $\mathcal{O}\,(d)$. Take $A \subset B$ the subgroup of diagonal matrices (which is a maximal reductive subgroup). We call a B-module W *elementary* if W is completely reducible as an A-module.

Theorem 9 (McKay [58])**.** *Every B-submodule and quotient B-module of an elementary B-module is elementary. Every indecomposable elementary B-module is of the form $\mathcal{O}\,(d) \otimes W$ for W a uniquely determined irreducible representation of*

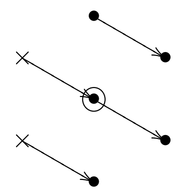


FIGURE 1. Computing the pull back of $T\mathbb{P}^2$ to a line in the projective plane.

$\mathrm{SL}(2,\mathbb{C})$. *We can draw the indecomposable elementary B-modules by identifying the weight lattice of* $\mathrm{SL}(2,\mathbb{C})$ *with* $\mathbb{Z}$*, drawing any elementary B-module as a sequence of integers, each larger by 2 than the previous. Representations of* $\mathrm{SL}(2,\mathbb{C})$ *are those which are symmetric under $x \mapsto -x$. Tensoring an indecomposable elementary B-module with $\mathcal{O}(d)$ shifts it to the left by d (careful: that makes $\mathcal{O}(2)$ sit at $x = -2$). Submodules are drawn by cutting off nodes from the left side, and quotients by cutting off the complementary nodes from the right side.*

Some notation: let $P \subset G$ be a parabolic subgroup of a semisimple Lie group, with Lie algebras $\mathfrak{p} \subset \mathfrak{g}$, and α a root of $\mathfrak{g}$ and not a root of $\mathfrak{p}$. As usual, write $\mathfrak{sl}(2,\mathbb{C})_\alpha$ for the copy of $\mathfrak{sl}(2,\mathbb{C})$ inside $\mathfrak{g}$ containing $\mathfrak{g}^\alpha$, and $\mathrm{SL}(2,\mathbb{C})_\alpha \to G$ for a Lie group morphism inducing the Lie algebra inclusion $\mathfrak{sl}(2,\mathbb{C})_\alpha \to \mathfrak{g}$, and let $B_\alpha \subset \mathrm{SL}(2,\mathbb{C})_\alpha$ be the Borel subgroup whose Lie algebra contains $\mathfrak{g}^{-\alpha}$. Let $\mathbb{P}^1_\alpha = \mathrm{SL}(2,\mathbb{C})_\alpha / B_\alpha$.

Corollary 10. *The B_α-module $\mathfrak{g}/\mathfrak{p}$ is elementary, and is thus a sum of indecomposable elementary B_α-modules, $\bigoplus \mathcal{O}(d_s)^{\oplus n_s}$, a sum over maximal α-strings of roots*

$$\beta, \beta - \alpha, \beta - 2\alpha, \dots$$

(possibly including one 0, but otherwise containing only proper roots), where the rank n_s is the number of roots of $\mathfrak{p}$ in each string, and the degree d_s is the number of roots in each string which are not roots of $\mathfrak{p}$. As a vector bundle

$$\begin{aligned} T(G/P)|_{\mathbb{P}^1_\alpha} &= \mathrm{SL}(2,\mathbb{C})_\alpha \times_{B_\alpha} (\mathfrak{g}/\mathfrak{p}) \\ &= \bigoplus \mathcal{O}(d_s)^{\oplus n_s}. \end{aligned}$$

Example 17. Look at figure 1, where $\mathbb{P}^1_\alpha$ is a line in the projective plane, taking α to be the uppermost of the two crossed roots. The ambient tangent bundle is $T\mathbb{P}^2\big|_{\mathbb{P}^1_\alpha} = \mathcal{O}(2) \oplus \mathcal{O}(1)$. The three strings compute out (from top to bottom): 0 copies of $\mathcal{O}(2)$, 1 copy of $\mathcal{O}(2)$, and 1 copy of $\mathcal{O}(1)$. In general, each string gives (number of crosses) copies of (number of dots) degree.

3.3. **The cominiscule varieties.** The classification of cominiscule varieties is in figure 2 on the facing page, with the following notation: let $\mathrm{Gr}\left(k, \mathbb{C}^{n+1}\right)$ be the Grassmannian of k-planes in $\mathbb{C}^{n+1}$, $Q^{2n-2} = \mathrm{Gr}_{\text{null}}\left(1, \mathbb{C}^{2n}\right)$ is the $2n-2$-dimensional smooth hyperquadric, i.e. the variety of null lines in $\mathbb{C}^{2n}$, $Q^{2n-1} = \mathrm{Gr}_{\text{null}}\left(1, \mathbb{C}^{2n+1}\right)$ is the $2n-1$-dimensional hyperquadric, i.e. the variety of null

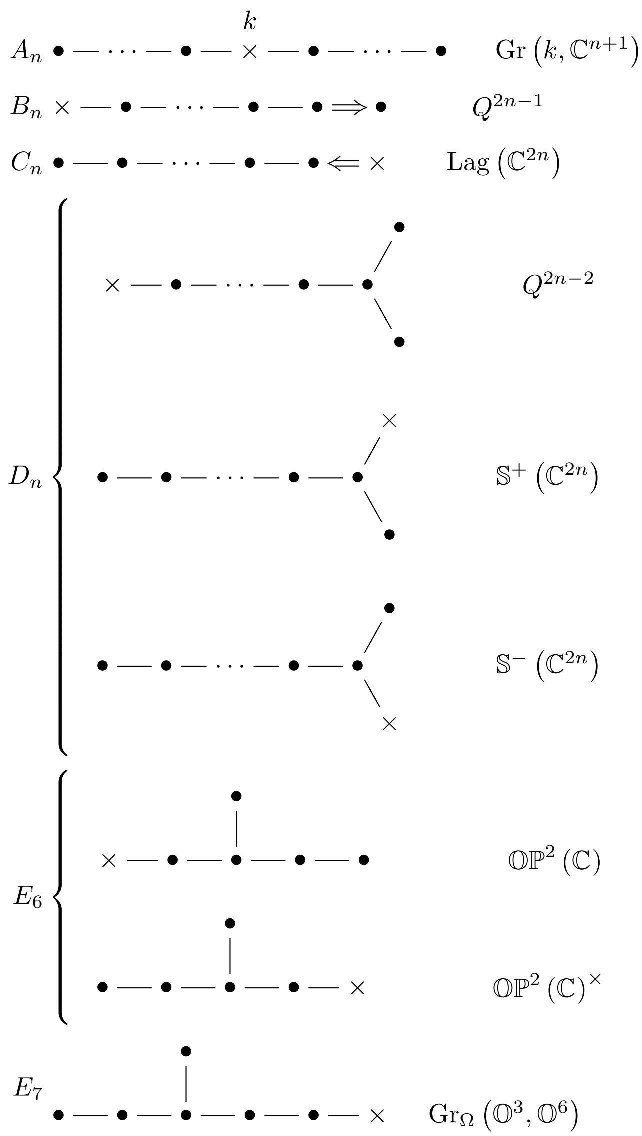


FIGURE 2. The cominiscule varieties

lines in $\mathbb{C}^{2n+1}$, $\mathbb{S}^+\left(\mathbb{C}^{2n}\right)$ is one component of the variety of maximal dimension null subspaces of $\mathbb{C}^{2n}$ and $\mathbb{S}^-\left(\mathbb{C}^{2n}\right)$ the other, $\mathbb{OP}^2(\mathbb{C})$ is the complexified octave projective plane (see Baez [3, 4]), $\mathbb{OP}^2(\mathbb{C})^\times$ its dual plane, and $\mathrm{Gr}_\Omega\left(\mathbb{O}^3, \mathbb{O}^6\right) \subset \mathbb{P}^{55}$ is the space of null octave 3-planes in octave 6-space (see Landsberg & Manivel [56]).

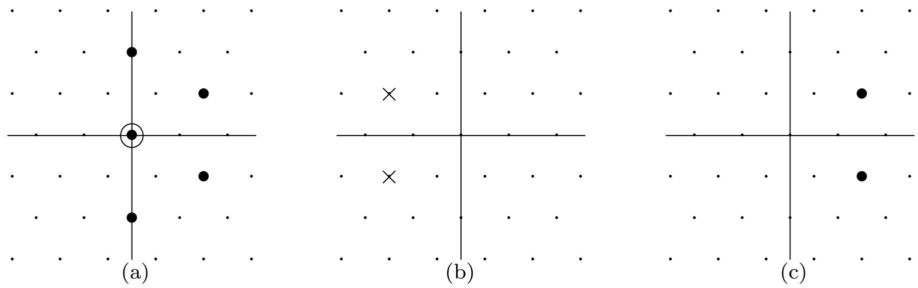


FIGURE 3. The root spaces of (a) the subalgebra $\mathfrak{p}$ fixing a point of the projective plane, (b) the $\mathfrak{p}$-module which solders the tangent bundle of the projective plane, (c) the $\mathfrak{p}$-module which solders the cotangent bundle.

4. Parabolic geometries

Example 18. Returning to example 13 on page 8, for any rational homogeneous variety G/P, we see that every torus quotient of the same dimension as G/P has a parabolic geometry modelled on G/P. Moreover, if we can find an abelian subalgebra $\Pi \subset \mathfrak{g}$ transverse to $\mathfrak{p}$, then we can construct a flat parabolic geometry on our torus. Consider the subalgebra $\mathfrak{n}^-$ spanned by the root spaces $\mathfrak{g}^\alpha$ which do not lie in $\mathfrak{p}$. This subalgebra is clearly nilpotent.

If $\mathfrak{n}^-$ is an irreducible P-module, then G/P is cominiscule. More generally, if $\mathfrak{n}^-$ is abelian, then G/P is a product of cominiscule varieties. In this case, we can clearly let $\Pi = \mathfrak{n}^-$. Moreover, the induced Cartan geometry is precisely the flat one induced on the open Bruhat cell in G/P (see Fulton and Harris [32] p. 396). We can also deform this flat parabolic geometry into a curved $\mathfrak{n}^-$ invariant geometry, as described in example 13 on page 8. Therefore every product of cominiscule varieties is the model for flat and nonflat parabolic geometries on every torus quotient of the appropriate dimension. In the paper [59], I proved that all parabolic geometries on complex tori are translation invariant, a complete classification.

Theorem 10. *Let $G/P_- \to G/P_+$ be rational homogeneous varieties. A parabolic geometry modelled on G/P_- drops to one modelled on G/P_+ just when its natural structure algebra contains the Lie algebra of P_+.*

Proof. Theorem 7 on page 12 and corollary 9 on page 17. □

Corollary 11 (McKay [58]). *A parabolic geometry is complete and flat just when it is isomorphic to its model.*

Corollary 12. *A parabolic geometry on a curve is isomorphic to its model just when the curve is rational.*

Proof. The only rational homogeneous variety which is a curve is a rational curve, hence simply connected. Any Cartan geometry on a curve is flat, because its curvature is a semibasic 2-form. □

5. Rational circles

Theorem 11. *Let $E \to M$ be a parabolic geometry modelled on G/P. The natural structure group $P^{nat} \subset G$ is the parabolic subgroup whose Lie algebra $\mathfrak{p}^{\text{nat}}$ is the sum of root spaces*

$$\mathfrak{p}^{\text{nat}} = \bigoplus_{\alpha} \mathfrak{g}^{\alpha}$$

of the roots α whose α-circles are rational curves.

