COMPLETE CARTAN CONNECTIONS

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Abstract. We prove that the completeness of a Cartan geometry on a manifold is equivalent to all curves in the manifold developing to the model.

1. Introduction

Definition 1. Let $H \subset G$ be a closed subgroup of a Lie group, with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. A G/H-geometry (a.k.a. a *Cartan geometry* modelled on G/H) on a manifold M is a principal right H-bundle $E \to M$ and a 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$ called the *Cartan connection*, which satisfies the following conditions:

(1) Denote the right action of $h \in H$ on $e \in E$ by $r_h e$. The Cartan connection transforms in the adjoint representation:
$$r_h^* \omega = \mathrm{Ad}_h^{-1} \omega.$$
(2) $\omega_e : T_e E \to \mathfrak{g}$ is a linear isomorphism at each point $e \in E$.
(3) For each $A \in \mathfrak{g}$, define a vector field $\vec{A}$ on E by the equation $\vec{A} \lrcorner \omega = A$. For $A \in \mathfrak{h}$, the vector fields $\vec{A}$ generate the right H-action:
$$\vec{A}(e) = \left. \frac{d}{dt} r_{e^{tA}} e \right|_{t=0}.$$

All statements in this paper hold equally true for real or holomorphic Cartan geometries.

Example 1. The bundle $G \to G/H$ is a G/H-geometry, called the *model*, with Cartan connection $\omega = g^{-1}\, dg$ the left invariant Maurer–Cartan 1-form on G.

Definition 2. A Cartan geometry is *complete* if all of the vector fields $\vec{A}$ are complete for all $A \in \mathfrak{g}$.

Example 2. The model is complete, as the $\vec{A}$ are the right invariant vector fields on G.

Lemma 1 (Sharpe [3] p. 188, theorem 3.15). The Cartan connection of any Cartan geometry $\pi : E \to M$ determines isomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \pi'(e) & \longrightarrow & T_e E & \longrightarrow & T_m M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{h} & \longrightarrow & 0
\end{array}$$

for any points $m \in M$ and $e \in E_m$; thus
$$TM = E \times_H (\mathfrak{g}/\mathfrak{h}).$$

Date: May 1, 2008.
1991 *Mathematics Subject Classification.* MA53A40.

Definition 3. Suppose that $E_0 \to M_0$ and $E_1 \to M_1$ are two G/H-geometries, with Cartan connections ω_0 and ω_1, and X is manifold, perhaps with boundary and corners. A smooth map $\phi_1 : X \to M_1$ is a *development* of a smooth map $\phi_0 : X \to M_0$ if there exists a smooth isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* E_1$ of principal H-bundles identifying the 1-forms ω_0 with ω_1.

Development is an equivalence relation. For example, a development of an open set is precisely a local isomorphism. The graph of Φ is an integral manifold of the Pfaffian system $\omega_0 = \omega_1$ on $\phi_0^* E_0 \times E_1$, and so Φ is the solution of a system of (determined or overdetermined) differential equations, and conversely solutions to those equations determine developments. By lemma 1 on the preceding page the developing map ϕ_1 has differential ϕ_1' of the same rank as ϕ_0' at each point of X.

Definition 4. Suppose that $E_0 \to M_0$ and $E_1 \to M_1$ are G/H-geometries. Suppose that $\phi_1 : X \to M_1$ is a development of a smooth map $\phi_0 : X \to M_0$ with isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* E_1$. By analogy with Cartan's method of the moving frame, we will call e_0 and e_1 *frames* of the development if $\Phi(e_0) = e_1$.

Definition 5. Suppose that $E_0 \to M_0$ and $E_1 \to M_1$ are G/H-geometries. We will say that M_1 *rolls freely* on curves in M_0 to mean that every curve $\phi_0 : C \to M_0$ has an unramified covering which admits a development with any chosen frames $e_0 \in \phi_0^* E_0$ and $e_1 \in E_1$.

Note that we get to pick $e_1 \in E_1$ arbitrarily, and then we can construct a development $\phi_1 : C \to M_1$ with e_1 lying in $\phi_1^* E_1$.

