

COMPLEX NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS AND GEOMETRY

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1. INTRODUCTION

1.1. Differential equations and curves. This paper belongs to the mathematics of the 17th-19th centuries. More specifically, I want to think about ordinary differential equations in complex variables. As the principal example which we will frequently return to, consider

$$\frac{d^2w}{dz^2} = 0,$$

a differential equation in two complex variables z and w . The solutions are of course the linear functions, $w = a + bz$. Lets keep this example in back of our minds, while we consider the general theory of second order ordinary differential equations

$$\frac{d^2w}{dz^2} = f\left(z, w, \frac{dw}{dz}\right).$$

Solutions of any complex ordinary differential equation are complex analytic functions. But I want to picture them geometrically, as curves in the z, w -plane. In my example, those curves will be the straight lines. Of course, the graph of a complex analytic function is a surface in a 4-dimensional space, not a curve in a plane. But it is convenient to think of the graph a complex analytic function as a “complex curve.”

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1.2. Projective geometry as analogue. Two further subtleties arise when we think about our example more carefully. First, if we think about lines in the plane, we include among them the vertical lines. These are not solutions of our differential equation. However, geometrically the lines which are neither horizontal nor vertical are solutions to either of the equations

$$\frac{d^2 w}{dz^2} = 0 \text{ or } \frac{d^2 z}{dw^2} = 0.$$

So we can imagine that an ordinary differential equation might have the same generic solution as some other equation in perhaps a different choice of variables, i.e. after a coordinate change. In our example, this is the coordinate change $(z, w) \rightarrow (w, z)$. We want to consider such a pair of equations to be equivalent, and “glue them together”, just as we would “glue together” the various branches of a multivalued analytic function.

There is a further subtlety, of the same kind, occurring in our example. Consider the coordinate change taking a point (z, w) to the point

$$(\zeta, \omega) = \left(\frac{1}{z}, \frac{w}{z} \right).$$

The lines still become lines:

$$w = a + bz \rightarrow \omega = a\zeta + b.$$

This allows us to include on each line all of the usual points, as well as the point $\zeta = 0, \omega = b$, which does not correspond to a point of the line in the z, w -plane. It is the point at infinity. Moreover, in the ζ, ω -plane, there is an additional line: $\zeta = 0$, which does not correspond to any line in the z, w -plane. Putting this all together, we have added a point “at infinity” on each line, and these points at infinity form a line at infinity.

We should certainly not expect to be able to attach points “at infinity” to the solutions of arbitrary ordinary differential equations. Indeed, it seems to be a rare phenomenon that one could sensibly attach such points, via some clever coordinate change that took us to a new ordinary differential equation near infinity. However, the main observation here is that the solutions of our example equation close up at infinity, and the equation continues to make sense across infinity, as long as we make the right choice of coordinate change. We want to classify differential equations exhibiting this phenomenon. The complex plane with a point at infinity attached is called the *Riemann sphere*, and is topologically a sphere.

1.3. The problem. Our problem: classify the second order ordinary differential equations whose solutions are Riemann spheres. More precisely, we want to determine which equations admit analytic continuation along each solution, so that (after possibly some coordinate changes) the solutions close up to become smooth complex curves, which are topologically spheres. We will call these equations *straight*.

The reader might well wonder whether there could be a better choice of complex curve to build a theory on than the Riemann sphere. Perhaps there is something simpler. Rest assured that the function theory of the Riemann sphere is the simplest possible, by far simpler than even the complex plane, or any other complex curve. The curious reader is strongly encouraged to read the beautiful books of Clemens [3] and Forster [5]. Analysts speak of the Riemann sphere (or the “extended complex plane”), but geometers always refer to rational curves to mean complex curves

which are topologically spheres. It turns out that rational curves can always be identified with the Riemann sphere by some homeomorphism which matches up the local and global function theory.

1.4. The solution.

Definition 1. For a second order ordinary differential equation

$$\frac{d^2w}{dz^2} = f\left(z, w, \frac{dw}{dz}\right),$$

(with f complex analytic) define the *torsion* [8, 2, 1]:

$$\frac{d^2}{dz^2} \frac{\partial^2 f}{\partial \dot{w}^2} - 4 \frac{d}{dz} \frac{\partial^2 f}{\partial w \partial \dot{w}} + \frac{\partial f}{\partial \dot{w}} \left(4 \frac{\partial^2 f}{\partial w \partial \dot{w}} - \frac{d}{dz} \frac{\partial^2 f}{\partial \dot{w}^2} \right) - 3 \frac{\partial f}{\partial w} \frac{\partial^2 f}{\partial \dot{w}^2} + 6 \frac{\partial^2 f}{\partial w^2}.$$

The torsion depends on derivatives of fourth order of the function f .