Remark 20. Circles are by definition the solutions of a system of ordinary differential equations. Thus we have reduced the problem of dropping to the study of trajectories of certain continuous dynamical systems.

Proof. If α is a root of $\mathfrak{p}^{\text{nat}}$, then every α-circle is a rational curve: an orbit of $\text{SL}\left(2,\mathbb{C}\right)_{\alpha} \subset P^{\text{nat}}$.

Conversely, suppose that the α-circles are rational curves, for some negative root α which is not a root of $\mathfrak{p}$. Take any α-circle $\phi : C \to E/B_{\alpha}$. The pullback bundle $\phi^* E \to C$ is a principal right B_{α}-bundle, and ω pulls back to that bundle to be the Cartan connection 1-form of a Cartan geometry modelled on $\text{SL}\left(2,\mathbb{C}\right)_{\alpha} \to \mathbb{P}^1$. The induced Cartan geometry on any α-circle C is isomorphic to the model by theorem 5 on page 10. Let $\phi : \text{SL}\left(2,\mathbb{C}\right)_{\alpha} \to G$ be the Lie group morphism associated the the Lie algebra morphism $\mathfrak{sl}\left(2,\mathbb{C}\right)_{\alpha} \subset \mathfrak{g}$. Thus we identify $C = \text{SL}\left(2,\mathbb{C}\right)_{\alpha}/B_{\alpha}$. Under the maps

$$\begin{array}{ccc} \text{SL}\left(2,\mathbb{C}\right)_{\alpha} & \longrightarrow & E \\ \downarrow & & \downarrow \\ C & \longrightarrow & E/B_{\alpha} \end{array}$$

the vector field $\vec{A}$ on E, for $A \in \mathfrak{sl}\left(2,\mathbb{C}\right)_{\alpha}$, is tangent to the (immersed) image of $\text{SL}\left(2,\mathbb{C}\right)_{\alpha}$. Moreover the vector field $\vec{A}$ on $\text{SL}\left(2,\mathbb{C}\right)_{\alpha}$ is left invariant so complete. Clearly E is foliated by copies of $\text{SL}\left(2,\mathbb{C}\right)_{\alpha}$ on which each $\vec{A}$ is complete, and therefore $\vec{A}$ is complete on E.

We still have to check the equation $\kappa(A, B) = 0$ for all $A \in \mathfrak{sl}\left(2,\mathbb{C}\right)_{\alpha}$ and $B \in \mathfrak{g}$. Recall that α is a negative root and not a root of $\mathfrak{p}$. So $-\alpha$ is a positive root, so a root of $\mathfrak{p}$. The curvature κ quotients under P action to a section of the curvature bundle

$$E \times_P \left(\mathfrak{g} \otimes \Lambda^2 \left(\mathfrak{g}/\mathfrak{p}\right)^*\right).$$

Take the obvious injection $\mathfrak{sl}\left(2,\mathbb{C}\right)_{\alpha}/\mathfrak{b}_{\alpha} \to \mathfrak{g}/\mathfrak{p}$ and surjection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ and transpose them to get $\left(\mathfrak{g}/\mathfrak{p}\right)^* \to \left(\mathfrak{sl}\left(2,\mathbb{C}\right)_{\alpha}/\mathfrak{b}_{\alpha}\right)^*$ and $\left(\mathfrak{g}/\mathfrak{p}\right)^* \to \mathfrak{g}^*$ and then tensor those together to get

$$\left(\mathfrak{g}/\mathfrak{p}\right)^* \otimes \left(\mathfrak{g}/\mathfrak{p}\right)^* \to \left(\mathfrak{sl}\left(2,\mathbb{C}\right)_{\alpha}/\mathfrak{b}_{\alpha}\right)^* \otimes \mathfrak{g}^*.$$

Take the inclusion $\Lambda^2 \left(\mathfrak{g}/\mathfrak{p}\right)^* \to \left(\mathfrak{g}/\mathfrak{p}\right)^* \otimes \left(\mathfrak{g}/\mathfrak{p}\right)^*$ to get

$$\Lambda^2 \left(\mathfrak{g}/\mathfrak{p}\right)^* \to \left(\mathfrak{sl}\left(2,\mathbb{C}\right)_{\alpha}/\mathfrak{b}_{\alpha}\right)^* \otimes \mathfrak{g}^*.$$

Tensor with $\mathfrak{g}$ to get a map

$$\kappa \in \mathfrak{g} \otimes \Lambda^2 \left(\mathfrak{g}/\mathfrak{p}\right)^* \mapsto \bar{\kappa} \in \mathfrak{g} \otimes \left(\mathfrak{sl}\left(2,\mathbb{C}\right)_{\alpha}/\mathfrak{b}_{\alpha}\right)^* \otimes \mathfrak{g}^*.$$

So the curvature κ at each point of E gives an element $\bar{\kappa}$ in this B_{α}-module. By corollary 10 on page 18, we see that the vector bundle associated to this B_{α}-module on any α-circle C is

$$\mathrm{SL}\left(2,\mathbb{C}\right)_\alpha \times_{B_\alpha} \left(\mathfrak{g}\otimes\mathfrak{g}^{\alpha *}\otimes\mathfrak{g}^*\right) = \mathcal{O}^{\oplus \dim \mathfrak{g}} \otimes \mathcal{O}\left(-2\right) \otimes \mathcal{O}^{\oplus \dim \mathfrak{g}},$$

a sum of line bundles of negative degrees, so has no nonzero global sections. Our $\bar\kappa$ quotients under B_α-action to a section of this bundle, so vanishes. Therefore $\kappa(A,B) = \bar\kappa(A,B) = 0$ for $A \in \mathfrak{g}^\alpha = \mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha / \mathfrak{b}_\alpha$ and for $B \in \mathfrak{g}$. Therefore α is a root of $\mathfrak{p}^{\mathrm{nat}}$. □

Theorem 1 on page 5 follows.

Lemma 11. *Suppose that $E \to M$ is a parabolic geometry modelled on G/P. Suppose that α is not a root of $\mathfrak{p}$ and the α-circles of M are rational curves. Write $\alpha = -\sum A_j \alpha_j$, where α_j are the positive simple roots, and $A_j \ge 0$ are integers. If $A_j > 0$ then $-\alpha_j$ is a root of $\mathfrak{p}^{\mathrm{nat}}$.*

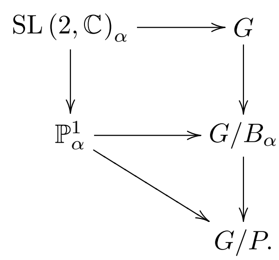
Proof. The root α must be negative, because it is not a root of $\mathfrak{p}$. Following Serre [67] p. 32, we can write $\alpha = -\left(\alpha_{i_1} + \dots + \alpha_{i_N}\right)$ where the partial sums $-\left(\alpha_{i_1} + \dots + \alpha_{i_j}\right)$ are also negative roots. But α is a root of $\mathfrak{p}^{\mathrm{nat}}$, as are the α_{i_N}, so therefore $\alpha + \alpha_{i_N}$ is too. Inductively, all of the partial sums are roots of $\mathfrak{p}^{\mathrm{nat}}$. Moreover the positive partial sums $\alpha_{i_1} + \dots + \alpha_{i_{j-1}}$ are positive roots, and therefore $-\alpha_{i_j}$ is a root of $\mathfrak{p}^{\mathrm{nat}}$. □

Remark 21. Thus we need only check the rationality of circles associated to a small collection of roots, in order to conclude rationality of many other circles associated to many other roots, and dot certain crosses α_j in the Dynkin diagram.

Corollary 13. *If the circles of a lowest root are rational, then the parabolic geometry drops, removing an entire component of the Dynkin diagram.*

Corollary 14 (McKay [58])**.** *A parabolic geometry is complete just when it is isomorphic to its model.*

Proof. Take $E \to M$ a complete parabolic geometry, say modelled on G/P, with Cartan connection 1-form ω. Take any root α of G, which is not a root of P, and consider the maps



The map on the bottom, call it $\phi_0 : \mathbb{P}^1_\alpha \to G/P$, is not constant, because α is not a root of P. It develops to a curve $\phi_1 : \mathbb{P}^1_\alpha \to M$, passing through any chosen point $m \in M$, and with an isomorphism $\Phi : \phi_0^* G \to \phi_1^* E$ so that $\Phi^*\omega = g^{-1}\,dg$, and so that $\Phi(1) = e$ for any chosen point $e \in E$ in the fiber over $m \in M$. Let $\Psi = \left.\Phi\right|_{\mathrm{SL}(2,\mathbb{C})_\alpha} : \mathrm{SL}\left(2,\mathbb{C}\right)_\alpha \to E$. Clearly $\Psi^*\omega = 0 \pmod{\mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha}$, and Ψ is B_α-equivariant. Lets write ψ for the quotient map of Ψ by the B_α-action, so $\psi : \mathbb{P}^1_\alpha \to E/B_\alpha$, and ψ is an integral curve of $\omega = 0 \pmod{\mathfrak{sl}\left(2,\mathbb{C}\right)_\alpha}$. We have constructed one rational α-circle, $\psi : \mathbb{P}^1_\alpha \to E/B_\alpha$. But the choice of the point $e \in E$, and hence of point of E/B_α through which our α-circle passes, is arbitrary.

The α-circles form a foliation of E/B_α, and therefore every maximal connected integral curve of that foliation is a rational curve. □

6. Developing rational curves

Definition 23. An immersed rational curve $C \subset M$ in a complex manifold is called *standard* if $TM|_C = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus p} \oplus \mathcal{O}^{\oplus q}$, for some integers $p, q \geq 0$.

Remark 22. The definition is motivated by work of Mori [61] (also see Hwang & Mok [40], Hwang [38]).

Remark 23. The vector bundle morphisms $\mathcal{O}(p) \to \mathcal{O}(q)$ are the global sections of $\mathcal{O}(p)^* \otimes \mathcal{O}(q) = \mathcal{O}(q-p)$, so vanish if $p > q$ and are uniquely determined up to scaling if $p = q$. So roughly speaking, degrees can't go down under vector bundle maps. In particular, there is precisely one vector bundle map

$$\mathcal{O}(2) \to \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus p} \oplus \mathcal{O}^{\oplus q}$$

up to scalar multiple, for any integers p and q.

Remark 24. Any rational curve $\phi : \mathbb{P}^1 \to M$ in a complex manifold M has ambient tangent bundle

$$\phi^* TM = \bigoplus \mathcal{O}(d_j)^{\oplus p_j}$$

a sum of line bundles of various degrees d_j. A rational curve is free just when none of the degrees are negative.

We now prove theorem 2 on page 6.

Proof. Let $\check{\phi} : C \to G/G_0$ be a development of $\phi : C \to M$.

$$\begin{aligned} \phi^* TM &= \phi^* \left(E \times_{G_0} (\mathfrak{g}/\mathfrak{g}_0)\right) \\ &= \check{\phi}^* \left(G \times_{G_0} (\mathfrak{g}/\mathfrak{g}_0)\right) \\ &= \check{\phi}^* T (G/G_0) . \end{aligned}$$

The tangent bundle of G/G_0 is spanned by its global sections. □

We now prove corollary 3 on page 6.