We obtain the same development if we replace the frames by $e_0 h$ and $e_1 h$, for any $h \in H$. The curve mentioned could be a real curve, or (if the manifolds and Cartan geometries are complex analytic) a complex curve, with boundary and corners and arbitrary complex analytic singularities. The frames determine the development, and any frames yield a local development, because the equation $\omega_0 = \omega_1$ is a system of first order ordinary differential equations in local coordinates.

Example 3. A Riemannian geometry is a Cartan geometry modelled on Euclidean space. As a Riemannian manifold, the unit sphere rolls freely on curves in the plane. Rolling in this context has its obvious intuitive meaning (see Sharpe [3] pp. 375–390). The development of a geodesic on the plane is a geodesic on the sphere, so a portion of a great circle. The upper half of the unit sphere does *not* roll freely on the plane (by our definition), because if we draw any straight line segment in the plane of length more than π, we can't develop all of it to the upper half of the sphere.

Theorem 1. Let M_1 be a manifold with a Cartan geometry. The following are equivalent:

(1) M_1 is complete.
(2) M_1 rolls freely on curves in its model.
(3) M_1 rolls freely on curves in any Cartan geometry with the same model.
(4) M_1 rolls freely on curves in some Cartan geometry with the same model.

Remark 1. Kobayashi [2] stated a weaker result; the proof has never appeared.

2. Proof of the theorem

Proof. Suppose that $E_1 \to M_1$ is complete. The local existence and uniqueness of a development is clear by applying the Frobenius theorem to the Pfaffian system

$\omega_1 = \omega_0$ on $\phi_0^* E_0 \times E_1$. (The curvature does not affect the involutivity of this Pfaffian system.) The maximal connected integral manifolds project locally diffeomorphically to $\phi_0^* E_0$, because ω_0 is a coframing on them. The problem is reduced to proving that these maximal connected integral manifolds project onto $\phi_0^* E_0$.

If $C = \mathbb{CP}^1$, we can first try to prove existence and uniqueness of a development of $C \setminus \text{pt}$, for two different choices of $\text{pt} \in C$. If we can do that, then we can clearly prove existence and uniqueness of a development for all of C. Therefore lets assume that C is a real or complex curve which is not $\mathbb{CP}^1$.

After replacing C with a covering space, we can assume that $\phi_0^* E_0 \to C$ is a trivial bundle, with a global section s_0. If we can develop, then this global section is identified via the isomorphism Φ with a global section $s_1 : C \to \phi_1^* E_1$ so that

$$s_1^* \omega_1 = s_0^* \omega_0. \tag{2.1}$$

Conversely, if we can solve this equation, then there is a unique isomorphism Φ for which

$$\Phi(s_0 h) = s_1 h$$

for all $h \in H$, by triviality of the bundles. So it suffices to solve equation 2.1.

If $E_1 \to M_1$ is isomorphic to the model, then it is elementary to solve equation 2.1, as in Bryant [1] or Sharpe [4], i.e. it is elementary to develop to the model. Indeed equation 2.1 is an ordinary differential equation of Lie type,

$$g^{-1}\, dg = s_0^* \omega_0,$$

so has a unique global solution with given initial condition $g = g_0$ at $t = t_0$. For example, if G is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ for some n, then global existence and uniqueness of a development follow from writing the ordinary differential equations of Lie type as a linear ordinary differential equation:

$$dg = g\, s_0^* \omega_0.$$

More generally, one glues together local solutions by using the group action of G to make two local solutions match up at some point. Then the local solutions match near this point by uniqueness. Any compact simply connected subset of C is covered by finitely many domains of such local solutions, which thereby must patch to a global solution. This proves Kobayashi's unproven theorem.

On the other hand, if we can solve equation 2.1 when $E_0 \to M_0$ is isomorphic to the model, i.e. develop curves from the model, then we can first develop to the model and then develop from the model. So it suffices to assume that $E_0 \to M_0$ is the model.