Example 1. The Lienard equation

$$\frac{d^2w}{dz^2} + w \frac{dw}{dz} + \frac{1}{9}w^3 = 0$$

has vanishing torsion.

Theorem 1 (McKay [6]). *A second order ordinary differential equation is torsion-free (i.e. the torsion vanishes) just when it is straight.*

2. TORSION

Tresse [8] discovered his torsion expression by asking a simpler question: what are the invariants of an ordinary differential equation of second order, under change of variables? In fact, he considered a somewhat more general notion of changing variables, on the phase space of an ordinary differential equation.

To motivate this idea, lets return once again to our example of lines in the plane. We can consider the plane itself as the “configuration space,” in the terminology of physicists. The “phase space” is the space of pointed lines, i.e. of choices of a line in the plane, and a point on that line. This phase space is clearly three dimensional, since there is a two-dimensional space of points (the plane), and through each point, a one-dimensional space of lines (determined by angle). Of course, once again we are being cagey about whether we are working with real or complex variables, in using a term like “angle”. But since the intuition comes from real variables, we can work with real variables for the moment.

So lets imagine that we have (x, y) coordinates on the plane, and then (x, y, θ) coordinates on the phase space. Tresse’s geometric picture is that the phase space is a three-dimensional block, but sliced into ribbons by the curves we obtain by varying θ and fixing x and y . These curves are in 1-1 correspondence with the points of the plane. There is however another collection of curves slicing up the three-dimensional block, in another direction: the curves we obtain by fixing a choice of line, and moving (x, y) along that line. This, obviously, fixes the angle θ . The reader might wish to try to draw for her own understanding the two different slicings of the three-dimensional phase space.

This is only for our example, so far. But every ordinary differential equation

$$\frac{d^2w}{dz^2} = f\left(z, w, \frac{dw}{dz}\right).$$

has a phase space, which for us is parameterized locally by variables (z, w, p) , say, so that z and w parameterize the z, w -plane, and p is some additional variable. We can view the first collection of curves slicing up the phase space as the curves of constant z, w -values, varying only p . We can view the other collection of curves as the solutions of the system

$$\begin{aligned}\frac{dw}{dz} &= p, \\ \frac{dp}{dz} &= f(z, w, p).\end{aligned}$$

This is an old trick, which goes by many names, and is usually presented to students in their first course in differential equations.

Tresse considers the general problem of when a three-dimensional space with two families of curves slicing it up comes about from an ordinary differential equation in this way. His first observation is that (1) the curves from each of the two slicings-up must smoothly cut up the three-dimensional space, with no two curves from the same family passing through the same point. His second is that (2) if a curve from one family strikes the same point as a curve from the other family, then the two curves cannot be tangent. Finally, he notes that (3) any vector field perpendicular to the curves of both families must have nonzero curl. These conditions can be checked explicitly. Tresse then uses the theorem of Frobenius (see Spivak [7]) to prove that these three conditions precisely characterize the geometry of a second order ordinary differential equation. One can recover the equation from the two families of curves directly.

Tresse made a monumental computation to find which combinations of derivatives of the function f above will be invariant under coordinate changes. In fact, he does not find the torsion this way, but rather discovers in his amazing long calculations that the torsion changes from one coordinate system to another, but in the process get multiplied by a nonzero factor. Hence the torsion can change under coordinate changes, but vanishing of the torsion holds in one choice of coordinates just when it vanishes in any.

3. RATIONALITY OF SOLUTIONS

3.1. Global function theory of rational curves.

Theorem 2. *Every holomorphic function on the Riemann sphere is constant.*

Proof. Such a function f must have a maximum modulus $|f|$ at some point of the Riemann sphere, by compactness. But by the maximum modulus theorem, only constant functions have interior maxima of modulus. Every point of the sphere is interior. \square

3.2. Vanishing torsion from rationality. Unfortunately, torsion is *not* a function. One can compute it as a function in one coordinate system, but then we get different values if we change coordinates. So this elementary argument won't work as it stands. Indeed the argument really requires that we have a function defined all the way around the Riemann sphere, right out to infinity.