Proof. Given a complex submanifold X of a complex manifold M, the blow-up $B_X M$ is a complex manifold with map $\pi : B_X M \to M$ which is a biholomorphism over $M \backslash X$, and with stalk over a point $x \in X$ given by $B_X M_x = \mathbb{P}(\nu_x X)$, where $\nu_x X = T_x M / T_x X$ is the normal bundle. For $\ell \in \mathbb{P}(\nu_x X)$, if we write $\pi_x : T_x M \to \nu_x X$, then

$$T_\ell B_X M = \pi_x^{-1} \ell \oplus \left(\ell^* \otimes (\nu_x X / \ell)\right).$$

Pick a point $x \in X$, a 2-plane $P \subset \nu_x X$, and a linear splitting $T_x M = T_x X \oplus \nu_x X$. Let $\mathbb{P}^1 = \mathbb{P}(P)$. Then clearly

$$TB_X M|_{\mathbb{P}^1} = \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus \dim X} \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus \dim M - \dim X - 2}.$$

So there is a rational curve on which the ambient tangent bundle contains a negative line bundle. □

We now prove corollary 2 on page 6.

Proof. In a complex manifold M with Cartan geometry, rational curves are free. Thus the ambient tangent bundle TM on any rational curve pulls back to a sum of nonnegative line bundles, at least one of which (the tangent line of the rational curve itself) is positive. So the determinant of the ambient tangent bundle will restrict to a positive line bundle, and therefore $K_M = \Lambda^{\text{top}}(T^*M)$ will restrict to a negative line bundle. Hence all positive tensor powers of K_M will restrict to negative line bundles. □

7. Bend-and-break

Definition 24. A map $\mathbb{P}^1 \to M$ to a complex manifold M is *multiply covered* if it factors $\mathbb{P}^1 \to \mathbb{P}^1 \to M$ through a map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree greater than 1. A map which is not multiply covered is called *simply covered*. Take any homology class $A \in H_2(M, \mathbb{Z})$. Let $M(A, J)$ be the set of all simply covered maps $\phi : \mathbb{P}^1 \to M$ with $\phi_* \left[\mathbb{P}^1\right] = A$, following McDuff and Salamon [57]. (The letter J, for us, just means that these maps are assumed holomorphic.) We define the *space of deformations* of a simply covered rational curve $\phi : \mathbb{P}^1 \to M$ in a complex manifold M to be the path component (in the Gromov–Hausdorff topology) in the space $M(A, J)/\mathbb{P}\operatorname{SL}(2, \mathbb{R})$ containing the point associated to $\phi : \mathbb{P}^1 \to M$, where A is the homology class $A = \phi_* \left[\mathbb{P}^1\right]$.

Definition 25. We will say that a rational curve in a complex manifold is *unbreakable* if its space of deformations is a compact smooth manifold, and each point in that space represents an irreducible rational curve.

Lemma 12. *Let M be a connected closed Kähler manifold containing a rational curve and bearing a Cartan geometry. Then through every point of M there passes an unbreakable rational curve.*

Proof. Take $\phi : C \to M$ a rational curve of minimal positive degree. In particular C cannot be deformed into a reducible curve. By Gromov's compactness theorem, every sequence of rational curves has a convergent subsequence, possibly converging to a reducible curve consisting of finitely many rational curves whose homology classes add together to the required homology class $\phi_* [C]$. Consider a sequence of deformations of ϕ. Any convergent subsequence must converge to an irreducible curve, so the space of deformations is compact in the Gromov–Hausdorff metric (McDuff and Salamon [57] theorem 1.4.1, p. 6).

Each point of the $M(A, J)$ is a smooth point by freedom of rational curves, so the moduli space is naturally a manifold (McDuff and Salamon [57] lemma 3.5.1, p. 38). It is naturally a complex manifold (McDuff and Salamon [57] p. 33). By freedom, we can move C around to pass through an open set of points of M. By compactness, the set of points of M which we can reach is compact. □

Lemma 13. *Let M be a closed Kähler manifold containing a rational curve and bearing a Cartan geometry. Fix a point $m \in M$, and an unbreakable rational curve $\phi_0 : \mathbb{P}^1 \to M$ with $\phi_0(0) = m$. Consider the moduli space Y parameterizing all deformations $\phi : \mathbb{P}^1 \to M$ of ϕ_0 for which $\phi(0) = m$, modulo reparameterization fixing* 0*. Then Y is a smooth closed Kähler manifold. Let $\check{Y}$ be the moduli space of deformations $\psi : \mathbb{P}^1 \to G/G_0$ of the development $\check{\phi}_0$ for which $\psi(0) = G_0 \in G/G_0$. Map $Y \to \check{Y}$, taking each such curve ϕ to its development $\check{\phi}$. If ϕ_0 is unbreakable, then this map is a biholomorphism, and both Y and $\check{Y}$ are closed Kähler manifolds.*

Proof. Again by Gromov compactness the space Y is a compact complex manifold. (See Hwang and Mok [41] p. 401 for details of the deformation theory, providing a proof without Gromov compactness that Y is a closed complex manifold.) The development takes every free curve to a free curve (identifying the ambient tangent bundles), so takes every point of Y to a smooth point of $\check{Y}$, near which $\check{Y}$ has the same dimension as Y. Development with given frames is clearly holomorphic and injective, and therefore a biholomorphism to its image. By compactness of Y, $Y \to \check{Y}$ is a biholomorphism. □

Remark 25. We can write $Y = Y(m, \phi_0)$ and $\check{Y} = Y\left(G_0, \check{\phi}_0\right)$. In lemma 13, the development $\check{\phi}$ depends not only on the choice of the point m and the curve ϕ, but also of the frames we choose in $\phi^* E$ and in G to carry out development. We will always pick $1 \in G$ as a frame, but in $\phi^* E$ there is no natural choice of frame.

Lemma 14. *Let M be a closed Kähler manifold bearing a parabolic geometry. Every minimal rational curve in M is unbreakable, and every unbreakable rational curve is standard. If M is connected and contains a rational curve, then M contains a minimal degree rational curve through every point.*

Proof. Let C be the unbreakable curve, and $\check{C}$ its development. Since G/P is a projective variety, we can apply Mori's bend–and–break II [54] p.11 to $\check{C}$. Bend–and–break II tells us that if $\check{C}$ is free but not standard, then it can be deformed into a reducible curve while fixing a point. This contradicts compactness of the moduli space $\check{Y}$ from lemma 13. □

Lemma 15. *Suppose that M is a closed Kähler manifold containing a rational curve, and bearing a parabolic geometry modelled on G/P. Pick $\phi_t : \mathbb{P}^1 \to M$ any family of unbreakable rational curves, and pick a family e_t of frames, i.e. $e_t \in \phi_t^* E$, say e_t lying in the fiber of $\phi_t^* E$ above $m_t \in M$. Suppose that ϕ_t, e_t and m_t depend continuously on t. Let $\check{\phi}_t$ be the development of ϕ_t. Then the deformation space*

$$\check{Y} = Y\left(P, \check{\phi}_t\right).$$

is independent of t. All of the deformation spaces $Y(m_t, \phi_t)$ are identified up to biholomorphism via development $Y(m_t, \phi_t) \to \check{Y}$.

Proof. By construction, the curves $\check{\phi}_t$ all pass through $P \in G/P$, since we fixed the choice of frame on G/P. Thus all of the $\check{\phi}_t$ are deformations of $\check{\phi}_0$, and so all are elements of $\check{Y} = Y\left(1, \check{\phi}_0\right)$. Hence all of the $\check{Y}$ spaces are just the same space. Each is identified by biholomorphism with $Y(m_t, \phi_t)$. □

8. Apparent structure algebras

Definition 26. Consider a parabolic geometry $\pi : E \to M$ modelled on G/P. Pick a point $m \in M$ and a point $e \in E_m$. Then $\omega_e : T_e E \to \mathfrak{g}$ and $\omega_e : \ker \pi'(e) \to \mathfrak{p}$ are isomorphisms. Define an isomorphism $\omega_e : T_m M = T_e E / \ker \pi'(e) \to \mathfrak{g}/\mathfrak{p}$.

Lemma 16. *Suppose that M is a closed Kähler manifold bearing a parabolic geometry with model G/P. On any standard rational curve $\phi : \mathbb{P}^1 \to M$, we have a vector subbundle*

$$\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus p} \subset \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus p} \oplus \mathcal{O}^{\oplus q} = \phi^* TM,$$

called the positive degree subbundle*, written* $\phi^* TM_+$*. Pick any point* $e \in \phi^* E$*, say in some fiber* E_m *for some point* $m \in M$*. Then* $\omega_e : T_m M \to \mathfrak{g}/\mathfrak{p}$ *is an isomorphism which restricts to a monomorphism* $\omega_e : T_m M_+ \hookrightarrow \mathfrak{p}^{\text{app}}(e)/\mathfrak{p}$*.*

Proof. Consider the curvature κ, a section of the curvature bundle

$$E \times_P \left(\mathfrak{g} \otimes \Lambda^2 \left(\mathfrak{g}/\mathfrak{p}\right)^*\right).$$

On G/P, the bundle $G \times_P \mathfrak{g}$ is trivial, because $\mathfrak{g}$ is a G-module. Therefore on the development $\check{\phi}$, the bundle $\check{\phi}^* G \times_P \mathfrak{g}$ is trivial. By development, the isomorphic bundle $\phi^* E \times_P \mathfrak{g}$ is also trivial. The ambient tangent bundle is

$$\begin{aligned} \phi^* TM &= \phi^* E \times_P (\mathfrak{g}/\mathfrak{p}) \\ &= \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus p} \oplus \mathcal{O}^{\oplus q}. \end{aligned}$$

Therefore the pullback curvature bundle is

$$\phi^* E \times_P \left(\mathfrak{g} \otimes \Lambda^2 \left(\mathfrak{g}/\mathfrak{p}\right)^*\right) = \mathcal{O}^{\oplus \dim \mathfrak{g}} \otimes \Lambda^2 \left(\mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus p} \oplus \mathcal{O}^{\oplus q}\right).$$

Apply the vector bundle map

$$\phi^* E \times_P \left(\mathfrak{g} \otimes \Lambda^2 \left(\mathfrak{g}/\mathfrak{p}\right)^*\right) \to \phi^* E \times_P \mathfrak{g} \otimes \phi^* TM_+^* \otimes \phi^* E \times_P \left(\mathfrak{g}/\mathfrak{p}\right)^*$$

to the pullback of the curvature; call the result $\bar{\kappa}$. Clearly $\bar{\kappa}$ quotients to a section of a vector bundle which is a sum of negative degree line bundles, so $\bar{\kappa} = 0$. Each vector in the positive degree subbundle extends to a vector field on the standard rational curve, by positivity of the line bundles. If we plug in a section s of the positive degree subbundle into the pullback of the curvature, $\kappa(s, \cdot) = \bar{\kappa}(s, \cdot) = 0$. □