Since local existence and local uniqueness is assured, global uniqueness is assured. To ensure global existence, we need only prove that we can extend the local solution along every map $[0,1] \to C$. So it suffices to prove the result for C a real curve. Take any smooth curve $g(t) \in G, 0 \le t \le 1$. Let

$$A(t) = g^{-1}(t) \frac{dg}{dt}.$$

We need to construct a curve $s_1 : [0,1] \to E_1$ so that

$$\frac{ds_1}{dt} \lrcorner\, \omega_1 = A(t).$$

For each $\varepsilon > 0$, construct a piecewise constant function $A^\varepsilon : [0,1] \to \mathfrak{g}$ so that $A^\varepsilon \to A$ uniformly as $\varepsilon \to 0$. The time-varying vector field $\vec{A}^\varepsilon$ on E_1 (and on G)

is clearly complete, because its flow is just the composition of the flows of various complete vector fields. The time-varying vector fields $\vec{A}^{\varepsilon}$ converge uniformly on compact sets of E_1 to the smooth time-varying vector field $\vec{A}$. Therefore $\vec{A}$ is complete. Its flow through $e_1 \in E_1$ is our required curve $s_1(t)$.

Conversely, suppose that M_1 rolls freely on curves in all Cartan geometries modelled on G/H. In particular we can take $M_0 = G/H$ the model. The one-parameter subgroups of G that don't lie in H quotient to entire curves on G/H, which develop to entire curves in M_1, with isomorphism Φ. The infinitesimal generator $\vec{A}$ of the one-parameter subgroup is tangent to the one-parameter subgroup, and therefore to the bundles that the isomorphism is identifying, and thus the isomorphism identifies the flows of $\vec{A}$ on G and on E_1. Therefore $\vec{A}$ is complete on E_1.

Finally, suppose that a Cartan geometry $\pi_1 : E_1 \to M_1$ rolls freely on curves in some Cartan geometry $\pi_0 : E_0 \to M_0$. Take points $e_0 \in E_0, m_0 \in M_0, e_1 \in E_1$ and $m_1 \in M_1$ with e_j in the fiber of E_j above m_j $(j = 0, 1)$. Take a map $\phi_0 : C \to M_0$ of a curve C with $\phi_0(c) = m_0$, some $c \in C$. Freedom of rolling means that there is a map $\phi_1 : C \to M_1$ and isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* E_1$ so that $\Phi(e_0) = e_1$ and $\omega_0 = \omega_1$ on the graph of Φ. Pick any $A \in \mathfrak{g}$. Consider a flow line $\Psi_0(t)$ of $\vec{A}$, so $\Psi_0(0) = e_0$ and $\Psi_0'(t) \lrcorner\, \omega_0 = A$. Let $\phi_0(t) = \pi_0(\Psi_0(t))$. Construct the curve ϕ_1 and isomorphism Φ. Then let $\Psi_1(t) = \Phi(\Psi_0(t))$. Clearly $\Psi_1(0) = e_1$ and $\Psi_1'(t) \lrcorner\, \omega_1 = A$, so Ψ_1 is a flow line of $\vec{A}$ on E_1. Therefore *every* flow line of $\vec{A}$ on E_1 is defined for as long a time as *any* flow line of $\vec{A}$ on E_0 through any point. In particular, since any flow line of $\vec{A}$ on E_0 must be defined for some positive time T, with $0 < T \le \infty$, every flow line of E_1 must be defined for time at least T. If a flow line on E_1 is defined for a finite time only, then after we flow for almost all of that time, we must reach a point where the flow is defined only for a short time, less than T. Therefore all flow lines of all elements $A \in \mathfrak{g}$ are defined for all time on E_1, i.e. E_1 is complete. □

3. Monodromy

Definition 6. Take any connected manifold X. Let $\pi : \tilde{X} \to X$ be the universal covering map. Suppose that $\phi_0 : X \to M_0$ is any smooth map, and M_0 and M_1 bear G/H-geometries. Let $\tilde{\phi}_0 = \phi_0 \circ \pi : \tilde{X} \to M_0$. Lets assume that there is a development $\tilde{\phi}_1 : \tilde{X} \to M_1$. Then we have an isomorphism $\tilde{\phi}_0^* E_0 = \tilde{\phi}_1^* E_1$. The map $\tilde{\phi}_0^* E_0 \to \phi_0^* E_0$ is a covering map, with covering group $\pi_1(X)$. Moreover, $\pi_1(X)$ acts on $\tilde{\phi}_0^* E_0 = \tilde{\phi}_1^* E_1$ as bundle automorphisms over the deck transformations of $\tilde{X}$. We refer to this action as the *monodromy* of the development.