What is the torsion, if not a function? Lets consider a simpler problem: what is an integral in complex analysis? We can integrate things that look like $f(z) dz$. Geometers call these 1-forms. They transform under a change of variables in the

only way they can: by the change of variables formula for integrals. So, for example, if we make a new variable $\zeta = \frac{1}{z}$, then we change the 1-form according to

$$f(z) dz = -\frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right) d\zeta.$$

This is just undergraduate calculus substitution in an integral. If we have a 1-form defined on the entire Riemann sphere, then

$$f\left(\frac{1}{\zeta}\right) \cong \zeta^2,$$

so

$$f(z) \cong \frac{1}{z^2},$$

decays quadratically at infinity. Therefore $f(z)$ must extend to the entire Riemann sphere continuously, setting $f(\infty) = 0$. But by the theorem above, $f(z)$ must be constant, and since $f(\infty) = 0$, we must have $f(z) = 0$ everywhere.

Torsion behaves in a very similar way under coordinate changes: if the solution of an ordinary differential equation closes up to a rational curve, then torsion is forced to vanish everywhere on that curve. Therefore if the generic solution is rational, then the torsion vanishes everywhere.

4. DUALITY

Tresse's geometric picture of second order ordinary differential equations has a natural symmetry to it: there are two families of curves in phase space, and if we change our minds about which is which, we still get a second order ordinary differential equation, by Tresse's characterization. This trick interchanges the configuration space with the space of solutions, yielding an ordinary differential equation on the space of solutions. For example (a simple, but not very serious, example): if we start with

$$\frac{d^2w}{dz^2} = -\left(\frac{dw}{dz}\right)^2,$$

the solutions are

$$w = \log(z + a) + b.$$

Now fix z and w , and think of a and b as variables. (They parameterize the space of solutions.) Differentiate once in a :

$$0 = \frac{1}{z + a} + \frac{db}{da},$$

and once again:

$$0 = -\frac{1}{(z + a)^2} + \frac{d^2b}{da^2}.$$

Solve for all z and w variables appearing:

$$0 = -\left(\frac{db}{da}\right)^2 + \frac{d^2b}{da^2}.$$

This is the *dual* ordinary differential equation, obtaining by switching the choices of families of curves in phase space.

As an easier example, our standard example equation

$$\frac{d^2w}{dz^2} = 0$$

is self-dual in this sense.

Duality shows us that the invariants under coordinate changes of an equation and of its dual must get interchanged by duality. Cartan [2] creates a complicated dictionary explaining how to compute the invariants of the dual from the invariants of the original equation. The power of duality comes from the fact that Tresse's complicated nonlinear torsion is interchanged under duality with the simple linear expression

$$\frac{d^4 f}{d\dot{w}^4}.$$

In particular, an equation has vanishing torsion just when its dual equation has the form

$$\frac{d^2 w}{dz^2} = \sum_{k=0}^3 a_k(z) \left(\frac{dw}{dz} \right)^k.$$

4.1. Rationality from vanishing torsion. The vanishing of torsion implies that the dual equation has this very simple form. We now want to see why this forces rationality of the solutions of the original equation. But in terms of the dual equation, we are asking about rationality of the curves obtained by varying p and fixing z, w . The curves represent the various directions (angles) at the point z, w in configuration space. Since the equation has this simple form, we can check that the equation extends to all directions, by analytic continuation (just changing variables from (z, w) to (w, z)). Therefore there is a unique solution in each direction. The phase space above the configuration point (z, w) is identified analytically with the "circle" of directions through the point (z, w) , which is a rational curve.

5. CONCLUSIONS AND DIRECTIONS FOR FURTHER RESEARCH

Cartan [2] proves that the generic torsion-free equation is an integrable system. The examples that have arisen are (perhaps not surprisingly) rather dull. All of the torsion-free ordinary differential equations found so far turn out to be integrable, by always by some elementary trick. The author has checked all of the equations given in all of the large handbooks in his library, coming up with about a dozen examples that are torsion-free.

The same ideas work in the context of 3rd order equations [6], and for higher order as well [4]. It seems possibly that they might work for complex curves somewhat more complicated than rational curves. For equations whose solutions are more complicated complex curves, the moduli ([3]) of the curves provide conservation laws, so I anticipate that the equations will be integrable, but through a more exciting mechanism.

A few advanced remarks: The requirement that the phase space of a second order complex ordinary differential equation be a closed Kähler manifold turns out to produce topological constraints. Since Mori theory (after Siu) tells us so much about three-dimensional closed Kähler manifolds, it should be possible to classify the phase spaces, and the equations. These would be very special equations, with a kind of nonlinear Painlevé property.

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