Corollary 15. *Suppose that* M *is a connected closed Kähler manifold containing a standard rational curve. Then every parabolic geometry on* M *drops locally to a parabolic geometry with a lower dimensional model.*

Proof. We need only show that $\mathfrak{p}^{\text{app}} \neq \mathfrak{p}$, i.e. that $\mathfrak{p}^{\text{app}}(e)/\mathfrak{p}$ contains some vector independent of e. We know that $\mathfrak{p}^{\text{app}}(e)/\mathfrak{p}$ contains the positive degree subspace of any standard rational curve $\phi : C \to M$, for every point $e \in \phi^* E$. Consider the deformation space Y of all deformations of C passing through the given point m. All of these curves have positive degree subbundles of the same rank, because the topological type of the ambient tangent bundle $\phi^* TM$ does not change under deformation. Let $V \subset \mathfrak{g}/\mathfrak{p}$ be the span of the images under ω_e of the fibers at m of the positive degree subbundles of all deformations of C passing through m. This subspace V has positive dimension and is identified under development with the subspace spanned by the fibers at P of the positive degree subbundles of all deformations of the development $\check{C} \subset G/P$ passing through P. By lemma 15 on the previous page, V is independent of $e \in E$. □

9. Snakes

9.1. **Defining snakes.**

Definition 27. Let $\phi_0 : \mathbb{P}^1 \to M$ be an unbreakable rational curve in a complex manifold. A *snake* modelled on ϕ_0 is a curve, $\phi : \mathbb{P}^1 \sqcup \mathbb{P}^1 \cdots \sqcup \mathbb{P}^1 \to M$ whose components are each deformations of ϕ_0, with choice of a pair of points a_i, b_i on each component, so that $\phi(b_i) = \phi(a_{i+1})$. Let Z_p be the set of p-component snakes modelled on ϕ_0. We will call a_1 the head, and b_p the tail.

The concept of snake is a minor alteration of the concept of a cusp curve in a framed class (see McDuff and Salamon [57] pg. 63).

Lemma 17. *Let* $\phi_0 : \mathbb{P}^1 \to M$ *be an unbreakable curve, with moduli space*

$$\begin{array}{ccc} Z & \longrightarrow & M \\ \downarrow & & \\ Y. & & \end{array}$$

(So now, changing notation, $Z = M(A, J)$ *and* Y *represents all deformations of* ϕ_0*, not just those passing through a given point of* M*.) The space* Z_p *of* p*-component snakes modelled on* ϕ_0 *is a closed analytic subvariety of* Z^{2p}*.*

Proof. Points of Z are deformations with marked point. Each component of a snake has two marked points, in a particular ordering, so can be thought of as two points of Z which happen to project to the same point of Y. □

Definition 28. A snake is *regular* if the marked points on every component are distinct: $a_i \neq b_i$.

Lemma 18. *The regular snakes with* p *components form a smooth manifold, and a Zariski dense open subset of* Z_p*.*

Proof. We can perturb the marked points on the first component, which might break the snake up into disconnected pieces. Then we grab the next component, and slide it over (by freedom of unbreakable curves) to touch the first component, etc., until a small perturbation has moved apart the points on every component. Therefore regular snakes are dense, and clearly Zariski open.

The ambient tangent bundle on each component is $TM|_{\mathbb{P}^1} = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus p} \oplus \mathcal{O}^{\oplus q}$, a holomorphic vector bundle which is invariant under biholomorphism of $\mathbb{P}^1$. The biholomorphism group of $\mathbb{P}^1$ is two-point transitive. Therefore the rank of the equations in Z^{2p} cutting out Z_p is constant on the regular snakes. □

Definition 29. Let $\phi_0 : \mathbb{P}^1 \to M$ be an unbreakable rational curve in a complex manifold. Take any point $m \in M$, and let $Z_p(m)$ be the set of snakes modelled on ϕ_0 with tail at m. Let $R_p(m)$ be the set of heads of snakes in $Z_p(m)$ (the *reachable set* from m), and let

$$R(m) = \bigcup_p R_p(m).$$

The *radius* of M (with respect to ϕ_0) is the smallest integer p so that for every point m we find $R(m) = R_p(m)$. (Set the radius to be ∞ if there is no such value of p.)

Lemma 19. *Every closed Kähler manifold* M *containing an unbreakable rational curve has finite radius.*

Proof. The reachable sets $R_p(m)$ are holomorphic images of closed analytic subvarieties $Z_p \subset Z^{2p}$, so are closed analytic subvarieties of M. Indeed they are the images in M of the level sets of the tail map, under the head map. By density of the regular snakes, all rank calculations are independent of choice of the point m. In particular, the dimension of $R_p(m)$ is independent of the point m. Since the head map is surjective, the reachable sets locally foliate M near any regular snake. Suppose that we take p large enough that the reachable sets all attain maximal dimension. If there is a deformation of ϕ_0 that passes through a smooth point of some reachable set $R_p(m)$, and is not tangent to $R_p(m)$, then $R_{p+1}(m)$ will have larger dimension than $R_p(m)$. Therefore all snakes must be tangent to all reachable sets $R_p(m)$ for the given value of p. Suppose that some point lies on the head of a regular snake. Because these sets $R_p(m)$ foliate M near a regular snake, all snakes passing through p must lie entirely inside one of these reachable sets. For a snake passing through a singular point of a reachable set, we can perturb the snake slightly to a smooth point of the reachable set. Consequently $R_{p+1}(m) = R_p(m)$ for every point m. □

9.2. **Snakes in a parabolic geometry.**

Definition 30. If M is a complex manifold bearing a parabolic geometry modelled on G/P, and $\phi_0 : \mathbb{P}^1 \to M$ is an unbreakable curve, then a *development* of a snake $\psi_0 : \mathbb{P}^1 \sqcup \mathbb{P}^1 \cdots \sqcup \mathbb{P}^1 \to M$ to G/P is a snake $\psi_1 : \mathbb{P}^1 \sqcup \mathbb{P}^1 \cdots \sqcup \mathbb{P}^1 \to G/P$ modelled on a development $\phi_1 : \mathbb{P}^1 \to G/P$ of ϕ_0, with isomorphism $\Psi : \psi_0^* E \to \psi_1^* G$ so that Ψ is the identity map on each pair of fibers

$$\psi_0^* E_{b_i} = \psi_0^* E_{a_{i+1}}.$$

Every development of a snake is obtained in the obvious way:

1. developing the tail to G/P, using any frame at the tail point
2. successively developing one component at a time, choosing any frame e_i at the point b_i and the corresponding frame $\Phi(e_i) \in G$ as the choice of frame to start with at the point a_{i+1}.

We will always choose to start with the frame $1 \in G$ at the tail point.

Lemma 20. *Suppose that M is a complex manifold bearing a parabolic geometry, modelled on some rational homogeneous variety G/P_-, and containing an unbreakable rational curve $\phi_0 : \mathbb{P}^1 \to M$. Let $\phi_1 : \mathbb{P}^1 \to G/P_-$ be the development of ϕ_0. The reachable set of ϕ_1 in G/P is $R(P_-) = P_+/P_-$ where $P_+ \subset P^{app}$ is a parabolic subgroup containing P_-.*

Proof. The deformation class of the unbreakable curve is G-invariant, since G is connected. Therefore the tangent cones of the reachable sets $R(gP_-)$ are all identified by G-action. Hence $R(gP_-)$ is a smooth subvariety, with tangent spaces all lying in the same G-orbit. In particular, the tangent space at $P_- \in G/P_-$ must be a P_--invariant subspace of $\mathfrak{g}/\mathfrak{p}_-$, which we can write as $\mathfrak{p}_+/\mathfrak{p}_-$, for $\mathfrak{p}_+$ a P_--invariant subspace of $\mathfrak{g}$ containing $\mathfrak{p}_-$. We need to see why $\mathfrak{p}_+$ is a Lie subalgebra. Clearly the preimage P_+ of $R(P_-)$ in G is an integral manifold of the exterior differential system

$$\omega = 0 \mod \mathfrak{p}_+.$$

The Maurer–Cartan equation

$$d\omega = -\frac{1}{2}[\omega, \omega]$$

ensures that P_+ is a subgroup with Lie algebra $\mathfrak{p}_+$.

All unbreakable curves through P_- are tangent to P^{app}/P_-. If we take the smallest subalgebra $\mathfrak{q} \subset \mathfrak{g}$ containing $\mathfrak{p}_-$ and for which $\mathfrak{q}/\mathfrak{p}_-$ contains the tangent lines of the unbreakable curves through P_-, then all unbreakable curves through P_- must be contained in Q/P_-, by G-invariance of the family of unbreakable curves. Therefore $P_+ = Q \subset P^{\text{app}}$.

Since Q contains all unbreakable curves, we find $R(P_-) \subset Q/P_- = P_+/P_-$. So $R(P_-)$ is a smooth submanifold of P_+/P_-, and has tangent space $\mathfrak{p}_+/\mathfrak{p}_-$ at P_-. Therefore $R(P_-) \subset P_+/P_-$ is an open subset, but is also compact. □

Lemma 21. *Suppose that M is a closed Kähler manifold containing an unbreakable rational curve C. Suppose that M bears a parabolic geometry modelled on a rational homogeneous variety G/P_-. Then for each point $m \in M$, there is a parabolic subgroup $P_+ \subset G$ and a holomorphic injection $P_+/P_- \to M$ whose image passes through m, so that rational circles in P_+/P_- are rational circles in M. The dimension of P_+/P_- is at least as large as the rank of the positive degree subbundle of the unbreakable curve.*

Proof. Define P_+ as in the previous lemma. The Pfaffian system

$$\omega = 0 \mod \mathfrak{p}_+$$

is defined on M modulo Cauchy characteristics and is holonomic because $\mathfrak{p}_+ \subset \mathfrak{p}^{\text{app}}$. Take $\Lambda \subset M$ the maximal connected integral manifold through $m \in M$. Let $\tilde{\Lambda}$ be the universal covering of Λ. Each snake in M with tail at m lives entirely inside Λ, and is simply connected, so lifts to a connected curve in $\tilde{\Lambda}$. Moreover, each snake in M develops to a snake in the model, with tail at $P_- \in G/P_-$. So the space of snakes $Z_p(m)$ with tail at m maps to P_+/P_-. This map is onto if p is large enough, and a submersion near a generic point. Under development, Λ is locally identified with P_+/P_- all the way along each snake. So the space of snakes $Z_p(m)$ maps to Λ. We will henceforth take p large enough so that $Z_p(m) \to \Lambda$ is a submersion near a generic point. The image in Λ is a Zariski closed subset of Λ because $Z_p(m)$ is compact. The image contains an open set, and therefore contains a Zariski open set of Λ. Since Λ is connected, the map $Z_p(m) \to \Lambda$ is onto. Therefore Λ is compact. The map $Z_p(m) \to \Lambda$ is covered by a unique map $Z_p(m) \to \tilde{\Lambda}$, lifting each snake to the covering space (since each snake is simply connected). The map $Z_p(m) \to \tilde{\Lambda}$ is a submersion near the generic point of $Z_p(m)$. By compactness of $Z_p(m)$, the map $Z_p(m) \to \tilde{\Lambda}$ is onto. Therefore $\tilde{\Lambda}$ is closed, and so $\tilde{\Lambda} \to P_+/P_-$ is a local diffeomorphism, and onto. Being a rational homogeneous variety, P_+/P_- is simply connected, so $\tilde{\Lambda} = P_+/P_-$. □

We now prove theorem 3 on page 6.