Picking a frame $\tilde{e}_0 \in \tilde{\phi}_0^* E_0$ and corresponding $\tilde{e}_1 \in \tilde{\phi}_1^* E_1$, and corresponding points $e_0 \in \phi_0^* E_0$, $\tilde{x} \in \tilde{X}$ and $x \in X$, we will examine the monodromy orbit of $\tilde{e}_1$.

Lemma 2. The monodromy of the development is a free and proper action preserving $\tilde{\phi}_0^* \omega$. Two elements of $\pi_1(X)$ have the same monodromy action on all of $\tilde{\phi}_1^* E_1$ just when they have the same monodromy action on some element of $\tilde{\phi}_1^* E_1$.

Proof. If two elements have the same effect on $\tilde{e}_1$, then composing one with the inverse of the other produces an element γ fixing $\tilde{e}_1$. The action of $\pi_1(X)$ commutes with the action of H, so γ fixes every element of the H-orbit through $\tilde{e}_1$. Moreover, γ fixes $\tilde{\phi}_1^* \omega$.

Take any path $p(t)$ in $\tilde{\phi}_1^* E_1$ with $p(0) = \tilde{e}_1$. Let $A(t) = \dot{p}(t) \lrcorner \tilde{\phi}_1^* \omega$. The only solution $q(t)$ to $\dot{q}(t) \lrcorner \tilde{\phi}_1^* \omega = A(t)$ satisfying $q(0) = \tilde{e}_1$ is $p(t)$. This differential equation and initial condition are γ invariant, and therefore all points of $p(t)$ are fixed by γ. Therefore γ fixes every point in the path component of the H-orbit of $\tilde{e}_1$, i.e. γ acts trivially on $\tilde{\phi}_1^* E_1$. □

Theorem 2. Take any connected manifold X with universal covering space $\tilde{X}$. Suppose that M_0 and M_1 are manifolds bearing G/H-geometries. Take $\phi_0 : X \to M_0$ any smooth map. Assume that there is a development of some covering space of X to M_1. Then a (possibly different) covering space $\hat{X} \to X$ develops from M_0 to M_1 just when the monodromy orbit of $\pi_1 \left(\hat{X} \right)$ on some point $\tilde{e}_1 \in \tilde{\phi}_1^* E_1$ maps to a single point in E_1.

Proof. We can replace X by $\hat{X}$ if needed to arrange that $X = \hat{X}$ without loss of generality. Clearly X develops from M_0 to M_1 just when $\tilde{\phi}_1$ is $\pi_1(X)$-invariant. Moreover, if X develops, then the monodromy orbit of $\pi_1(X)$ on any point $\tilde{e}_1$ must map to a single point in E_1. Suppose that the monodromy orbit of $\pi_1(X)$ on $\tilde{e}_1$ maps to a single point of E_1. Let $x \in X$ be the point of X which is the image of $\tilde{e}_1$ under the obvious bundle map $\tilde{\phi}_1^* E_1 \to X$. The map $\tilde{\phi}_1^* E_1 \to E_1$ is H-equivariant, so every monodromy orbit of $\pi_1(X)$ above x is mapped to a single point of E_1. Take a point $\tilde{e}_1 \in \tilde{\phi}_1^* E_1$, and suppose that $\tilde{e}_1$ maps to $e_1 \in E_1$. Take any smooth path $p(t)$ in $\tilde{\phi}_1^* E_1$ with $p(0) = \tilde{e}_1$. Let $A(t) = \dot{p}(t) \lrcorner \tilde{\phi}_1^* \omega$. The only solution $q(t)$ to $\dot{q}(t) \lrcorner \tilde{\phi}_1^* \omega = A(t)$ satisfying $q(0) = \tilde{e}_1$ is $p(t)$. Therefore in E_1, the only solution $q(t)$ to $\dot{q}(t) \lrcorner \omega = A(t)$ satisfying $q(0) = e_1$ is the image in E_1 of $p(t)$. The monodromy group will move $\tilde{e}_1$ around the monodromy orbit, moving $p(t)$ to another curve, but yielding the same solution curve $q(t)$ in E_1. Therefore the map $\tilde{\phi}_1^* E_1 \to E_1$ is invariant under $\pi_1(X)$ and drops to a smooth map on the quotient: $\pi_1(X) \backslash \tilde{\phi}_1^* E_1 \to E_1$.