Proof. Combine theorem 1 on page 5 with lemma 14 and lemma 21. □

Remark 26. If M is a closed Kähler manifold bearing a rational curve and a parabolic geometry, then the MRC-fibration of M (see Kollár, Miyaoka and Mori [53]) must be a smooth fiber bundle, with rational homogeneous fibers, dropping M.

Example 19 (Projective connections). The only connected closed Kähler manifold bearing a projective connection and containing a rational curve is $\mathbb{P}^n$, and the only projective connection it bears is the standard flat one. This generalizes the results of Jahnke and Radloff [43]. (They require that the projective connection be torsion-free; see Gunning [36].)

We now prove corollary 6 on page 7.

Proof. There are rational curves (the various curves given by the projections $\mathbb{P}^1_\alpha \to G/B_\alpha \to G/P$) whose tangent lines span the tangent space. So even after dropping, some of those rational curves project to rational curves, unless we drop to a point. □

There are numerous techniques to find rigid rational curves (to obstruct parabolic geometries); see Kollár and Mori [54].

10. Bend–and–break in higher genus

Theorem 12. *Suppose that M is a smooth projective variety bearing a parabolic geometry. If M contains a smooth closed curve C which develops to a closed curve in the model, then M drops.*

Proof. Every curve in G/P_- admits a deformation fixing a point: just translate the curve to pass through $P_- \in G/P_-$, and employ a 1-parameter subgroup of P_-. Lemma 10 on page 17 ensures that this deformation is not trivial.

Start with our curve $C \subset M$, and develop it into a curve $\check{C} \subset G/P_-$. The infinitesimal motion of $\check{C}$ fixing the given point induces an infinitesimal motion of C fixing the corresponding point, via the isomorphism $TM|_C = T(G/P_-)|_{\check{C}}$. Because the right hand side is a vector bundle spanned by its global sections, it has trivial first cohomology, so the deformation theory is unobstructed. Moreover, the deformation theory fixing a point is also unobstructed. By bend–and–break I ([54] p. 9), there is a rational curve in M. □

Remark 27. Note that we didn't need to use positive characteristic methods. Mori employs positive characteristic methods only to obtain an infinitesimal deformation fixing a point, but we already have one. Nevertheless, we can't justify this theorem for closed Kähler manifolds, because the bend–and–break proof requires that the inclusion $C \to M$ extend to a map $D \times C \to M$, for some proper positive dimensional variety D, giving a deformation of C fixing a point. Kollár and Mori [54] p. 10 give an example to show that deformations of a curve on a closed Kähler manifold fixing a point might not be even locally isomorphic to a deformation given by such a map.

Remark 28. The previous theorem is important because it signals the end of development of closed curves as a tool in the study of complex parabolic geometries. Henceforth research will necessarily focus on geometries that are not lifted from lower dimension. This theorem tells us not to expect any closed curves to develop to the model from any of these geometries.

11. Regularity

We draw the definition of regularity from Calderbank and Diemer [13].

Definition 31. Take $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra of a semisimple Lie algebra. Write $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$, the Langlands decomposition. The *canonical filtration* of $\mathfrak{g}/\mathfrak{p}$ is the filtration of $\mathfrak{g}/\mathfrak{p}$

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset \mathfrak{g}/\mathfrak{p},$$

with $F_0 = 0$ and

$$F_{j+1} = \{ A \in \mathfrak{g}/\mathfrak{p} \mid \mathrm{Ad}_B A \in F_j \text{ for any } B \in \mathfrak{n} \} .$$

Example 20. Every cominiscule variety has $F_1 = \mathfrak{g}/\mathfrak{p}$.

Definition 32. Take the product filtration on the curvature module $\mathfrak{g} \otimes \Lambda^2 (\mathfrak{g}/\mathfrak{p})^*$. A parabolic geometry is *regular* if its curvature takes values only in the positive weight submodule of the curvature module.

Regularity of a parabolic geometry is also defined and discussed in Čap [15]. Unfortunately there is no elementary definition of regularity, but the condition of regularity is satisfied in all geometric constructions of parabolic geometries; the paper Čap and Schichl [16] provides a precise statement of this fact, and a proof.

Definition 33. Suppose that X is a noncompact Hermitian symmetric space. By the Borel theorem, X is equivariantly embedded $X \subset G/P$ into the compact dual Hermitian symmetric space, so that the biholomorphism group of X is the subgroup of G fixing X. Suppose that M is a quotient of X. The embedding $X \subset G/P$ pulls back to a canonical flat parabolic geometry on M.

Corollary 16. *Suppose that M is a closed quotient of a noncompact Hermitian symmetric space X with compact dual G/P. Then the canonical flat parabolic geometry on M is the only regular G/P-geometry.*

Proof. Klingler [45] proves the uniqueness of the reduction of structure group of the first and second order frame bundles. Čap and Schichl [16] prove that the reduction of the second order frame bundle determines the regular parabolic geometry. □

12. Nonholonomic obstruction theory

Definition 34. Suppose that $P \subset G$ is a parabolic subgroup of a semisimple Lie group, with Lie algebras $\mathfrak{p} \subset \mathfrak{g}$. We will call the vector bundle $G \times_P F_k \subset T(G/P)$ the *degree k plane field.*

Example 21. On a cominiscule rational homogeneous variety G/P (or a product of cominiscule rational homogeneous varieties), the degree 1 plane field is $T(G/P)$.

Definition 35. A plane field (i.e. a vector subbundle of the tangent bundle) is called *holonomic* if any local sections X and Y of the plane field give rise to a Lie bracket $[X, Y]$ which is also a local section. We will pass freely without mention between a Pfaffian system and the plane field of its integral vectors.

Lemma 22. *The degree k plane field on a rational homogeneous variety is holonomic just when it is the* 0 *bundle or the tangent bundle.*

Remark 29. Thus every rational homogeneous variety which is *not* a product of cominiscule varieties has a nonholonomic degree k plane field, for some value of k (indeed for every k strictly between 0 and the degree of $\mathfrak{g}/\mathfrak{p}$).

Proof. Degrees are invariant under decomposition into root spaces. Under Langlands decomposition, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$, degree is unchanged by bracketing with $\mathfrak{l} \oplus \mathfrak{a}$ and goes down under bracketing with $\mathfrak{n}$, so that degree of a root space is counted by expressing your root as a sum of negatives of simple roots,

$$\alpha = -\sum_{\alpha_i} m_i(\alpha)\, \alpha_i$$

and adding up the contributions from simple roots whose root spaces lie in $\mathfrak{n}$:

$$\deg \mathfrak{g}^\alpha = \sum_{\mathfrak{g}^{-\alpha_i} \not\subseteq \mathfrak{p}} m_i(\alpha) .$$

Bracketing the vector fields on G/P which arise from left invariant vector fields on G clearly adds degrees. In particular, holonomy of these plane fields occurs precisely when the tangent bundle has degree 1. For a cominiscule variety, the degree of the tangent bundle obviously can't exceed 1, because $\mathfrak{g}/\mathfrak{p}$ is an irreducible P-module. Consequently the tangent bundle of any product of cominiscule varieties has degree 1. We need only show that $\mathfrak{g}/\mathfrak{p}$ has degree more than 1 if G/P is not cominiscule. We can easily restrict to the case when G is simple, for which the proof is due to Kostant; see [55] p. 104. □

Remark 30. Any G-invariant plane field will give a P-invariant plane on $\mathfrak{g}/\mathfrak{p}$. Looking at the action of the Cartan subgroup on the root spaces, we see that the plane must be a sum of root spaces.

Proposition 3. *In any regular parabolic geometry, the plane fields of all degrees are nonholonomic, except for the* 0 *bundle and the tangent bundle.*

Remark 31. Thus every regular parabolic geometry whose model is *not* a product of cominiscule varieties has a nonholonomic degree k plane field, for some k (indeed for every k strictly between 0 and the degree of $\mathfrak{g}/\mathfrak{p}$).

Proof. See the (unnumbered) proposition on page 9 of Čap [15].

□

Definition 36. A line bundle L on a Kähler manifold is *pseudoeffective* if $c_1(L)$ can be represented by a closed positive $(1,1)$-current. (See Demailly [24] for more information.)

Lemma 23. *Let M be a closed Kähler manifold with nonholonomic plane field* Π. *The canonical bundle of M is* not *pseudoeffective.*

Proof. If Π has rank k, and M has dimension n, then let $\theta \in \Omega^{n-k}(M) \otimes \det(TM/\Pi)$, defined by

$$\theta(v_1, v_2, \dots, v_{n-k}) = (v_1 + \Pi) \wedge \dots \wedge (v_{n-k} + \Pi).$$

By Demailly [25] p. 1, Main Theorem applied to θ, the canonical bundle of M is not pseudoeffective. □

Corollary 17. *Suppose that G/P is* not *a product of cominiscule rational homogeneous varieties. There is no regular G/P-geometry on any closed Kähler manifold of negative Ricci curvature.*

Proof. The canonical bundle is effective. □

Theorem 13. *Suppose that G/P is an irreducible rational homogeneous variety. Suppose that M is a closed Kähler manifold with $c_1(M) < 0$. Suppose that M has a regular parabolic geometry modelled on G/P. Then G/P is a Hermitian symmetric space, M is covered by the noncompact dual symmetric space of G/P, and the parabolic geometry on M is the flat one obtained by the Borel embedding of the noncompact dual inside G/P.*

Proof. The manifold M must be a smooth projective variety because $c_1(M) < 0$. We have seen that G/P must be a product of cominiscules, and being irreducible by hypothesis must be cominiscule, i.e. a compact Hermitian symmetric space. Kobayashi and Ochiai [51] prove that M must be covered by the dual symmetric space to G/P. Klingler [45] proves that there is only one holomorphic reduction of the first and second order frame bundles of M to the structure group P. Finally, Čap and Schichl [16] prove that a regular parabolic geometry of second order is determined by its reduction of second order structure group, and that Hermitian symmetric spaces are of second order. □

Definition 37. A parabolic geometry $E \to M$ is *holonomic* if for any P-invariant subspace $V \subset \mathfrak{g}/\mathfrak{p}$, the plane field $E \times_P V \subset TM$ is holonomic.

Lemma 24. *A smooth projective variety with a nonholonomic parabolic geometry contains a rational curve.*

Proof. By Boucksom et. al. [9], a smooth projective variety is uniruled just when it has nonpseudoeffective canonical bundle. □

Theorem 14. *Let G/P be a rational homogeneous variety of positive dimension which is* not *a product of cominiscule varieties. Any regular parabolic geometry modelled on G/P on any smooth projective variety drops to a lower dimensional parabolic geometry.*

Corollary 18. *Fundamental parabolic geometries on smooth projective varieties are holonomic.*

Remark 32. Differentiating the holonomy condition uncovers global conditions on curvature. Therefore, even though we defined fundamental geometries by the open condition that they don't drop, we have found a global set of partial differential equations (a closed condition) that all fundamental geometries must all satisfy on all smooth projective varieties.

Corollary 19. *Every flat parabolic geometry with nonholonomic model on any smooth projective variety drops to a flat parabolic geometry, on a smooth projective variety, whose model is a product of cominiscule varieties.*

13. Automorphism groups

Definition 38. The *Kodaira dimension* κ_M of a closed complex manifold M is the smallest integer κ for which

$$\frac{\dim H^0\left(M, K_M^{\otimes n}\right)}{n^\kappa}$$

is finite for $n > 0$ (where K_M is the canonical bundle of M), but set $\kappa_M = -1$ if $\dim H^0\left(M, K^{\otimes n}\right) = 0$ for all but finitely many $n > 0$.

Theorem 15. *Let M be a connected smooth projective variety bearing a regular parabolic geometry modelled on G/P. Then M either contains no rational curves (in which case we set $\bar{M} = M$) or M is the total space of a bundle of rational homogeneous varieties over a base $\bar{M}$ which contains no rational curves. The parabolic geometry is lifted to M from a regular parabolic geometry on $\bar{M}$. Suppose that G is a simple Lie group. The maximal connected subgroup $\operatorname{Aut}^0 M \subset \operatorname{Aut} M$ is an abelian variety, and*

$$\dim \operatorname{Aut} M \le \dim \bar{M} - \kappa_{\bar{M}},$$

where $\kappa_{\bar{M}}$ is the Kodaira dimension of $\bar{M}$.

Remark 33. We conjecture that $\kappa_{\bar{M}} \ge 0$.

Remark 34. This theorem is similar to results of Frances [30] and Bader, Frances and Melnick [2].

Proof. Clearly we can drop until our parabolic geometry is fundamental, and then it contains no rational curves. We can then reconstruct our original manifold M as a lift of some $\bar{M}$, with $\bar{M}$ a closed Kähler manifold containing no rational curves. The automorphism groups of the parabolic geometries on M and on $\bar{M}$ are identical: biholomorphisms $E \to E$ preserving the Cartan connection 1-form. Since G is simple, the model on $\bar{M}$ has discrete kernel, so we can replace $E \to \bar{M}$ by its reduction without altering the set of infinitesimal automorphisms. The automorphisms of any smooth projective variety which is not uniruled satisfy

$$\dim \operatorname{Aut} \bar{M} \le \dim \bar{M} - \kappa_{\bar{M}},$$

where $\kappa_{\bar{M}}$ is the Kodaira dimension of $\bar{M}$; moreover, the identity component of $\operatorname{Aut} \bar{M}$ is an abelian variety (for proof see [39] p. 6 and [65]). $\square$

14. Applications

14.1. **Second order ordinary differential equations.** A path geometry is a geometric description of a system of 2nd order ordinary differential equations. To be precise, a *path geometry* is a choice of 2 nowhere tangent foliations on a manifold M^{2n+1}, the first by curves, called *integral curves*, and the second by submanifolds of dimension n, called *stalks*, both foliations being tangent to a (necessarily uniquely determined) contact plane field. Near any point there are local coordinates x, $y, \dot{y}$, with $x \in \mathbb{C}, y, \dot{y} \in \mathbb{C}^{n+1}$, and functions $f\left(x, y, \dot{y}\right)$ for which the integral curves are the solutions of

$$dy = \dot{y}\, dx,\ d\dot{y} = f\, dx,$$

while the stalks are the solutions of $dx = dy = 0$. Tanaka [72] proved that a path geometry on a manifold M induces a regular parabolic geometry $E \to M$ modelled on the flag variety $\mathbb{F}\left(0, 1|\mathbb{P}^{n+1}\right)$. Moreover, the integral curves of the model are the fibers of the diagram

$$\begin{array}{ccccccccccccc} \times & \Rightarrow & \times & - & \times & - & \bullet & - & \cdots & - & \bullet \\ & & & & & & \downarrow & & & & \\ & & \bullet & - & \times & - & \bullet & - & \cdots & - & \bullet \end{array}$$

while the stalks of the model are the fibers of the diagram

$$\begin{array}{ccccccccccccccccccc} \times & - & \bullet & - & \cdots & - & \bullet & \Rightarrow & \times & - & \times & - & \bullet & - & \cdots & - & \bullet \\ & & & & & & & & & & & & \downarrow & & & & \\ & & & & & & & & \times & - & \bullet & - & \bullet & - & \cdots & - & \bullet \end{array}.$$

In each commutative diagram, the upper and lower Dynkin diagrams differ in a single root; lets call it α. Rationality of the circles associated to α is equivalent to dropping down from the model at the top of the diagram to the model at the bottom.

Consider a system of 2nd order ordinary differential equations

$$\frac{d^2 y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right).$$

Let M be the "phase space" of these equations, i.e. a complex manifold with path geometry locally given by this system in some open set. If M is a 3-fold (i.e. $y \in \mathbb{C}$), then define the *Tresse torsion* of this particular coordinate chart to be

$$\Phi = \frac{d^2}{dx^2}\frac{\partial^2 f}{\partial \dot{y}^2} - 4\frac{d}{dx}\frac{\partial^2 f}{\partial y \partial \dot{y}} + \frac{\partial f}{\partial \dot{y}}\left(4\frac{\partial^2 f}{\partial y \partial \dot{y}} - \frac{d}{dx}\frac{\partial^2 f}{\partial \dot{y}^2}\right) - 3\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial \dot{y}^2} + 6\frac{\partial^2 f}{\partial y^2}.$$

If $y \in \mathbb{C}^n$ for some $n > 1$, i.e. M is a $(2n+1)$-fold for some $n > 1$, then define the *Fels torsion* of this coordinate chart to be

$$\Phi^I_J = \phi^I_J - \frac{1}{n}\phi^K_K \delta^I_J$$

where

$$\phi^I_J = \frac{1}{2}\frac{d}{dx}\frac{\partial f^I}{\partial \dot{y}^J} - \frac{\partial f^I}{\partial y^J} - \frac{1}{4}\frac{\partial f^I}{\partial \dot{y}^K}\frac{\partial f^K}{\partial \dot{y}^J},$$

$I, J, K, L = 1, \dots, n$.

Theorem 16. *Let M be a complex manifold of dimension $2n+1$ carrying a path geometry.*

(1) *The following are equivalent*
   (a) *The path geometry is locally isomorphic to a path geometry all of whose integral curves are rational.*
   (b) *The Tresse [Fels] torsion vanishes.*
   (c) *The equations arise locally as the equations of circles on a parabolic geometry modelled on a Grassmannian $\mathbb{F}\left(1|\mathbb{P}^{n+1}\right)$.*
(2) *The following are equivalent*
   (a) *The integral curves are all rational.*
   (b) *The path geometry is the lift of a parabolic geometry modelled on a Grassmannian $\mathbb{F}\left(1|\mathbb{P}^{n+1}\right)$.*
(3) *The following are equivalent*
   (a)
$$\frac{\partial^4 f}{\partial \dot{y}^4} = 0$$
   (b) *The path geometry is locally the path geometry of the geodesic equation of a normal parabolic geometry modelled on projective space (i.e. a normal projective connection).*
(4) *The following are equivalent*
   (a) *The stalks are biholomorphic to projective space.*
   (b) *The path geometry is the path geometry of the geodesic equation of a normal projective connection.*
   (c) *Each stalk is a closed Kähler manifold, and one of the stalks contains a rational curve.*
(5) *The following are equivalent*
   (a) *The Tresse/Fels torsion vanishes as does $\frac{\partial^4 f}{\partial \dot{y}^4}$.*
   (b) *The equations are locally point equivalent to*
$$\frac{d^2 y}{dx^2} = 0.$$
   (c) *The path geometry can is locally isomorphic to a path geometry whose integral curves and stalks are rational.*

Remark 35. A weaker result in the same direction appeared in McKay [60]. In that paper, we showed that the class of second order ordinary differential equations with vanishing Tresse or Fels torsion includes all of the classical ordinary differential equations of mathematical physics which were discovered before the work of Painlevé. Identification of the Tresse–Fels torsion with the relevant circle torsion is a calculation from the structure equations of Fels [29]. This calculation involves a change of coframing to reach the parabolic coframing of Tanaka [72], which is not Fels' coframing.

Proof. Each stalk bears a normal projective connection (which we see from the diagram), flat just when the second order equation comes from a normal projective connection (see Hitchin [37]). A projective connection on a connected closed Kähler manifold is isomorphic to its model just when the manifold contains a rational curve, because there is only one root in the Dynkin diagram whose root space does not lie in the structure algebra, so the geometry can only drop to a point. The rest follows from the theorems above. □

Theorem 17. *Every path geometry on a smooth projective variety drops to either a parabolic geometry modelled on a Grassmann variety or to a projective connection.*

In particular, in the lowest dimensional case of a path geometry, that of path geometries on 3-folds:

Theorem 18. *The only smooth projective 3-folds bearing a path geometry are the projectivized tangent bundles of*

(1) $\mathbb{P}^2$,
(2) *3-folds with unramified covering by the ball in* $\mathbb{C}^3$ *or*
(3) *3-folds with unramified covering by an abelian 3-fold.*

These path geometries are the lifts of the projective connections. In the first two cases the projective connections are the obvious flat ones, while in the third case the projective connections are translation invariant.

Remark 36. These are the first global theorems on the algebraic geometry of the phase space of a nonlinear 2nd order ordinary differential equation.

Proof. The model is not cominiscule and the parabolic geometry is regular. Therefore the parabolic geometry must drop to a projective connection on a surface. The surfaces which bear projective connections have been classified [49]. □

We can obtain some results without regularity.

Theorem 19. *The only parabolic geometries modelled on* $\times — \times$ *on smooth projective 3-folds are*

(1) *those lifted from projective connections on surfaces (as described in theorem 18) and*
(2) *translation invariant holonomic geometries on torus quotients.*

Proof. The nonholonomic geometries drop to projective connections, i.e. to geometries modelled on $\bullet — \times$ or $\times — \bullet$ (which are the same Dynkin diagram, so the same class of geometries). We need to find the holonomic geometries. Write out the structure equations, which we can put in the form

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + a^i_j \omega^2_1 \wedge \omega^3_1 + b^i_j \omega^2_1 \wedge \omega^3_2 + c^i_j \omega^3_1 \wedge \omega^3_2$$

$(i, j = 1, 2, 3)$. From the structure equations, the degree 1 plane field (a 2-plane field) is precisely the reduction modulo Cauchy characteristics of the plane field

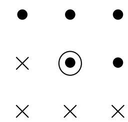


FIGURE 4. The root spaces of the Borel subalgebra of B_2.

$\left(\omega_1^3 = 0\right)$. Therefore the degree 1 plane field is holonomic just when $b_1^3 = -1$. Assume that $b_1^3 = -1$. Take exterior derivatives of all structure equations to find that $db_1^1 = \omega_3^1, db_1^2 = \omega_3^2, db_2^3 = \omega_2^1$ modulo semibasic 1-forms. The equations $b_1^1 = b_1^2 = b_2^3 = 0$ cut out a smooth subbundle, on which the structure group reduces from the parabolic subgroup down to the maximal reductive subgroup LA of the parabolic subgroup. (For proof, see Gardner [33], lecture 4.). The linear splitting

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is invariant under LA, and therefore on the subbundle the Cartan connection splits into a sum of 1-form s corresponding to this splitting. The $\mathfrak{l} \oplus \mathfrak{a}$ part of the Cartan connection is a connection for the subbundle. Therefore the Atiyah class of the subbundle vanishes, ensuring the vanishing of the characteristic classes (see McKay [59] for a more detailed explanation) and the complex 3-fold on which the geometry is defined is a torus quotient. Every parabolic geometry on a torus quotient is translation invariant (see McKay [59], theorem 4, p.26). $\square$

Remark 37. This argument provides evidence for conjecture 3 on page 44.