The development $\tilde{\phi}_0^* E_0 = \tilde{\phi}_1^* E_1$ identifies $\phi_0^* E_0 = \pi_1(X) \backslash \tilde{\phi}_0^* E_0 = \pi_1(X) \backslash \tilde{\phi}_1^* E_1$. By equivariance under the action of H, our map descends to a smooth map $\phi_1 : X = \pi_1(X) \backslash \tilde{\phi}_1^* E_1 / H \to E_1 / H = M_1$. We need to define a development of $\phi_1 : X \to M_1$. We already have a development $\tilde{\Phi} : \tilde{\phi}_0^* E_0 \to \tilde{\phi}_1^* E_1$, which is $\pi_1(X)$-equivariant. Define $\Phi : \phi_0^* E_0 \to \phi_1^* E_1$ by quotienting $\tilde{\Phi}$ to $\pi(X)$ orbits. Because $\pi(X)$ acts freely and properly, the quotient map Φ is a smooth map. Moreover, locally each sheet of $\tilde{\phi}_0^* E_0$ is identified with $\phi_0^* E_0$ and each sheet of $\tilde{\phi}_1^* E_1$ is identified with $\phi_1^* E_1$. Locally, on each sheet, Φ is identified with $\tilde{\Phi}$. Clearly Φ is a local diffeomorphism, an H-bundle isomorphism, and satisfies $\omega_0 = \omega_1$, so a development. □

4. Rolling along Lipschitz CW-complexes

Definition 7. A locally Lipschitz/C^k/smooth/analytic (etc.) CW-complex is a CW-complex whose attaching maps are locally Lipschitz/C^k/smooth/analytic (etc.). A map between CW-complexes is locally Lipschitz/C^k/smooth/analytic (etc.) just when all of its restrictions to each simplex are.

For example, a manifold or real or complex analytic variety, possibly with boundary and corners, is a locally Lipschitz CW-complex. Clearly the proofs above only require that X be connected by Lipschitz curves, so hold equally well with X any connected locally Lipschitz CW-complex, rather than a manifold, the developed map ϕ_0 locally Lipschitz, and the isomorphism Φ locally essentially bounded. In particular, in theorem 1 on page 2, the curves we roll on can have arbitrary analytic singularities. One could instead try to uniformize the curve first, but then it would not be so clear that the development could be "deuniformized". Locally Lipschitz development is likely to be useful in developing calibrated cycles in studying Cartan geometries modelled on symmetric spaces.

5. Applications

Theorem 3. Suppose that M_0 and M_1 are manifolds and bear real/complex analytic G/H-geometries. Suppose that M_1 is complete. Every local development of a real/complex analytic map $\phi_0 : X \to M_0$ from a simply connected analytic variety X extends to a real/complex analytic development $\phi_1 : X \to M_1$. Moreover $\phi_0^* TM_0 = \phi_1^* TM_1$ are isomorphic vector bundles on X.

Proof. Clearly it is enough to extend the development along all curves. The local obstructions vanish by analyticity. The vector bundle isomorphism follows from lemma 1 on page 1. □

Corollary 1. Let M be a complex manifold bearing a complete complex analytic Cartan geometry. Each point of M lies on an immersed affine line. Moreover M contains a rational curve through every point just if the model G/H contains a rational curve, and otherwise M contains no rational curves.

For example, Kobayashi hyperbolic complex manifolds admit no complete complex analytic Cartan geometries.

Proof. We need only prove that G has a complex subgroup whose orbit in G/H is a complex curve. This is easy to see if G contains a semisimple group, since G then contains $\mathrm{SL}(2, \mathbb{C})$, and we can just examine the homogeneous spaces of $\mathrm{SL}(2, \mathbb{C})$ by examining closed subgroups. If G contains no semisimple group, then G is solvable and contains a complex abelian subgroup of positive dimension, and the result is obvious. □

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