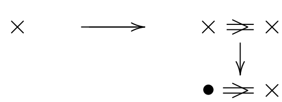
14.2. **Third order ordinary differential equations.** Sato & Yoshikawa have shown that, similarly to the results above on second order systems of ordinary differential equations, every third order ordinary differential equation for one function of one variable, say

$$\frac{d^3y}{dx^3} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right),$$

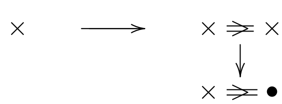
determines a parabolic geometry on the manifold M^4 whose coordinates are x, y, p, q, equipped with the exterior differential system

$$dy = p\,dx,\ dp = q\,dx,\ dq = f(x, y, p, q)\,dx.$$

The parabolic geometry is modelled on $\mathrm{Sp}\left(4, \mathbb{C}\right)/B = C_2/B = B_2/B = \mathbb{P}\mathrm{O}\left(5, \mathbb{C}\right)/B$, B the Borel subgroup, drawn in figure 4. The Dynkin diagram of the model is $\times \Rrightarrow \times$. Lets refer to a parabolic geometry with this model which arises locally from a 3rd order ordinary differential equation as a *3rd order ODE geometry.* The fibers of the bundle map



are the e_3-circles of the model (in the terminology of Sato & Yoshikawa) while those of



are the integral curves of the model (which Sato & Yoshikawa call e_4-circles).

Theorem 20. *Consider a manifold with a 3rd order ODE geometry. The following are equivalent:*

(1) *The 3rd order ODE geometry is locally isomorphic to a 3rd order ODE geometry whose integral curves are rational curves.*
(2) *The 3rd order ODE geometry locally drops to a parabolic geometry modelled on* $\times \Rrightarrow \bullet$ *, the 3-dimensional hyperquadric* $\mathbb{P}\mathrm{O}(5,\mathbb{C})/P$.
(3) *The 3rd order ODE geometry is locally isomorphic to the 3rd order ODE geometry of the equation of circles of a unique holomorphic conformal structure on a 3-dimensional manifold.*
(4) *The Chern invariant:*

$$I = -\frac{\partial f}{\partial y} - \frac{1}{3}\frac{\partial f}{\partial y'}\frac{\partial f}{\partial y''} - \frac{2}{27}\left(\frac{\partial f}{\partial y''}\right)^2 + \frac{1}{2}\frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{1}{3}\frac{\partial f}{\partial y''}\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{1}{6}\frac{d^2}{dx^2}\frac{\partial f}{\partial y''}$$

vanishes.

Remark 38. The construction of the 3rd order ordinary differential equation out of the conformal structure has been well known since work of Wünschmann (see Chern [20, 21], Dunajski & Tod [27], Frittelli, Newman & Nurowski [31], Sato & Yoshikawa [66], Silva-Ortigoza & García-Godínez [69], Wünschmann [74]). Identification of the curvature obstruction to dropping with the Chern invariant is a long but straightforward calculation (see Sato & Yoshikawa [66]). Hitchin [37] pointed out that a rational curve on a surface with appropriate topological constraint on its normal bundle must lie in the moduli space of rational curves constituting the integral curves of a unique third order ordinary differential equation with vanishing Chern invariant.

Theorem 21. *The only 3rd order ODE geometries on smooth projective varieties are the equations of circles of conformal geometries on smooth projective 3-folds. The smooth projective 3-folds which admit conformal geometries are precisely*

(1) *the quadric,*
(2) *3-folds with unramified covering by an abelian 3-fold,*
(3) *3-folds covered by the Lie ball.*

Proof. The model for 3rd order ODE geometries is not cominiscule. The geometries are regular. Therefore they drop, either to geometries modelled on $\bullet \Rrightarrow \times$, which are called contact projective geometries, or to geometries modelled on $\times \Rrightarrow \bullet$, (the hyperquadric) which are called conformal geometries. Contact projective connections are never holonomic, as we see from the structure equations of Sato and Yoshikawa, [66] p. 1000. Therefore they are isomorphic to their model. The classification of smooth projective 3-folds admitting conformal geometries is due to Jahnke and Radloff [42]. Jahnke and Radloff use a definition of conformal geometry which is equivalent to our definition with the additional hypothesis that Jahnke and Radloff require the vanishing of part of the curvature called the *torsion* (see Čap [15]). Caution is required to ensure that the dropping ensures torsion-freedom of the conformal geometry, a simple local calculation, so that the results of Jahnke and Radloff apply. □

For closed Kähler manifolds, we have a slightly weaker result.

Theorem 22. *If a closed Kähler manifold bears a 3rd order ODE geometry and contains a rational curve, then the 3rd order ODE geometry is the 3rd order ODE geometry of the equation of circles of a conformal geometry on a smooth 3-fold, or of circles in a contact projective structure on a smooth 3-fold.*

Remark 39. This is the first global theorem on the geometry of the phase space of nonlinear 3rd order ordinary differential equations.

Example 22. Hyperbolas in the plane are the solutions (along with some degenerate conics) of the differential equation

$$\frac{d^3y}{dx^3} = \frac{3\frac{d^2y}{dx^2}}{2\frac{dy}{dx}}.$$

The Chern invariant vanishes. Therefore the phase space of the problem can be extended along each hyperbola, so that the equation (perhaps in new coordinates) remains smooth, and the hyperbola closes up to a rational curve. (The obvious idea of treating x and y as an affine chart on the projective plane doesn't allow the equation to remain smooth across the hyperplane at infinity.)

Example 23. Circles in the plane are the solutions (along with some degenerate conics) of the differential equation

$$\frac{d^3y}{dx^3} = \frac{3\frac{dy}{dx}\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}.$$

The Chern invariant vanishes and the same remarks apply.

Example 24. The equation of "prescribed projective acceleration" (Cartan [19]) (also called "prescribed Schwarzian")

$$\{y, x\} = K(x),$$

(where

$$\{y, x\} = \frac{\frac{d^3y}{dx^3}}{\frac{dy}{dx}} - \frac{3}{2}\left(\frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}}\right)^2$$

is the Schwarzian derivative) has vanishing Chern invariant for any function $K(x)$, so once again the same remarks apply.

Example 25. The Chern invariant of a linear equation with constant coefficients

$$\frac{d^3y}{dx^3} = ay + b\frac{dy}{dx} + c\frac{d^2y}{dx^2}$$

is $I = -\left(a + \frac{1}{3}bc + \frac{2}{27}c^3\right)$.

Example 26. The *Jordan-Pochhammer* equation

$$x\frac{d^3y}{dx^3} = \left(\frac{x}{\kappa} + \beta\right)\frac{d^2y}{dx^2} + \left(1 + \frac{\gamma}{\kappa}\right)\frac{dy}{dx} + \frac{y}{\kappa}$$

and the *Falkner-Skan* equation

$$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} + \beta\left(1 - \left(\frac{dy}{dx}\right)^2\right) = 0$$

cannot have everywhere vanishing Chern invariant, for any choices of the various constants. Therefore their 3rd order ODE geometries are not locally isomorphic to any 3rd order ODE geometries with any rational integral curves. The same is true of the Chazy equations of types IX, X and XI (and probably of many other Chazy equations); see Cosgrove [23] for definitions of those equations.

In fact, we can do better for this specific model.

Theorem 23. *The only parabolic geometries modelled on* $\times \Rightarrow \times$ *on any smooth projective 4-fold are (1) the model and (2) translation invariant holonomic geometries on torus quotients.*

Proof. Sato and Yoshikawa [66] p. 1000 write out the complete structure equations of any regular parabolic geometry $E \to M$ with this model, in terms of 10 1-forms $\omega^1, \dots, \omega^{10}$, forming a coframing of the 10-dimensional manifold E. The structure equations, even for an irregular parabolic geometry with this model, are of course the same except for curvature terms. We use the notation $\omega^{ij} = \omega^i \wedge \omega^j$. The 1-forms $\omega^1, \omega^2, \omega^3$ and ω^4 span the semibasic 1-forms, i.e. span the conormals of the fibers of $E \to M$. The structure equations as computed by Sato and Yoshikawa say that the 2-forms

$$
\begin{aligned}
\nabla\omega^1 &= d\omega^1 + \omega^{15} + \omega^{24} \\
\nabla\omega^2 &= d\omega^2 + \omega^{16} + \omega^{27} + \omega^{34} \\
\nabla\omega^3 &= d\omega^3 + \omega^{26} + \omega^{35} + 2\,\omega^{37} \\
\nabla\omega^4 &= d\omega^4 + \omega^{18} + \omega^{29} + \omega^{45} - \omega^{47} \\
\nabla\omega^5 &= d\omega^5 + 2\,\omega^{1,10} + \omega^{28} - \omega^{46} \\
\nabla\omega^6 &= d\omega^6 + \omega^{2,10} + \omega^{38} + \omega^{56} + \omega^{67} \\
\nabla\omega^7 &= d\omega^7 + \omega^{1,10} + \omega^{28} + \omega^{39} \\
\nabla\omega^8 &= d\omega^8 + \omega^{4,10} + \omega^{69} + \omega^{78} \\
\nabla\omega^9 &= d\omega^9 + \omega^{48} - \omega^{59} + 2\,\omega^{79} \\
\nabla\omega^{10} &= d\omega^{10} + \omega^{5,10} + \omega^{68}
\end{aligned}
$$

are semibasic, i.e. $\nabla\omega^I = K^I_{ij}\omega^{ij}$, where $I = 1, \dots, 10$ and $i, j = 1, \dots, 4$. Although Sato and Yoshikawa assume regularity, this much of their argument is independent of regularity.

The Pfaffian system $\left(\omega^1 = \omega^2 = 0\right)$ has fibers of $E \to M$ as Cauchy characteristics, and projects to the degree 1 plane field on M. Similarly, the Pfaffian system $\left(\omega^1 = 0\right)$ has the fibers of $E \to M$ as Cauchy characteristics, and projects to the degree 2 plane field. Clearly from the structure equations, the degree 2 plane field is holonomic just when $K^1_{24} = 1, K^1_{23} = 0, K^1_{34} = 0$. Similarly the degree 1 plane field is holonomic just when $K^2_{34} = 1$.

Assume that these equations hold. Take the exterior derivatives of both sides of all of the structure equations. Apply Cartan's lemma to the resulting equations for

dK^I_{ij}. You will find that if you set

$$\begin{aligned}
\nabla K^1_{12} &= dK^1_{12} - \omega^8 + K^1_{12}\omega^7 + K^1_{13}\omega^6 + K^1_{14}\omega^9 \\
\nabla K^1_{13} &= dK^1_{13} - K^1_{13}\omega^5 + 2\,K^1_{13}\omega^7 \\
\nabla K^1_{14} &= dK^1_{14} + \omega^6 + K^1_{14}\left(\omega^5 + \omega^9 - \omega^7\right) \\
\nabla K^6_{34} &= dK^6_{34} - \omega^{10} + \left(K^7_{34} - K^5_{34}\right)\omega^6 - K^3_{34}\omega^8 + K^6_{34}\omega^5,
\end{aligned}$$

then all of these 1-forms are forced to be semibasic. For instance, you find

$$0 = \nabla K^1_{12} \wedge \omega^{12} + \nabla K^1_{13} \wedge \omega^{13} + \nabla K^1_{14} \wedge \omega^{14},$$

which forces $\nabla K^1_{12}, \nabla K^1_{13}, \nabla K^1_{14}$ to be semibasic 1-forms. By the same argument as in theorem 19 on page 36, the subset where $K^1_{12} = K^1_{13} = K^1_{14} = K^6_{34} = 0$ is a smooth subbundle, whose structure group is the maximal reductive subgroup of our parabolic structure group P. Therefore by the same argument again, the parabolic geometry is translation invariant, on a torus. □

Remark 40. This argument is further evidence for conjecture 3 on page 44.

14.3. **Parabolic geometries on 3-folds.**

Theorem 24. *The only regular parabolic geometry modelled on* $\bullet \Rrightarrow \times$ *on any smooth projective variety is the model* $\mathbb{P}^3$.

Proof. The model is not a product of cominiscules, so any regular parabolic geometry with that model must drop. □

Theorem 25. *All curves bear parabolic geometries modelled on* $\mathbb{P}^1$. *The only smooth projective surfaces bearing parabolic geometries with connected Dynkin diagrams are* $\mathbb{P}^2$ *and ball and torus quotients. The only smooth projective 3-folds bearing regular parabolic geometries with connected Dynkin diagrams are*

(1) $\times - \times$ *path geometries lifted from projective connections: on projectivized tangent bundles of* $\mathbb{P}^2$ *and ball and torus quotients,*
(2) $\times - \bullet - \bullet$ *projective connections: on* $\mathbb{P}^3$*, ball and torus quotients, and étale quotients of smooth modular families of false elliptic curves,*
(3) $\times \Rrightarrow \bullet$ *conformal connections: on the hyperquadric* Q^3*, torus quotients, and quotients of the bounded symmetric domain dual to the hyperquadric (a.k.a. the* Lie ball*),*
(4) $\bullet \Rrightarrow \times$ *symplectic connections: on* $\mathbb{P}^3$ *(the model).*

Remark 41. We assumed that the Dykin diagram of the model is connected. Lets briefly consider disconnected Dynkin diagrams. There is one rational homogeneous surface with disconnected Dynkin diagram: $\times \quad \times$, i.e. $\mathbb{P}^1 \times \mathbb{P}^1$. The associated parabolic geometries are conformal connections on surfaces. There are two 3-dimensional rational homogeneous varieties with disconnected Dynkin diagrams: $\times \quad \times \quad \times = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\times \quad \times - \bullet = \mathbb{P}^1 \times \mathbb{P}^2$. See below for more about the possibilities with disconnected Dynkin diagrams.

Proof. All of the cases except the last are due to Kobayashi and Ochiai [49] or Jahnke and Radloff [43, 42]. The classification of 3-dimensional rational homogeneous varieties follows immediately from the classification of simple Lie groups in low dimensions. □

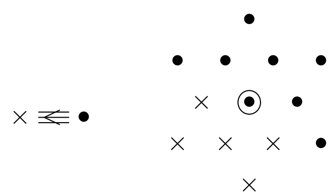


FIGURE 5. The parabolic subgroup $P_1 \subset G_2$

14.4. **Disconnected Dynkin diagrams.** If a Dynkin diagram has more than one component, then the tangent bundle of any parabolic geometry with that model splits into a sum of corresponding vector bundles. There is a great deal known about closed Kähler manifolds with split tangent bundle; see Brunella et. al. [10].

Definition 39. Lets say that a splitting of the tangent bundle is *ordinary* if some covering space is a product, with the pulled back subbundles of the splitting identified with the tangent bundles of the factors.

Beauville conjectured that a splitting of the tangent bundle of a closed Kähler manifold is ordinary just when the summands of the splitting are holonomic (i.e. closed under Lie bracket). This conjecture is so far confirmed [10] only if there are precisely two summands, one of which is a line bundle.

Theorem 26. *Consider a Dynkin diagram with one or more $\times$ components. If a parabolic geometry on a complex manifold M has such a model, then the associated splitting of the tangent bundle is ordinary.*

Proof. By [10] theorem 1.1, p. 242, either the splitting is ordinary or the circles of some $\times$ are rational. But then the geometry drops, ensuring that it is a product, so the splitting is ordinary. □

In particular, the closed Kähler manifolds admitting geometries modelled on

$$\times \quad \times \quad \cdots \quad \times$$

are precisely the quotients of products of curves.

Remark 42. It is not known if parabolic geometries modelled on $\times \quad \times\!-\!\!\bullet$ on smooth projective 3-folds must be locally products of parabolic geometries modelled on $\times$ and $\times\!-\!\!\bullet$.

14.5. **Nondegenerate 2-plane fields on 5-folds.** Given a plane field Π on a manifold M, let Π' be the sheaf of $\mathcal{O}$-modules spanned by local sections of Π and brackets of pairs of local sections of Π. Declare a 2-plane field Π on a 5-manifold M to be *nondegenerate* if Π' is a 3-plane field (i.e. locally free of rank 3), and $\Pi'' = TM$.

The group G_2 has root lattice drawn in figure 5, with the roots of a parabolic subgroup P_1 drawn as dots. From the root lattice, one sees that the two crosses closest to the center of the diagram (i.e. to the Cartan subalgebra) span a 2-plane field in the tangent bundle of G_2/P_1, which is a P_1-invariant subspace, and therefore a G_2-invariant 2-plane field.

It is well known (see [17, 71, 72]) that a nondegenerate 2-plane field on a 5-dimensional manifold determines and is determined by a parabolic geometry modelled on G_2/P_1.

Theorem 27. *The only nondegenerate 2-plane field on any smooth projective 5-fold is the standard one on* G_2/P_1.

Proof. The 2-plane field is nonholonomic, so the result follows from corollary 18 on page 33. □

We have weaker results for closed Kähler manifolds.

Theorem 28. *The only closed Kähler 5-fold containing a rational curve and bearing a nondegenerate 2-plane field is* G_2/P_1 *with the standard* G_2*-invariant 2-plane field.*

Theorem 29. *A nondegenerate 2-plane field on a 5-manifold is locally isomorphic to one whose bicharacteristic curves (the circles tangent to the 2-plane field) are rational just when the 2-plane field is locally isomorphic to the standard 2-plane field on* G_2/P_1.

Remark 43. These are the first global theorems on nondegenerate 2-plane fields.

14.6. **Nondegenerate 3-planes on 6-folds and 4-plane fields on 7-folds.** Similarly, declare a 3-plane field on a 6-manifold M to be *nondegenerate* if $\Pi' = TM$. Finally, we won't give the elaborate analogous condition on a 4-plane field to be nondegenerate (see Biquard [7]). It is well known (see Čap [15], [7]) that nondegenerate 3-plane fields on 6-folds and nondegenerate 4-plane fields on 7-folds determine regular parabolic geometries modelled on

$$\bullet - \bullet \Rightarrow \times$$

and

$$\bullet - \times \Leftarrow \bullet \, .$$

The results above for 2-plane fields on 5-manifolds have obvious generalizations:

Theorem 30. *The only nondegenerate* $2, 3, 4$*-plane field on any smooth projective 5,6,7-fold is the standard one on* $G_2/P_1, \mathbb{P}\mathrm{O}\left(6, \mathbb{C}\right)/P_3, \mathrm{Sp}\left(\mathbb{C}^6\right)/P_2$.

Proof. The plane field is nonholonomic, so the result follows from corollary 18 on page 33. □

Theorem 31. *The only closed Kähler 5,6,7-fold containing a rational curve and bearing a nondegenerate 2,3,4-plane field is* $G_2/P_1, \mathbb{P}\mathrm{O}\left(6, \mathbb{C}\right)/P_3, \mathrm{Sp}\left(\mathbb{C}^6\right)/P_2$ *with the standard* $G_2, \mathbb{P}\mathrm{O}\left(6, \mathbb{C}\right), \mathrm{Sp}\left(\mathbb{C}^6\right)$*-invariant 2,3,4-plane field.*

Theorem 32. *A nondegenerate 2,3,4-plane field on a 5,6,7-manifold is locally isomorphic to one whose bicharacteristic curves (the circles tangent to the 2,3,4-plane field) are rational just when the 2,3,4-plane field is locally isomorphic to the standard 2,3,4-plane field on* $G_2/P_1, \mathbb{P}\mathrm{O}\left(6, \mathbb{C}\right)/P_3, \mathrm{Sp}\left(\mathbb{C}^6\right)/P_2$.

These are the first global theorems on nondegenerate 2,3,4-plane fields on 5,6,7-folds.

15. Summary

We have found that rationality of certain circles enables us to expand the structure group of a parabolic geometry, into a natural structure group. This applies to ordinary differential equations of second and third order, to pick out some very special equations (including those of mathematical physics) whose solutions close up to become rational curves. We have an explicit mechanism for testing rationality of solutions of ordinary differential equations, and of circles of arbitrary parabolic geometries.

Characteristic class obstructions to Cartan geometries follow from arguments like those of Gunning [36], Kobayashi [48], and Jahnke and Radloff [43, 42]; see McKay [59].

Conjecture 1 (Dumitrescu). Holomorphic rigid geometric structures (in the sense of Gromov [35]) on closed Kähler manifolds are locally homogeneous.

Remark 44. Rigid geometric structures include Cartan geometries (see Quiroga-Barranco and Candel [64]).

Dumitrescu (personal communication) gave an example of an affine connection on a closed complex surface which is nowhere locally homogeneous, so the conjecture can not extend beyond the category of Kähler manifolds. Robert Bryant (personal communication) gave an example of an inhomogeneous holomorphic G-structure of infinite type on a smooth projective variety.

Conjecture 2. All results proven in this paper for smooth projective varieties hold equally for closed complex manifolds.

One could weaken this conjecture to closed Kähler manifolds instead of closed complex manifolds.

Conjecture 3. If G is simple, and G/P is not cominiscule, then every fundamental parabolic geometry modelled on G/P on any closed complex manifold is a translation invariant parabolic geometry on a torus, modulo replacing by a finite covering space.

Conjecture 4. Regular parabolic geometries on closed complex manifolds drop to holonomic parabolic geometries and can be deformed into flat parabolic geometries.

One could try to weaken this conjecture to closed Kähler manifolds, or to smooth projective varieties, instead of closed complex manifolds.

Rather than developing rational curves, we could develop complex affine lines, giving Nevanlinna theory of complex parabolic geometries, to see when they might admit a "large family" of complex lines, without admitting rational curves. Similarly, we could develop complex disks, giving various Kobayashi pseudometrics.

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University College Cork, National University of Ireland